CHARACTERIZATIONS OF THE FUNCTIONS WITH BOUNDED VARIATION

by
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Abstract. The present study concerns the class of the functions with bounded variations and its relations with other classes of well-known characterized functions.

Definition 1.1. A function f : [a,b] $\to$ **R** is called with bounded variation on [a,b] if there is M > 0 such that for any partition Δ = ($a = x_o < x_1 < \dots < x_n = b$) of the interval [a,b] we have:

$$\bigvee_a^b (f) = \sup \{V_\Delta(f) \mid \Delta \text{ division of } [a,b]\}$$

is called the total variation of the function f on the interval [a,b].

Remarks.

i) the concept of the function with bounded variation has sense only on compact intervals;
ii) the definition can easily be extended when the function takes values in a metric space.

For a better understanding of the class of functions with bounded variation one finds appropriate to list a series of well-known results, see for instance [1], the proof being reserved only for no classical results.

1.PROPERTIES OF THE FUNCTIONS WITH BOUNDED VARIATION

Proposition 2.1. A function f : [a,b] $\to$ **R** is constant if and only if f is a function with bounded variation and $\bigvee_a^b (f) = 0$.

Proposition 2.2. A function f : [a,b] $\to$ **R** is monotonic if and only if f is a function with bounded variation and $\bigvee_a^b (f) = |f(b) - f(a)|$.

Theorem 2.1. The set of the functions with bounded variation on a given compact interval forms an algebra which is not closed. Furthermore, the set of functions with bounded variation and with nonzero values on a given compact interval forms a commutative field.

Proposition 2.3. Let f : [a,b] $\rightarrow$ **R** be a function with bounded variation on [a,b] and let V: [a,b] $\rightarrow$ **R** be a function defined as:

$$V(x) = \bigvee_a^x (f) \text{ for any } x \in [a,b].$$

Then V and V – f are increasing functions and satisfy the following inequality:

$$\int_a^b V^2(x)dx - \frac{1}{b-a}\left(\int_a^b V(x)dx\right)^2 \geq \int_a^b V(x)f(x)dx - \frac{1}{b-a}\left(\int_a^b V(x)dx\right)\left(\int_a^b f(x)dx\right)$$

Proof. Let x, y $\in$ [a,b] such that x > y. Then V(x) – V(y) = $\bigvee_y^x (f)$ $\geq$ 0 and V(x) – f(x) – V(y) + f(y) = $\bigvee_y^x (f)$ – f(x) + f(y) $\geq$ 0. Hence, V and V – f are increasing functions. Applying the Cebîşev's inequality for the functions V and V – f results in:

$$\int_a^b (V(x) - f(x))V(x)dx \;\geq\; \frac{1}{b-a}\left(\int_a^b V(x)dx\right)\left(\int V(x)dx - \int_a^b f(x)dx\right)$$

which is in fact the requested inequality.

Theorem 2.2. (Jordan) A function is with bounded variation if and only if it can be represented as the difference of two increasing (decreasing) functions.

Remarks.

i) the two monotonic functions can be taken positive (by adding a sufficiently large constant) and continuous if in addition the initial function is continuous;

ii) the decomposition is not unique.

Corollary 2.1. If a function is with bounded variation is not continuous it has only discontinuities of the first kind.

Corollary 2.2. (Froda) The set of the discontinuity points of a function with bounded variation is at most countable.

Corollary 2.3. A function with bounded variation is Riemann integrable. The reciprocal is false and a contraexemple is given by the function:

$$f : [0,2] \to \mathbf{R}, \quad f(x) = \begin{cases} x \sin(\pi/x) & \text{if} \quad 0 < x \le 2 \\ 0 & \text{if} \quad x = 0 \end{cases}$$

which is Riemann integrable on [0,2] being continuous, but is not with bounded variation on [0,2].

Corollary 2.4. (Lebesque) A function with bounded variation is almost everywhere derivable.

Observation. Based on corollary 2.4., a class of continuous functions which are not (even locally) with bounded variation is the class of continuous functions nowhere derivable.

Proposition 2.4. A function with with bounded variation is bounded. The reciprocal is false and a contraexemple is given by the Dirichlet function, namely

$$f : [a,b] \to \mathbf{R}, \quad f(x) = \begin{cases} 0, & \text{if} \quad x \in [a,b] \cap Q \\ 1, & \text{if} \quad x \in [a,b] \cap (R - Q) \end{cases}$$

which is bounded, but is not with bounded variation from corollary 2.2.

