SUBCLASSES OF CONVEX FUNCTIONS ASSOCIATED WITH SOME HYPERBOLA

Mugur Acu

Abstract.In this paper we define some subclasses of convex functions associated with some hyperbola by using a generalized Sălăgean operator and we give some properties regarding these classes.

2000 *Mathematics Subject Classification*: 30C45.

Key words and phrases: Convex functions, Libera-Pascu integral operator, Briot-Bouquet differential subordination, generalized Sălăgean operator

1. Introduction

Let $\mathcal{H}(U)$ be the set of functions which are regular in the unit disc U, $A = \{f \in \mathcal{H}(U) : f(0) = f'(0) - 1 = 0\}$, $\mathcal{H}_u(U) = \{f \in \mathcal{H}(U) : f \text{ is univalent in } U\}$ and $S = \{f \in A : f \text{ is univalent in } U\}$.

Let D^n be the Sălăgean differential operator (see [12]) defined as:

$$D^n : A \to A \ , \quad n \in \mathbf{N} \ \text{and} \ D^0 f(z) = f(z)$$
$$D^1 f(z) = Df(z) = zf'(z) \ , \quad D^n f(z) = D(D^{n-1} f(z)).$$

Remark. *If* $f \in S$, $f(z) = z + \sum\limits_{j=2}^{\infty} a_j z^j$, $z \in U$ *then* $D^n f(z) = z + \sum\limits_{j=2}^{\infty} j^n a_j z^j$.

We recall here the definition of the well - known class of convex functions

$$CV = S^c = \left\{ f \in A : \mathrm{Re}\left(\frac{zf''(z)}{f'(z)} + 1 \right) > 0 \ , \ z \in U \right\}.$$

Let consider the Libera-Pascu integral operator $L_a : A \to A$ defined as:

$$f(z) = L_a F(z) = \frac{1+a}{z^a} \int\limits_0^z F(t) \cdot t^{a-1} dt \ , \quad a \in \mathbf{C} \ , \quad \text{Re}\, a \geq 0. \qquad (1)$$

Generalizations of the Libera-Pascu integral operator was studied by many mathematicians such are P.T. Mocanu in [8], E. Drăghici in [7] and D. Breaz in [6].

Definition 1.1.*Let $n \in \mathbf{N}$ and $\lambda \geq 0$. We denote with D_λ^n the operator defined by*

$$D_\lambda^n : A \to A \,,$$
$$D_\lambda^0 f(z) = f(z) \ , \ D_\lambda^1 f(z) = (1-\lambda) f(z) + \lambda z f(z) = D_\lambda f(z) \,,$$
$$D_\lambda^n f(z) = D_\lambda D_\lambda^{n-1} f(z) \,.$$

Remark 1.2. *We observe that D_λ^n is a linear operator and for $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$ we have*

$$D_\lambda^n f(z) = z + \sum_{j=2}^{\infty} \left(1 + (j-1)\lambda\right)^n a_j z^j \,.$$

Also, it is easy to observe that if we consider $\lambda = 1$ in the above definition we obtain the Sălăgean differential operator.

The next theorem is result of the so called "admissible functions method" introduced by P.T. Mocanu and S.S. Miller (see [9], [10], [11]).

Theorem 1.1. *Let h convex in U and $\text{Re}[\beta h(z) + \gamma] > 0$, $z \in U$. If $p \in H(U)$ with $p(0) = h(0)$ and p satisfied the Briot-Bouquet differential subordination*

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h(z), \quad \textit{then} \ \ p(z) \prec h(z).$$

In [4] is introduced the following operator:

Definition 1.2. *Let* $\beta, \lambda \in \mathbf{R}$, $\beta \geq 0$, $\lambda \geq 0$ *and* $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$. *We denote by* D_λ^β *the linear operator defined by*

$$D_\lambda^\beta : A \to A\,,$$

$$D_\lambda^\beta f(z) = z + \sum_{j=2}^{\infty} (1 + (j-1)\lambda)^\beta\, a_j z^j\,.$$

Remark 1.3. *It is easy to observe that for* $\beta = n \in \mathbf{N}$ *we obtain the Al-Oboudi operator* D_λ^n *and for* $\beta = n \in \mathbf{N}$, $\lambda = 1$ *we obtain the Sălăgean operator* D^n.

