R-SEQUENCES AND APPLICATIONS

by
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Abstract. In this paper the *R*-sequences are defined. The main result shows that the sequence $(a_n)_{n\geq 1}, a_n = \frac{1}{n^p}$, where p > 0, defining the Riemann zeta function, is not an R-sequence.

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A sequence $(a_n)_{n\geq 1}$ of real numbers is called a **rational sequence** or shortly *R*-sequence if there exists a rational function *R* such that for any positive integer $n \geq 1$ the following relation holds

$$a_1 + a_2 + ... + a_n = R(n) \qquad (1)$$

Example 1. The sequence $(x_n)_{n\geq 1}, x_n = \frac{1}{n(n+1)}$, is an *R*-sequence. Indeed, for any positive integer $n \geq 1$, we have $x_1 + x_2 + ... + x_n = R(n)$, where $R(x) = \frac{x}{x+1}$.

Example 2. (Romanian Mathematical Olympiad, [1, pp. 8 and 53], [4, pp. 170 and 514]) The sequence $(y_n)_{n\geq 1}, y_n = \frac{1}{n!}$, is not an *R*-sequence.

The argument follows by contradiction. Assume that for any positive integer $n \geq 1$

$$\frac{1}{1!} + \frac{1}{2!} + ... + \frac{1}{n!} = \frac{A(n)}{B(n)} = R(n)$$

where $A, B \in IR[x]$ and $\deg(A) = k, \deg(B) = m$. From $\lim_{n\to\infty} \frac{A(n)}{B(n)} = e$ it follows that $k = m$. Consider the polynomial function given by $Q(x) = A(x+1)B(x) - A(x)B(x+1)$. It is clear that

$$\lim_{x\to\infty} \frac{Q(x+1)}{Q(x)} = \lim_{n\to\infty} \frac{Q(n+1)}{Q(n)} = 1 \qquad (2)$$

Taking into account that $A(n+1) = R(n+1)B(n+1)$ and $A(n) = R(n)B(n)$, we obtain

$$\begin{aligned}Q(n) &= A(n+1)B(n) - B(n+1)A(n) = \\ &= R(n+1)B(n+1)B(n) - R(n)B(n)B(n+1) = \\ &= B(n)B(n+1)(R(n+1) - R(n)) = \frac{B(n)B(n+1)}{(n+1)!}\end{aligned}$$

Therefore

$$\frac{Q(n+1)}{Q(n)} = \frac{B(n+1)B(n+1)}{B(n)B(n+1)} \cdot \frac{(n+1)!}{(n+2)!} = \frac{B(n+2)}{B(n)} \cdot \frac{1}{n+2}$$

and

$$\lim_{n\to\infty} \frac{Q(n+1)}{Q(n)} = \lim_{n\to\infty} \frac{B(n+2)}{B(n)} \cdot \lim_{n\to\infty} \frac{1}{n+2} = 1 \cdot 0 = 0$$

relation which contradicts (2).

Example 3. The sequence $(z_n)_{n\geq 1}, z_n = \frac{1}{n}$, is not an R-sequence.

If for any positive integer $n \geq 1$

$$1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n} = \frac{P(n)}{Q(n)} \qquad (3)$$

where $P, Q \in IR[x]$, then from well-known relation

$$\lim_{n\to\infty}\left(1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n}\right) = +\infty$$

it follows $\deg(P) > \deg(Q)$, i.e. $\deg(P) \geq \deg(Q) + 1$.

On the other hand, from (3) one obtains

$$\frac{1 + \frac{1}{2} + ... + \frac{1}{n}}{n} = \frac{P(n)}{nQ(n)} \qquad (4)$$

By using Cesaro' Lemma we have

$$\lim_{n\to\infty} \frac{1 + \frac{1}{2} + ... + \frac{1}{n}}{n} = \lim_{n\to\infty} \frac{1}{n} = 0$$

and from (4) it follows $\lim_{n\to\infty}\frac{P(n)}{nQ(n)}=0$, i.e. $\deg(P)<\deg(Q)+1$ in contradiction with the relation $\deg(P)\geq\deg(Q)+1$.

The main purpose of this present paper is to prove that the sequence $(a_n)_{n\geq 1}, a_n=\frac{1}{n^p}$, where p > 0, is not an *R*-sequence.

It is well-known that $\sum_{n=1}^{\infty}\frac{1}{n^p}$ is divergent when $p\in(0,1]$ and

$$\sum_{n=1}^{\infty}\frac{1}{n^p}=\varsigma(p) \qquad (5)$$

where ς is the Riemann function, when $p>1$.