Observation. The function $f : [0,1] \to \mathbf{R}$

$$f(x) = \begin{cases} 1/x, & \text{if} \quad x \in (0,1] \\ 0, & \text{if} \quad x = 0 \end{cases}$$

is not with bounded variation being unbounded but this example shows that a monotonic function on a noncompact interval, namely $g : (0,1] \to \mathbf{R}$, $g(x)=1/x$ could be structurally far away from a function with bounded variation.

Proposition 2.5. Let $f,g : [a,b] \to \mathbf{R}$ and $K > 0$, $c \ge 1$ be with the property that

$$| f(x) - f(y) | \le K| g(x) - g(y) |^c \text{ for any } x, y \in [a,b].$$

If g is with bounded variation then f is with bounded variation.

Proof. From proposition 2.4. there is $M > 0$such that $|g(x)| < M$ for any $x \in [a,b]$. Then for any division $\Delta = (a = x_o < x_1 < \ldots < x_n = b)$ of the interval [a,b] we have:

$$V_\Delta(f) = \sum_{i=0}^{n-1} \left| f(x_{i-1}) - f(x_i) \right| \le \sum_{i=0}^{n-1} K \left| g(x_{i+1}) - g(x_i) \right|^c \le$$

$$\leq \text{K}\,(2\text{M})^{\text{c-1}}\sum_{i=0}^{n-1} K\left|g(x_{i+1}) - g(x_i)\right| \;\leq\; \text{K}\,(2\text{M})^{\text{c-1}}\bigvee_a^b (g).$$

Hence, f is with bounded variation on [a, b].

Corollary 2.5. A Lipschitz function is with bounded variation.

Corollary 2.6. Let Φ : [a,b] $\rightarrow$ **R** be a Riemann integrable function. Then the function f : [a,b] $\rightarrow$ **R** defined as:

$$\text{f(x)} = \int_a^x \Phi(t)dt \quad \text{for any x} \in \text{[a,b]}$$

is with bounded variation.

Corollary 2.7. A derivable function with bounded derivative on [a, b] is with bounded variation on [a, b].

Theorem 2.3. A derivable function f : [a,b] $\rightarrow$ **R** with integrable derivative is with bounded variation and $\bigvee_a^b (f) = \int_a^b |\, f'(x) \,|\, dx$.

2.RELATIONS WITH OTHER CLASSES OF FUNCTIONS

Proposition 3.1. Let [a,b] $\xrightarrow{\text{f}}$ [c,d] $\xrightarrow{\text{g}}$ **R** be functions with the properties that f is with bounded variation on [c, d] and g is monotonic on [a, b]. Then the composition $\text{f} \circ \text{g}$ is with bounded variation on [a, b].

Proof. Suppose g is an increasing function. Since f is with bounded variation, from Jordan's theorem there are Φ and Ψ increasing functions such that f = $\Phi - \Psi$. Then $\text{f} \circ \text{g} = \Phi \circ \text{g} - \Psi \circ \text{g}$ and $\Phi \circ \text{g}$ and $\Psi \circ \text{g}$ are increasing functions. Finally, from Jordan's theorem, results that $\text{f} \circ \text{g}$ is with bounded variation on [a, b].

Proposition 3.2. Let f : [a,b] $\rightarrow$ **R** be a function with Darboux property such that |f | is with bounded variation. Then f is continuous.

Proof. It is easy to observe that, since f has Darboux property, then |f | has this property as well. First we shall prove that |f | is continuous. Assume, by contradiction, that there is x_0 a discontinuity point for |f |.

Since $|f|$ is with bounded variation, from corollary 2.1., results that x_0 is a discontinuity point of the first kind for $|f|$. This conclusion is in contradiction with the fact that $|f|$ has Darboux property since such a function cannot have discontinuity of the first kind. Hence, $|f|$ is continuous. Assume again, by contradiction, that there is y_0 a discontinuity point for f. Since f has Darboux property results that y_0 is a point of discontinuity of the second kind for f and thus y_0 is yet a point of discontinuity for $|f|$ for which we have proved that is continuous. This is a contradiction and hence, f is continuous.