The purpose of this note is to define some subclasses of convex functions associated with some hyperbola by using the operator D_λ^β defined above and to obtain some properties regarding these classes.

2. Preliminary results

Definition 2.1. [1] *A function* $f \in A$ *is said to be in the class* $CVH(\alpha)$ *if it satisfies*

$$\left| \frac{zf''(z)}{f'(z)} - 2\alpha\left(\sqrt{2}-1\right) + 1 \right| < \mathrm{Re}\left\{ \sqrt{2}\,\frac{zf''(z)}{f'(z)} \right\} + 2\alpha\left(\sqrt{2}-1\right) + \sqrt{2}\,,$$

for some α $(\alpha > 0)$ *and for all* $z \in U$.

Remark 2.1. *Geometric interpretation: Let*

$$\Omega(\alpha) = \left\{ w = u + i \cdot v \,:\, v^2 < 4\alpha u + u^2\,,\, u > 0 \right\}.$$

Note that $\Omega(\alpha)$ *is the interior of a hyperbola in the right half-plane which is symmetric about the real axis and has vertex at the origin. Whit this notations we have* $f(z) \in CVH(\alpha)$ *if and only if* $\frac{zf''(z)}{f'(z)} + 1$ *take all values in the convex domain* $\Omega(\alpha)$ *contained in the right half-plane.*

Definition 2.2. [2] *Let* $f \in A$ *and* $\alpha > 0$. *We say that the function* f *is in the class* $CVH_n(\alpha)$, $n \in \mathbf{N}$, *if*

$$\left| \frac{D^{n+2}f(z)}{D^{n+1}f(z)} - 2\alpha\left(\sqrt{2}-1\right) \right| < \mathrm{Re}\left\{ \sqrt{2}\,\frac{D^{n+2}f(z)}{D^{n+1}f(z)} \right\} + 2\alpha\left(\sqrt{2}-1\right)\,,\ z \in U\,.$$

Remark 2.2. *Geometric interpretation: If we denote with* p_α *the analytic and univalent functions with the properties* $p_\alpha(0) = 1$, ${p'}_\alpha(0) > 0$ *and* $p_\alpha(U) = \Omega(\alpha)$ *(see Remark 2.1), then* $f \in CVH_n(\alpha)$ *if and only if* $\frac{D^{n+2}f(z)}{D^{n+1}f(z)} \prec p_\alpha(z)$*, where the symbol* $\prec$ *denotes the subordination in* U *. We have* $p_\alpha(z) = (1+2\alpha)\sqrt{\frac{1+bz}{1-z}} - 2\alpha$, $b = b(\alpha) = \frac{1+4\alpha-4\alpha^2}{(1+2\alpha)^2}$ *and the branch of the square root* $\sqrt{w}$ *is chosen so that* $\operatorname{Im}\sqrt{w} \geq 0$ *. If we consider* $p_\alpha(z) = 1 + C_1 z + \dots$, *we have* $C_1 = \frac{1+4\alpha}{1+2\alpha}$.

Theorem 2.1. [2] *If* $F(z) \in CVH_n(\alpha)$, $\alpha > 0$, $n \in \mathbf{N}$, *and* $f(z) = L_a F(z)$*, where* L_a *is the integral operator defined by (1), then* $f(z) \in CVH_n(\alpha)$, $\alpha > 0$, $n \in \mathbf{N}$.

Theorem 2.2. [2] *Let* $n \in \mathbf{N}$ *and* $\alpha > 0$*. If* $f \in CVH_{n+1}(\alpha)$ *then* $f \in CVH_n(\alpha)$.