Lemma 1. Consider $f:[1,\infty)\to[0,\infty)$ a continuous and decreasing function. Let *F* be a differentiable function such that its derivative is *f*. Then the sequence $(x_n)_{n\geq 1}$, given by

$$x_n=f(1)+f(2)+\ldots+f(n)-F(n), n\geq 1 \quad (6)$$

is convergent.

Proof. Applying Lagrange' Theorem to *F* on the interval $[k,k+1], k\geq 1$, it follows that there exists $c_k\in(k,k+1)$ such that

$$F(k+1)-F(k)=f(c_k)$$

By using the monotony of function *f* we obtain

$$f(k+1)\leq F(k+1)-F(k)\leq f(c_k) \qquad (7)$$

Taking *k = 1, 2, . . . , n* in (7) and adding all these inequalities we get

$$f(2)+f(3)+\ldots+f(n+1)\leq F(k+1)-F(k)\leq f(1)+f(2)+\ldots+f(n) \qquad (8)$$

On the other hand let us note that

$$x_{n+1}-x_n=F(n)-F(n+1)+f(n+1)\geq 0, n\geq 1$$

that is the sequence $(x_n)_{n\geq 1}$ is increasing.
From the left inequality in (8) we obtain

$$x_n=f(1)+f(2)+...+f(n)-F(n)\leq f(1)-F(1), n\geq 1$$

that is the sequence $(x_n)_{n\geq 1}$ is upper bounded.

Lemma 2. (Stolz-Cesaro' Theorem, the case 0/0).
Let $(\alpha_n)_{n\geq 1},(\beta_n)_{n\geq 1}$ be two se-quences of real numbers satisfying the following hypothesis:

1) $\lim_{n\to\infty}\alpha_n=0$ and $\lim_{n\to\infty}\beta_n=0$;

2) The sequence $(\beta_n)_{n\geq 1}$ is strict decreasing (or strict increasing);

3) There exists $\lim_{n\to\infty}\frac{\alpha_{n+1}-\alpha_n}{\beta_{n+1}-\beta_n}=A$ (finite or not).

Then $(\alpha_n/\beta_n)_{n\geq 1}$ is convergent and $\lim_{n\to\infty}\frac{\alpha_n}{\beta_n}=A$.

For the proof of this variant of well-known Stolz-Cesaro' theorem we refer to the papers
[3] and [2].
Let now state our main result.

Theorem. The sequence $(a_n)_{n\geq 1}, a_n=\frac{1}{n^p}$, where $p > 0$, is not an *R*-sequence.

Proof. We will consider few situations on $p > 0$.

Case 1: $p\in(0,1)$. Assume that there exists a rational function *R* such that for any $n\geq 1$

$$\frac{1}{1^p}+\frac{1}{2^p}+...+\frac{1}{n^p}=R(n)$$

From Lemma 1 we obtain that the sequence $(x_n)_{n\geq 1}$, where

$$x_n = \frac{1}{1^p} + \frac{1}{2^p} + ... + \frac{1}{n^p} - \frac{1}{(1-p)n^{p-1}}$$

is convergent. Therefore the limit

$$\lim_{n\to\infty}\left(R(n) - \frac{1}{(1-p)n^{p-1}} \right) \qquad (9)$$

is finite. But

$$\lim_{n\to\infty} R(n) = \lim_{n\to\infty}\left(\frac{1}{1^p} + \frac{1}{2^p} + ... + \frac{1}{n^p} \right) = +\infty$$

implies the relation $R = P + R_1$, where P is a polynomial function and R_1 is a rational function with $\lim_{n\to\infty} R_1(n) = 0$. It follows

$$\lim_{n\to\infty}\left(R(n) - \frac{1}{1-p} n^{p-1} \right) = \lim_{n\to\infty}\left(P(n) - \frac{1}{1-p} n^{p-1} \right) = +\infty$$

by contradicting the finiteness of limit (9).