Observation. We recall that a function with bounded variation and with Darboux property is necessary continuous. This observation will be referred as "analysis of discontinuities". The following corollaries can easily be proved from this analysis.

Corollary 3.1. Let a, x_0 and b be real numbers such that $a < x_0 < b$ and let $f : [a,b] \rightarrow \mathbf{R}$ be a function with the following properties:

i) f is local with bounded variation at x_0, i.e. there is a compact interval included in [a, b] and containing in interior the point x_0 on which f is with bounded variation.

ii) f possesses primitives on (a, x_0) and (x_0, b).

Then f possesses primitives on (a, b) if and only if f is continuous at x_0.

Corollary 3.2. Let $f : [a,b] \rightarrow I$ be a function, where I is an interval included in [a, b]. Then:

i)If $f \circ f \circ ... \circ f$ is discontinuous and with bounded variation then f has not Darboux property.

ii) If f has Darboux property and $f \circ f$ is with bounded variation then $f^{(2n)} = \underbrace{f \circ f \circ ... \circ f}_{(2n)\text{times}}$ is continuous.

Corollary 3.3. A function with bounded variation which can be represented as a ratio of two functions possessing primitives is continuous.

Lemma 3.1. Let $f : [a,b] \rightarrow \mathbf{R}$ be a continuous one to one function and let $g : [a,b] \rightarrow \mathbf{R}$ be a function possessing primitives. Then the product $f \cdot g$ possesses primitives.

Proof. Since f is one-to-one there is an left-inverse f^{-1}, say, such that $(f^{-1} \circ f)(x) = x$ for any $x \in [a, b]$. In addition, since f is continuous results that f^{-1} is continuous. Let G be a primitive of the function g, so G' = g. then the function $G \circ f^{-1}$ is continuous and hence possesses primitives. Let H be such a primitive, so $H' = G \circ f^{-1}$. We prove now that the function $f \cdot g$ possesses primitives by showing that the function $T : [a,b]$

$\to$ **R**, defined as T(x) = f(x)·G(x) – ($H \circ f$)(x) is derivable and its derivative is the function f · g. for this, let y be an arbitrary point in [a, b]. Then,

$$\lim_{x\to y}\frac{T(x)-T(y)}{x-y} =$$

$$\lim_{x\to y}\frac{f(x)G(x)-H(f(x))-f(y)G(y)+H(f(y))+f(x)G(y)-f(x)G(y)}{x-y} =$$

$$\lim_{x\to y} f(x)\cdot\left(\frac{G(x)-G(y)}{x-y}\right) + \lim_{x\to y}\left[G(y)\cdot\left(\frac{f(x)-f(y)}{x-y}\right)-\frac{H(f(x))-H(f(y))}{x-y}\right].$$

We apply now Lagrange's mean value theorem to the function H on the interval [f(x), f(y)]. Thus there is $\xi \in$ [f(x), f(y)] such that:

$$H(f(x)) - H(f(y)) = \left(f(x)-f(y)\right)\cdot G(f^{-1}(\xi)),$$

whence it exists $\eta_x \in$ [x, y] such that

$$\frac{H(f(x))-H(f(y))}{x-y} = G(\eta_x).$$

Then the limit calculated above becomes:

$$\lim_{x\to y}\frac{T(x)-T(y)}{x-y} = f(y)g(y)+ \lim_{x\to y}\left[\left(f(x)-f(y)\right)\cdot\frac{G(x)-G(\eta_x)}{x-y}\right] =$$

$$f(y)g(y)+ + \lim_{x\to y}\left(f(x)-f(y)\right)\cdot\frac{y-\eta_x}{x-y}\cdot g(y) = f(y)\cdot g(y).$$

So, T is derivable and T' = f · g, hence f · g possesses primitives.

Lemma 3.2. Let f : [a,b] $\to$ **R** be a continuous monotonic function and let g : [a,b] $\to$ **R** be a function possesses primitives.

Proof. Suppose f is an increasing function on [a, b]. Then the function h : [a,b] $\to$ **R**, defined as h(x) = f(x) + x, for any $x \in$ [a, b] is a strictly increasing function and hence a one-to-one function. From lemma 3.1. it results that the function p : [a,b] $\to$ **R** defined as:

$$p(x) = h(x)g(x) = (f(x) + x)g(x) \text{ for any } x \in [a, b]$$

possesses a primitive P, say, such that P' = p. Also, from lemma 3.1.. it results that the function q : [a,b] $\to$ **R** defined as:

$$q(x) = x \cdot g(x) \quad \text{for any } x \in [a, b]$$

possesses a primitive Q, say, such that Q' = q.