3. Main results

Definition 3.1. *Let* $\beta \geq 0$, $\lambda \geq 0$, $\alpha > 0$ *and* $p_\alpha(z) = (1+2\alpha)\sqrt{\frac{1+bz}{1-z}} - 2\alpha$, *where* $b = b(\alpha) = \frac{1+4\alpha-4\alpha^2}{(1+2\alpha)^2}$ *and the branch of the square root* $\sqrt{w}$ *is chosen so that* $\operatorname{Im}\sqrt{w} \geq 0$ *. We say that a function* $f(z) \in S$ *is in the class* $CVH_{\beta,\lambda}(\alpha)$ *if*

$$\frac{D_\lambda^{\beta+2}f(z)}{D_\lambda^{\beta+1}f(z)} \prec p_\alpha(z) \,,\; z \in U \,.$$

Remark 3.1. *Geometric interpretation:* $f(z) \in CVH_{\beta,\lambda}(\alpha)$ *if and only if* $\frac{D_\lambda^{\beta+2}f(z)}{D_\lambda^{\beta+1}f(z)}$ *take all values in the domain* $\Omega(\alpha)$ *which is the interior of a hyperbola in the right half-plane which is symmetric about the real axis and has vertex at the origin (see Remark 2.1 and Remark 2.2).*

Remark 3.2. *We observe that this class generalize the class* $CVH_n(\alpha)$ *studied in [2] and the class* $CVH(\alpha)$ *studied in [1].*

Theorem 3.1. *Let* $\beta \geq 0$, $\alpha > 0$ *and* $\lambda > 0$*. We have*

$$CVH_{\beta+1,\lambda}(\alpha) \subset CVH_{\beta,\lambda}(\alpha) \,.$$

Proof. Let $f(z) \in CVH_{\beta+1,\lambda}(\alpha)$.

With notation

$$p(z) = \frac{D_\lambda^{\beta+2} f(z)}{D_\lambda^{\beta+1} f(z)}\,,\; p(0) = 1\,,$$

we obtain

$$\frac{D_\lambda^{\beta+3} f(z)}{D_\lambda^{\beta+2} f(z)} = \frac{D_\lambda^{\beta+3} f(z)}{D_\lambda^{\beta+1} f(z)} \cdot \frac{D_\lambda^{\beta+1} f(z)}{D_\lambda^{\beta+2} f(z)} = \frac{1}{p(z)} \cdot \frac{D_\lambda^{\beta+3} f(z)}{D_\lambda^{\beta+1} f(z)} \tag{2}$$

Also, we have

$$\frac{D_\lambda^{\beta+3} f(z)}{D_\lambda^{\beta+1} f(z)} = \frac{z + \sum_{j=2}^{\infty} (1 + (j-1)\lambda)^{\beta+3} a_j z^j}{z + \sum_{j=2}^{\infty} (1 + (j-1)\lambda)^{\beta+1} a_j z^j}$$

and

$$zp'(z) = \frac{z\left(D_\lambda^{\beta+2} f(z)\right)'}{D_\lambda^{\beta+1} f(z)} - \frac{D_\lambda^{\beta+2} f(z)}{D_\lambda^{\beta+1} f(z)} \cdot \frac{z\left(D_\lambda^{\beta+1} f(z)\right)'}{D_\lambda^{\beta+1} f(z)} =$$

$$= \frac{z\left(1 + \sum_{j=2}^{\infty} (1 + (j-1)\lambda)^{\beta+2} j a_j z^{j-1}\right)}{D_\lambda^{\beta+1} f(z)} -$$

$$-p(z) \cdot \frac{z\left(1 + \sum_{j=2}^{\infty} (1 + (j-1)\lambda)^{\beta+1} j a_j z^{j-1}\right)}{D_\lambda^{\beta+1} f(z)}$$

or

$$zp'(z) = \frac{z + \sum_{j=2}^{\infty} j\,(1 + (j-1)\lambda)^{\beta+2} a_j z^j}{D_\lambda^{\beta+1} f(z)} - p(z) \cdot \frac{z + \sum_{j=2}^{\infty} j\,(1 + (j-1)\lambda)^{\beta+1} a_j z^j}{D_\lambda^{\beta+1} f(z)}\,. \tag{3}$$