Case 2: $p = 1$. We already proved this case in Example 3. Now we will indicate a different argument.
As in the previous case, the sequence $(y_n)_{n\geq 1}$

$$y_n = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n} - \ln n$$

is convergent. Therefore $\lim_{n\to\infty}(R(n) - \ln n)$ is finite. But

$$\lim_{n\to\infty}(R(n) - \ln n) = \lim_{n\to\infty}(P(n) - \ln n) = +\infty$$

which is a contradiction.
Case 3: $p > 1$, $p \notin Z_+$. In this situation, the sequence $(z_n)_{n\geq 1}$

$z_n = \frac{1}{1^p} + \frac{1}{2^p} + ... + \frac{1}{n^p}$ is convergent to $\varsigma(p)$. Then

$$\lim_{n\to\infty}(z_n - \varsigma(p)) = \lim_{n\to\infty}(R(n) - \varsigma(p)) = 0$$

It follows that for any $n \geq 1$ the relation holds

$$R(n) - \varsigma(p) = \frac{P_1(n)}{Q_1(n)}$$

where P_1, Q_1 are polynomial functions and $\deg(P_1) < \deg(Q_1)$.
On the other hand, there exists a positive integer *k* such that $\deg(x^k P_1) = \deg(Q_1)$.
It follows that the limit

$$\lim_{n\to\infty} \frac{n^k P_1(n)}{Q_1(n)}$$

is finite and different from zero. Therefore, the limit

$$\lim_{n\to\infty} n^k \left(1 + \frac{1}{2^p} + \ldots + \frac{1}{n^p} - \varsigma(p)\right) \qquad (10)$$

is finite and different from zero.
But, we can write the limit (10) in the following way by using Lemma 2:

$$\lim_{n\to\infty} n^k \left(1 + \frac{1}{2^p} + \ldots + \frac{1}{n^p} - \varsigma(p)\right) = \lim_{n\to\infty} \frac{\frac{1}{(n+1)^p}}{\frac{1}{(n+1)^k} - \frac{1}{n^k}} =$$

$$= -\lim_{n\to\infty} \frac{n^k (n+1)^k}{(n+1)^p \left[(n+1)^k - n^k\right]} = -\lim_{n\to\infty} \frac{n^k (n+1)^k}{(n+1)^p \left[\binom{k}{1} n^{k-1} + \ldots\right]}$$

The last limit is finite and different from zero if and only is $p + k - 1 = 2k$, i.e. if and only if $p = k + 1$. This relation is not possible since *p* is not an integer.

Case 4: *p > 1*, $p \in Z$. Suppose that for any integer $n \geq 1$

$$\frac{1}{1^p} + \frac{1}{2^p} + \ldots + \frac{1}{n^p} = R(n)$$

where $R = \frac{P}{Q}$ and $\gcd(P, Q) = 1$.

It follows $R(n+1) - R(n) = \frac{1}{(n+1)^p}$, i.e.

$\frac{P(n+1)}{Q(n+1)} - \frac{P(n)}{Q(n)} = \frac{1}{(n+1)^p}, n \geq 1$.

This is equivalent to

$$(n+1)^p \left(P(n+1)Q(n) - P(n)Q(n+1)\right) = Q(n)Q(n+1), n \geq 1.$$

It is necessary to have the equality

$$(x+1)^p\left(P(x+1)Q(x)-P(x)Q(x+1)\right)=Q(x)Q(x+1) \qquad (11)$$

for any $x\in R$.

Denote $U(x)=\gcd(Q(x),Q(x+1))$ and obtain

$$Q(x)=R_1(x)U(x),\quad Q(x+1)=R_2(x)U(x)$$

where . The relation (11) is equivalent to

$$(x+1)^p\left(P(x+1)R_1(x)-P(x)R_2(x+1)\right)=U(x)R_1(x)R_2(x)$$

Because R_1, R_2 are relatively prime, it follows that at least one of then is relatively prime to $(x+1)^p$. Let say that the polynomial R_1 has this property. Then R_1 divides the polynomial $P(x+1)R_1(x)-P(x)R_2(x)$, i.e. R_1 divides *P*. Taking into account that $R_1|Q$ and $\gcd(P,Q)=1$, it follows that R_1 is constant.
From the equality $Q(x)=R_1(U(x))$ it follows that $Q(x+1)=R_1U(x+1)$. Combining with $Q(x+1)=R_2(x)U(x)$ one obtains
the relation

$$U(x+1)=\frac{R_2(x)}{R_1}U(x).$$

But $\deg(U(x+1))=\deg(U(x))$ implies R_2 is also constant and one obtains $Q(x+1)=\alpha Q(x), x\in R$, where α is a constant. The last relation implies that *Q* is constant and we get

$$\frac{1}{1^p}+\frac{1}{2^p}+...+\frac{1}{n^p}=kP(n), n\geq 1$$

and

$$\lim_{n\to\infty}\left(1+\frac{1}{2^p}+...+\frac{1}{n^p}\right)=\pm\infty$$

a contradiction.

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