It is obvious now that f $\cdot$ g = p-q and thus a primitive of the product f $\cdot$ g is the derivable function P – Q.

Theorem 3.1. Let f : [a,b] $\to$ **R** be a continuous function with bounded variation and let g : [a,b] $\to$ **R** be a function possesses primitives. Then the product f $\cdot$ g possesses primitives.

Proof. Since f is a continuous function with bounded variation, from Jordan's theorem, there are Φ and Ψ two continuous increasing functions such that f = Φ – Ψ. Then f $\cdot$ g = Φ $\cdot$ g - Ψ $\cdot$ g. now using lemma 3.2 it results that Φ $\cdot$ g and Ψ $\cdot$ g possesses primitives and hence, product f $\cdot$ g possesses primitives.

Proposition 3.3. Let f : [a,b] $\to$ **R** be a function with bounded variation with the property that there is a function g : [a,b] $\to$ **R**\{0} possesses primitives such that the product f $\cdot$ g possesses primitives. Then the product f $\cdot$ h possesses primitives for any function h : [a,b] $\to$ **R** possessing primitives.

Proof. Let Φ : [a,b] $\to$ **R** be a function defined as

$$\Phi(x) = f(x) \cdot g(x) \quad \text{for any } x \in [a, b].$$

Then from hypothesis the functions g and Φ possess primitives and f is with bounded variation. Since g(x) $\neq 0$ for any $x \in [a, b]$ and f = Φ /g, from corollary 3.3 it results that f is continuous. Now the conclusion of the proposition is given by the proposition 3.1.

Definition 3.1. Two sets are said to be equipotent (or cardinal equivalent) if there is an univoc application between them.

Proposition 3.4. For a, b $\in$ **R** denote

$$BV[a, b] = \{ f : [a,b] \to \mathbf{R} \ \mid \ f \text{ is with bounded variation on } [a, b]\}$$

$$B[a, b] = \{ f : [a,b] \to \mathbf{R} \ \mid \ f \text{ is bounded on } [a, b]\}$$

$$DP[a, b] = \{ f : [a,b] \to \mathbf{R} \ \mid \ f \text{ has Darboux property on } [a, b]\}$$

Then BV[a, b] and B[a, b] are not equipotent and so there are BV[a, b] and DP[a, b].
Proof. Assume, by contradiction, that there is an univoc application Φ : B[a,b] $\to$ BV[a, b] and take f : [a,b] $\to$ **R** defined as

$$f(x) = \begin{cases} 0, \text{ if } & x \in [a,b] \cap Q \\ 1, \text{ if } & x \in [a,b] \cap (R - Q) \end{cases}.$$

Obviously, f $\in$ B[a, b] and

$$(\Phi \circ f)(x) = \begin{cases} \Phi(0), & \text{if} \quad x \in [a,b] \cap Q \\ \Phi(1), & \text{if} \quad x \in [a,b] \cap (R - Q) \end{cases}.$$

Recall now that Φ (f) is a Dirichlet-type function and it is with bounded variation if and only if is constant, but this in contradiction with Φ is a one-to-one function. Hence, BV[a, b] and B[a, b] are not equipotent.

For the second conclusion assume again, by contradiction, that there is an univoc application Ψ : BV[a,b] $\rightarrow$ DP[a, b] and take g : [a,b] $\rightarrow$ **R** defined as

$$g(x) = \begin{cases} 0, & \text{if} \quad x \in [a, (a+b)/2] \\ 1, & \text{if} \quad x \in [(a+b)/2, b] \end{cases}.$$

Obviously, $g \in$ BV[a, b] and

$$(\Psi \circ g)(x) =. \begin{cases} \Psi(0), & \text{if} \quad x \in [a, (a+b)/2] \\ \Psi(1), & \text{if} \quad x \in [(a+b)/2, b] \end{cases}$$

Again Ψ(g) is a Dirichlet-type function and has Darboux property if and only if is constant, but this in contradiction with Ψ is a one-to-one function. Hence, BV[a, b] and DP[a, b] are not equipotent.

REFERENCE

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