We have

$$z + \sum_{j=2}^{\infty} j\,(1 + (j-1)\lambda)^{\beta+2} a_j z^j = z + \sum_{j=2}^{\infty} ((j-1)+1)\,(1 + (j-1)\lambda)^{\beta+2} a_j z^j =$$

$$= z + \sum_{j=2}^{\infty} \left(1 + (j-1)\lambda\right)^{\beta+2} a_j z^j + \sum_{j=2}^{\infty} (j-1) \left(1 + (j-1)\lambda\right)^{\beta+2} a_j z^j =$$

$$= z + D_\lambda^{\beta+2} f(z) - z + \sum_{j=2}^{\infty} (j-1) \left(1 + (j-1)\lambda\right)^{\beta+2} a_j z^j =$$

$$= D_\lambda^{\beta+2} f(z) + \frac{1}{\lambda} \sum_{j=2}^{\infty} \left((j-1)\lambda\right) \left(1 + (j-1)\lambda\right)^{\beta+2} a_j z^j =$$

$$= D_\lambda^{\beta+2} f(z) + \frac{1}{\lambda} \sum_{j=2}^{\infty} \left(1 + (j-1)\lambda - 1\right) \left(1 + (j-1)\lambda\right)^{\beta+2} a_j z^j =$$

$$= D_\lambda^{\beta+2} f(z) - \frac{1}{\lambda} \sum_{j=2}^{\infty} \left(1 + (j-1)\lambda\right)^{\beta+2} a_j z^j + \frac{1}{\lambda} \sum_{j=2}^{\infty} \left(1 + (j-1)\lambda\right)^{\beta+3} a_j z^j =$$

$$= D_\lambda^{\beta+2} f(z) - \frac{1}{\lambda} \left(D_\lambda^{\beta+2} f(z) - z\right) + \frac{1}{\lambda} \left(D_\lambda^{\beta+3} f(z) - z\right) =$$

$$= D_\lambda^{\beta+2} f(z) - \frac{1}{\lambda} D_\lambda^{\beta+2} f(z) + \frac{z}{\lambda} + \frac{1}{\lambda} D_\lambda^{\beta+3} f(z) - \frac{z}{\lambda} =$$

$$= \frac{\lambda - 1}{\lambda} D_\lambda^{\beta+2} f(z) + \frac{1}{\lambda} D_\lambda^{\beta+3} f(z) =$$

$$= \frac{1}{\lambda} \left((\lambda - 1) D_\lambda^{\beta+2} f(z) + D_\lambda^{\beta+3} f(z)\right) .$$

Similarly we have

$$z + \sum_{j=2}^{\infty} j \left(1 + (j-1)\lambda\right)^{\beta+1} a_j z^j = \frac{1}{\lambda} \left((\lambda - 1) D_\lambda^{\beta+1} f(z) + D_\lambda^{\beta+2} f(z)\right) .$$

From (3) we obtain

$$zp'(z) = \frac{1}{\lambda} \left(\frac{(\lambda-1) D_\lambda^{\beta+2} f(z) + D_\lambda^{\beta+3} f(z)}{D_\lambda^{\beta+1} f(z)} - p(z) \frac{(\lambda-1) D_\lambda^{\beta+1} f(z) + D_\lambda^{\beta+2} f(z)}{D_\lambda^{\beta+1} f(z)} \right) =$$

$$= \frac{1}{\lambda} \left((\lambda - 1) p(z) + \frac{D_\lambda^{\beta+3} f(z)}{D_\lambda^{\beta+1} f(z)} - p(z) \left((\lambda - 1) + p(z)\right) \right) =$$

$$= \frac{1}{\lambda} \left(\frac{D_\lambda^{\beta+3} f(z)}{D_\lambda^{\beta+1} f(z)} - p(z)^2 \right)$$

Thus
$$\lambda z p'(z) = \frac{D_\lambda^{\beta+3} f(z)}{D_\lambda^{\beta+1} f(z)} - p(z)^2$$
or
$$\frac{D_\lambda^{\beta+3} f(z)}{D_\lambda^{\beta+1} f(z)} = p(z)^2 + \lambda z p'(z) \, .$$

From (2) we obtain
$$\frac{D_\lambda^{\beta+3} f(z)}{D_\lambda^{\beta+2} f(z)} = \frac{1}{p(z)} \left(p(z)^2 + \lambda z p'(z) \right) = p(z) + \lambda \frac{z p'(z)}{p(z)} \, ,$$
where $\lambda > 0$.

