A PRESERVING PROPERTY OF THE ALEXANDER OPERATOR

by

Mugur Acu and Dorin Blezu

Abstract. The $H(U)$ be the set of functions which are regular in the unit disc U and

$$A = \left\{ f \in H(U),\ f(0) = 0,\ f'(0) = 1 \right\}.$$

Let denote by I the Alexander operator $I : A \to H(U)$ as:

$$f(z) = IF(z) = \int_0^z \frac{F(t)}{t} dt \quad (1).$$

The purpose of this note is to show that the n-uniform starlike functions of order γ and type α and the n-uniform close to convex functions of order γ and type α are preserved by the Alexander operator.

1. Preliminary results

Definition 2.1. Let D^n be the Sălăgean differential operator defined as:

$D^n : A \to A, \quad n \in N$ and

$D^0 f(z) = f(z)$

$D^1 f(z) = Df(z) = zf'(z)$

$D^n f(z) = D\left(D^{n-1} f(z)\right).$

Definition 2.2. [2] Let $f \in A$, we say that f is n-uniform starlike function of order γ and type α if

$$\operatorname{Re}\left(\frac{D^{n+1} f(z)}{D^n f(z)}\right) \geq \alpha \cdot \left|\frac{D^{n+1} f(z)}{D^n f(z)} - 1\right| + \gamma, \quad z \in U$$

where $\alpha \geq 0$, $\gamma \in [-1, 1)$, $\alpha + \gamma \geq 0$, $n \in N$. We denote this class with $US_n(\alpha, \gamma)$.

Remark 2.1. Geometric interpretation: $f \in US_n(\alpha, \gamma)$ if and only if $\frac{D^{n+1} f(z)}{D^n f(z)}$ take all values in the convex domain included in right half plane $\Delta_{\alpha,\gamma}$, where $\Delta_{\alpha,\gamma}$ is a eliptic region for $\alpha > 1$, a parabolic region for $\alpha = 1$, a hyperbolic region for $0 < \alpha < 1$, the half plane $u > \gamma$ for $\alpha = 0$.

Definition 2.3. [1] Let $f \in A$, we say that f is n-uniform close to convex function of order γ and type α in respect to the functions n-uniform starlike of order γ and type α $g(z)$, where $\alpha \geq 0$, $\gamma \in [-1, 1)$, $\alpha + \gamma \geq 0$, if

$$\operatorname{Re}\left(\frac{D^{n+1} f(z)}{D^n g(z)}\right) \geq \alpha \cdot \left|\frac{D^{n+1} f(z)}{D^n g(z)} - 1\right| + \gamma , \quad z \in U$$

where $\alpha \geq 0$, $\gamma \in [-1, 1)$, $\alpha + \gamma \geq 0$, $n \in N$. We denote this class with $UCC_n(\alpha, \gamma)$.

Remark 2.2. Geometric interpretation: $f \in UCC_n(\alpha, \gamma)$ if and only if $\frac{D^{n+1} f(z)}{D^n g(z)}$ take all values in the convex domain included in right half plane $\Delta_{\alpha, \gamma}$, where $\Delta_{\alpha, \gamma}$ is a eliptic region for $\alpha > 1$, a parabolic region for $\alpha = 1$, a hiperbolic region for $0 < \alpha < 1$, the half plane $u > \gamma$ for $\alpha = 0$.

The next two theorems are results of the so called “admissible functions method” introduced by P. T. Mocanu and S. S. Miller (see [3], [4], [5]).

Theorem 2.1. Let h convex in U and $\operatorname{Re}[\beta h(z) + \gamma] > 0$, $z \in U$. If $p \in H(U)$ with $p(0) = h(0)$ and p satisfied the Briot-Bouquet differential subordination

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h(z), \text{ then } p(z) \prec h(z).$$

Theorem 2.2. Let q convex in U and $j : U \to C$ with $\operatorname{Re}[j(z)] > 0$. If $p \in H(U)$ and p satisfied $p(z) + j(z) \cdot zp'(z) \prec q(z)$, then $p(z) \prec q(z)$.

2. Main results

Theorem 3.1. If $F(z) \in US_n(\alpha, \gamma)$, with $\alpha \geq 0$ and $\gamma > 0$, then $f(z) = IF(z) \in US_n(\alpha, \gamma)$ with $\alpha \geq 0$ and $\gamma > 0$.

Proof. We know that $F(z) \in US_n(\alpha, \gamma)$ if and only if $\frac{D^{n+1} F(z)}{D^n F(z)}$ take all values in the convex domain included in right half plane $\Delta_{\alpha, \gamma}$.

By differentiating $f(z) = IF(z) = \int_0^z \frac{F(t)}{t} dt$ we obtain: $F(z) = zf'(z)$.

By means of the application of the linear operator D^{n+1} we obtain:

$$D^{n+1}F(z) = D^{n+1}\left(zf'(z)\right) \text{ or } D^{n+1}F(z) = D^{n+2}f(z).$$