From $f(z) \in CVH_{\beta+2,\lambda}(\alpha)$ we have
$$p(z) + \lambda \frac{z p'(z)}{p(z)} \prec p_\alpha(z) \, ,$$
with $p(0) = p_\alpha(0) = 1$, $\alpha > 0$, $\beta \geq 0$, $\lambda > 0$, and $\operatorname{Re} p_\alpha(z) > 0$ from here construction. In this conditions from Theorem 1.1, we obtain
$$p(z) \prec p_\alpha(z)$$
or
$$\frac{D_\lambda^{\beta+2} f(z)}{D_\lambda^{\beta+1} f(z)} \prec p_\alpha(z) \, .$$
This means $f(z) \in CVH_{\beta,\lambda}(\alpha)$.

Theorem 3.2. *Let $\beta \geq 0$, $\alpha > 0$ and $\lambda \geq 1$. If $F(z) \in CVH_{\beta,\lambda}(\alpha)$ then $f(z) = L_a F(z) \in CVH_{\beta,\lambda}(\alpha)$, where L_a is the Libera-Pascu integral operator defined by (1).*

Proof. From (1) we have
$$(1+a)F(z) = af(z) + zf'(z)$$
and, by using the linear operator $D_\lambda^{\beta+2}$, we obtain
$$(1+a) D_\lambda^{\beta+2} F(z) = a D_\lambda^{\beta+2} f(z) + D_\lambda^{\beta+2} \left(z + \sum_{j=2}^{\infty} j a_j z^j \right) =$$

$$= aD_\lambda^{\beta+2}f(z) + z + \sum_{j=2}^{\infty}(1+(j-1)\lambda)^{\beta+2}\, ja_j z^j$$

We have (see the proof of the above theorem)

$$z + \sum_{j=2}^{\infty} j\,(1+(j-1)\lambda)^{\beta+2}\, a_j z^j = \frac{1}{\lambda}\left((\lambda-1)D_\lambda^{\beta+2}f(z) + D_\lambda^{\beta+3}f(z)\right)$$

Thus

$$(1+a)D_\lambda^{\beta+2}F(z) = aD_\lambda^{\beta+2}f(z) + \frac{1}{\lambda}\left((\lambda-1)D_\lambda^{\beta+2}f(z) + D_\lambda^{\beta+3}f(z)\right) =$$

$$= \left(a + \frac{\lambda-1}{\lambda}\right)D_\lambda^{\beta+2}f(z) + \frac{1}{\lambda}D_\lambda^{\beta+3}f(z)$$

or

$$\lambda(1+a)D_\lambda^{\beta+2}F(z) = ((a+1)\lambda-1)\,D_\lambda^{\beta+2}f(z) + D_\lambda^{\beta+3}f(z)\,.$$

Similarly, we obtain

$$\lambda(1+a)D_\lambda^{\beta+1}F(z) = ((a+1)\lambda-1)\,D_\lambda^{\beta+1}f(z) + D_\lambda^{\beta+2}f(z)\,.$$

Then

$$\frac{D_\lambda^{\beta+2}F(z)}{D_\lambda^{\beta+1}F(z)} = \frac{\frac{D_\lambda^{\beta+3}f(z)}{D_\lambda^{\beta+2}f(z)}\cdot\frac{D_\lambda^{\beta+2}f(z)}{D_\lambda^{\beta+1}f(z)} + ((a+1)\lambda-1)\cdot\frac{D_\lambda^{\beta+2}f(z)}{D_\lambda^{\beta+1}f(z)}}{\frac{D_\lambda^{\beta+2}f(z)}{D_\lambda^{\beta+1}f(z)} + ((a+1)\lambda-1)}\,.$$

With notation

$$\frac{D_\lambda^{\beta+2}f(z)}{D_\lambda^{\beta+1}f(z)} = p(z)\,,\ p(0)=1\,,$$

we obtain

$$\frac{D_\lambda^{\beta+2}F(z)}{D_\lambda^{\beta+1}F(z)} = \frac{\frac{D_\lambda^{\beta+3}f(z)}{D_\lambda^{\beta+2}f(z)}\cdot p(z) + ((a+1)\lambda-1)\cdot p(z)}{p(z) + ((a+1)\lambda-1)} \tag{4}$$