Similarly, by means of the application of the linear operator D^n we obtain:

$$D^nF(z) = D^{n+1}f(z).$$

Thus

$$\frac{D^{n+1}F(z)}{D^nF(z)} = \frac{D^{n+2}f(z)}{D^{n+1}f(z)} \quad (2).$$

With notation $\frac{D^{n+1}f(z)}{D^nf(z)} = p(z)$, where $p(z) = 1 + p_1z + ...$ we have

$$zp'(z) = \frac{D^{n+2}f(z) \cdot D^nf(z) - \left(D^{n+1}f(z)\right)^2}{\left(D^nf(z)\right)^2}.$$

$$\frac{1}{p(z)} \cdot zp'(z) = \frac{D^{n+2}f(z)}{D^{n+1}f(z)} - p(z).$$

From here we obtain:

$$\frac{D^{n+2}f(z)}{D^{n+1}f(z)} = p(z) + \frac{1}{p(z)} \cdot zp'(z).$$

Thus from (2) we obtain:

$$\frac{D^{n+1}F(z)}{D^nF(z)} = p(z) + \frac{1}{p(z)} \cdot zp'(z) \quad (3).$$

If we consider $h \in H_u(U)$, with $h(0) = 1$, which maps the unit disc into the convex domain included in right half plane $\Delta_{\alpha,\gamma}$, then from $\frac{D^{n+1}F(z)}{D^nF(z)}$ take all values in $\Delta_{\alpha,\gamma}$, using (3) we obtain:

$$p(z) + \frac{1}{p(z)} \cdot zp'(z) \prec h(z)$$

where, from her construction, we have $\operatorname{Re} h(z) > 0$ and from theorem (2.1) we obtain $p(z) \prec h(z)$. From here follows that $\frac{D^{n+1} f(z)}{D^n f(z)}$ take all values in the convex domain included in right half plane $\Delta_{\alpha,\gamma}$, or $f(z) = IF(z) \in US_n(\alpha,\ \gamma)$, with $\alpha \geq 0$ and $\gamma > 0$.

Theorem 3.2. If $F(z) \in UCC_n(\alpha,\ \gamma)$, in respect to the function n-uniform starlike of order γ and type α $G(z)$, with $\alpha \geq 0$ and $\gamma > 0$, then $f(z) = IF(z) \in UCC_n(\alpha,\ \gamma)$ in respect to the function n-uniform starlike of order γ and type α, see theorem (3.1), $g(z) = IG(z)$ with $\alpha \geq 0$ and $\gamma > 0$.

Proof. We know that $F(z) \in UCC_n(\alpha,\ \gamma)$ if and only if $\frac{D^{n+1} F(z)}{D^n G(z)}$ take all values in the convex domain included in right half plane $\Delta_{\alpha,\gamma}$.

By differentiating (1) we obtain:

$$F(z) = zf'(z) \quad \text{and} \quad G(z) = zg'(z)$$

By means of the application of the linear operator D^{n+1} we obtain:

$$D^{n+1} F(z) = D^{n+2} f(z) \quad \text{and} \quad D^n G(z) = D^{n+1} g(z).$$

With simple calculation we obtain:

$$\frac{D^{n+1} F(z)}{D^n G(z)} = \frac{D^{n+2} f(z)}{D^{n+1} g(z)} \qquad (4).$$

With notation $\frac{D^{n+1} f(z)}{D^n g(z)} = p(z)$, and $\frac{D^{n+1} g(z)}{D^n g(z)} = h(z)$ it follows that:

$$\frac{D^{n+2} f(z)}{D^{n+1} g(z)} = p(z) + \frac{1}{h(z)} \cdot zp'(z).$$

Thus from (4) we obtain:

$$\frac{D^{n+1} F(z)}{D^n G(z)} = p(z) + \frac{1}{h(z)} \cdot zp'(z) \qquad (5).$$

If we consider q convex in unit disc U, which maps the unit disc into the convex domain included in right half plane $\Delta_{\alpha,\gamma}$, then from $\frac{D^{n+1} F(z)}{D^n G(z)}$ take all values in $\Delta_{\alpha,\gamma}$, using (5) we obtain:

$$p(z) + \frac{1}{h(z)} \cdot zp'(z) \prec q(z)$$

where, from her construction, we have $\operatorname{Re} h(z) > 0$. From here follows that $\operatorname{Re}\frac{1}{h(z)} > 0$. In this conditions from theorem (2.2) we obtain $p(z) \prec q(z)$. From here follows that $\frac{D^{n+1} f(z)}{D^n g(z)}$ take all values in the convex domain included in right half plane $\Delta_{\alpha,\gamma}$; or $f(z) = IF(z) \in UCC_n(\alpha, \gamma)$, in respect to $g(z) = IG(z) \in US_n(\alpha, \gamma)$ with $\alpha \geq 0$ and $\gamma > 0$.

3. Some particular cases

1. From theorem (3.1), for n=1, we obtain that the integral operator (1) preserved the class $US^c(\alpha, \gamma)$, with $\gamma > 0$, of uniform convex of type α and of order γ functions, introduced by I. Magdaş.
2. From theorem (3.1), for n=1, $\gamma = 0$ we obtain that the integral operator (1) preserved the class $US^c(\alpha)$, of uniform convex of type α functions, introduced by S. Kanas and A. Visniowska.
3. From theorem (3.1), for n=1, $\gamma = 0$, $\alpha = 1$, we obtain that the integral operator (1) preserved the class US^c, of uniform convex function, introduced by A. W. Goodman, and studied by W. Ma and D. Minda.
4. From theorem (3.1), for n=1, $\alpha = 1$, we obtain that the integral operator (1) preserved the class $US^c[\gamma]$, with $\gamma > 0$, of uniform convex of order γ functions, introduced by F. Ronning.
5. From theorem (3.1), for n=0, $\alpha = 1$, we obtain that the integral operator (1) preserved the class $SP\left(\frac{1-\gamma}{2}, \frac{1+\gamma}{2}\right)$, introduced by F. Ronning.
6. From theorem (3.2), for $\gamma = 0$, we obtain that the integral operator (1) preserved the class $UCC_n(\alpha)$, of n-uniform close to convex of type α in respect to a n-uniform starlike of type α function, introduced by D. Blezu.

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Authors:
Mugur Acu, Dorin Blezu, "Lucian Blaga" University of Sibiu