We have (see the proof of the above theorem)

$$\lambda z p'(z) = \frac{D_\lambda^{\beta+3}f(z)}{D_\lambda^{\beta+2}f(z)}\cdot\frac{D_\lambda^{\beta+2}f(z)}{D_\lambda^{\beta+1}f(z)} - p(z)^2 =$$

$$= \frac{D_\lambda^{\beta+3} f(z)}{D_\lambda^{\beta+2} f(z)} \cdot p(z) - p(z)^2 .$$

Thus

$$\frac{D_\lambda^{\beta+3} f(z)}{D_\lambda^{\beta+2} f(z)} = \frac{1}{p(z)} \cdot \Big(p(z)^2 + \lambda z p'(z) \Big) .$$

Then,from (4), we obtain

$$\frac{D_\lambda^{\beta+2} F(z)}{D_\lambda^{\beta+1} F(z)} = \frac{p(z)^2 + \lambda z p'(z) + ((a+1)\lambda - 1)\, p(z)}{p(z) + ((a+1)\lambda - 1)} =$$

$$= p(z) + \lambda \frac{z p'(z)}{p(z) + ((a+1)\lambda - 1)} ,$$

where $a \in \mathbf{C}$, $\operatorname{Re} a \geq 0$, $\beta \geq 0$, and $\lambda \geq 1$.

From $F(z) \in CVH_{\beta,\lambda}(\alpha)$ we have

$$p(z) + \frac{z p'(z)}{\frac{1}{\lambda} \left(p(z) + ((a+1)\lambda - 1) \right)} \prec p_\alpha(z) ,$$

where $a \in \mathbf{C}$, $\operatorname{Re} a \geq 0$, $\alpha > 0$, $\beta \geq 0$, $\lambda \geq 1$, and from her construction, we have $\operatorname{Re} p_\alpha(z) > 0$. In this conditions we have from Theorem 1.1 we obtain

$$p(z) \prec p_\alpha(z)$$

or

$$\frac{D_\lambda^{\beta+2} f(z)}{D_\lambda^{\beta+1} f(z)} \prec p_\alpha(z) .$$

This means $f(z) = L_a F(z) \in CVH_{\beta,\lambda}(\alpha)$.

Remark 2.3. *If we consider $\beta = n \in \mathbf{N}$ in the previously results we obtain the Theorem 3.1 and Theorem 3.2 from [3].*

References

[1] M. Acu and S. Owa, *Convex functions associated with some hyperbola*, Journal of Approximation Theory and Applications, Vol. 1(2005), (to appear).

[2] M. Acu, *On a subclass of n-convex functions associated with some hyperbola*, Annals of the Oradea Univ., Fascicola Matematics (2005), (to appear).

[3] M. Acu and S. Owa, *On n-convex functions associated with some hyperbola*, (to appear).

[4] M. Acu and S. Owa, *Note on a class of starlike functions*, (to appear).

[5] F.M. Al-Oboudi, *On univalent funtions defined by a generalized Sălăgean operator*, Ind. J. Math. Math. Sci. 2004, no. 25-28, 1429-1436.

[6] D. Breaz, *Operatori integrali pe spaţii de funcţii univalente*, Editura Academiei Române, Bucureşti 2004.

[7] E. Drăghici, *Elemente de teoria funcţiilor cu aplicaţii la operatori integrali univalen'ti*, Editura Constant, Sibiu 1996.

[8] P.T. Mocanu, *Classes of univalent integral operators*, J. Math. Anal. Appl. 157, 1(1991), 147-165.

[9] S. S. Miller and P. T. Mocanu, *Differential subordonations and univalent functions*, Mich. Math. 28 (1981), 157 - 171.

[10] S. S. Miller and P. T. Mocanu, *Univalent solution of Briot-Bouquet differential equations*, J. Differential Equations 56 (1985), 297 - 308.

[11] S. S. Miller and P. T. Mocanu, *On some classes of first-order differential subordinations*, Mich. Math. 32(1985), 185 - 195.

[12] Gr. Sălăgean, *Subclasses of univalent functions*, Complex Analysis. Fifth Roumanian-Finnish Seminar, Lectures Notes in Mathematics, 1013, Springer-Verlag, 1983, 362-372.

Author:

Mugur Acu
University "Lucian Blaga" of Sibiu
Department of Mathematics
Str. Dr. I. Raţiu, No. 5-7
550012 - Sibiu, Romania
e-mail address: acu_mugur@yahoo.com