MOMENT PRESERVING SPLINE APPROXIMATION ON FINITE INTERVALS AND CHAKALOV-POPOVICIU QUADRATURES

Ana Maria Acu

Abstract. In this paper we shall discuss the approximation of a function $f=f(t)$ on the interval $[0,1]$ by a monospline function of degree $n\geq 2$ with the nodes $\{t_k\}_{k=\overline{1,m}}$ and defects $z=(z_1,z_2,\dots,z_m)$, $1\leq z_k\leq n$, $k=\overline{1,m}$. The problem of approximating a function f has a unique solution if and only if certain Chakalov-Popoviciu quadrature of Radau and Lobatto type exist coresponding to measures depending on f .

2000 Mathematics Subject Classification: 26D15, 65D30, 65D32, 41A55

Keywords and phrases: quadrature rule, numerical integration, Gauss-Lobatto quadrature formulae, Gauss-Radau quadrature formulae, Chakalov-Popoviciu quadrature formulae, spline, monospline.

1. Introduction

We denote $\mathcal{P}_n$ the space of polynomials of degree most equal with n. Let $d\lambda$ be a nonnegative measure on the line $\mathbb{R}$, with compact or infinite support, for which all moments

$$\mu_k=\int_{\mathbb{R}} t^k d\lambda(t)\ ,\ k=0,1,\dots$$

exists and are finite, and $\mu_0>0$

Let $m \in \mathbb{N}$ and $\sigma = (s_1, s_2, \dots, s_m)$, where s_i , $i = \overline{1,m}$ are nonnegative integers. The nodes

$$x_1 < x_2 < \cdots < x_m \ , \ \ x_k \in \text{supp}(d\lambda) \ , \ k = \overline{1,m}$$

have the orders of multiplicity $2s_1 + 1, 2s_2 + 1, \dots, 2s_m + 1$, respectively.

Definition 1 *The polynomial*

$$\pi_{m,\sigma}(t) = \prod_{k=1}^{m} (x - x_k)$$

which satisfies the orthogonality conditions

$$\int_{\mathbb{R}} \prod_{k=1}^{m} (t - x_k)^{2s_k+1} t^k d\lambda(t) = 0 \, , k = 0, \dots , m-1 \tag{1}$$

is called σ**-orthogonal polynomial with respect to the measure** $d\lambda(t)$.

The quadrature formulae

$$\int_{\mathbb{R}} f(t) d\lambda(t) = \sum_{k=1}^{m} \sum_{i=0}^{2s_k} A_{ik}^G f^{(i)}(x_k) + \mathcal{R}_m[f] \tag{2}$$

has the maximum degree of exactness $r = 2\sum_{k=1}^{m} s_k + 2m - 1$ if and only if x_k , $k = \overline{1,m}$ are the roots of σ-orthogonal polynomial of degree m with respect to the measure $d\lambda(t)$

The quadrature formulae (2) is called **Chakalov-Popoviciu quadrature formulae**.

Let $[a, b]$ be the support of the nonnegative measure $d\lambda(t) = w(t)dt$, where $w(t)$ is the weight function. Let

$$\int_a^b f(t) d\lambda(t) = \sum_{k=1}^{m} \sum_{i=0}^{2s_k} A_{ik} f^{(i)}(x_k) + \sum_{k=1}^{q} \sum_{i=0}^{p_k} \alpha_{ik} f^{(i)}(y_k) + \mathcal{R}_{m,q}[f] \tag{3}$$

be a quadrature formulae, where $y_k \in \mathbb{R} \setminus (a,b)$, $k = \overline{1,q}$ are fixed, distinct nodes and outside of (a,b), and the nodes $x_k \in (a,b)$, $k = \overline{1,m}$ will be determined such that the quadrature formulae (3) to have maximal algebraic degree of precision. These quadrature formulas are so-called Gauss quadrature formulas with fixed nodes. To observe that the function f must be defined on the nodes from outside of (a,b).

The **Gauss-Radau quadrature formulas** are the Gauss quadrature formulas with one fixed node, namely $y_1 = a$ or $y_1 = b$.

The quadrature formulas

$$\int_a^b f(t)d\lambda(t) = \sum_{i=0}^{p} \alpha_{i1} f^{(i)}(a) + \sum_{k=1}^{m}\sum_{i=0}^{2s_k} A_{ik}^{\mathcal{R}} f^{(i)}(x_k) + \mathcal{R}_{m,p}^{\mathcal{R}}[f] \tag{4}$$

where $x_k \in (a,b)$, $k = \overline{1,m}$, $-\infty < a < +\infty$, $p \in \mathbb{N}$ with

$$\mathcal{R}_{m,p}^{\mathcal{R}}[f] = 0 \text{ for } f \in \mathcal{P}_{2(\sum_{k=1}^{m} s_k + m)+p}$$

and

$$\int_a^b f(t)d\lambda(t) = \sum_{i=0}^{p} \tilde{\alpha}_{i1} f^{(i)}(b) + \sum_{k=1}^{m}\sum_{i=0}^{2s_k} \tilde{A_{ik}}^{\mathcal{R}} f^{(i)}(\tilde{x_k}) + \tilde{\mathcal{R}}_{m,p}^{\mathcal{R}}[f] \tag{5}$$

where $\tilde{x_k} \in (a,b)$, $k = \overline{1,m}$, $-\infty < b < +\infty$, $p \in \mathbb{N}$ with

$$\tilde{\mathcal{R}}_{m,p}^{\mathcal{R}}[f] = 0 \text{ for } f \in \mathcal{P}_{2(\sum_{k=1}^{m} s_k + m)+p}$$

are called **Chakalov-Popoviciu quadrature formulas of Radau type.**

The **Gauss-Lobatto quadrature formulas** are the Gauss quadrature formulas with two fixed nodes, namely $y_1 = a$ and $y_2 = b$.

The quadrature formulae

$$\int_a^b f(t)d\lambda(t) = \sum_{i=0}^{p_1} \alpha_{i1} f^{(i)}(a) + \sum_{i=0}^{p_2} \alpha_{i2} f^{(i)}(b) + \sum_{k=1}^{m}\sum_{i=0}^{2s_k} A_{ik}^{L} f^{(i)}(x_k) + \mathcal{R}_{m,p_1,p_2}^{L}[f] \tag{6}$$

where $x_k \in (a,b)$, $k = \overline{1,m}$, $-\infty < a < b < +\infty$, p_1 , $p_2 \in \mathbb{N}$ with

$$\mathcal{R}^L_{m,p_1,p_2}[f] = 0 \text{ for } f \in \mathcal{P}_{2(\sum_{k=1}^m s_k + m)+p_1+p_2+1}$$

is called **Chakalov-Popoviciu quadrature formulae of Lobatto type.**

In [13], M.M. Spalević gives a method of approximating a function $f = f(t)$ on the interval $[0,1]$, by a spline function of degree $n \geq 2$ with the nodes $\{t_k\}_{k=\overline{1,m}}$ and defects $z = (z_1, z_2, \ldots, z_m)$, $1 \leq z_k \leq n$, $k = \overline{1,m}$.

M.M. Spalević shows that the problem has a unique solution if and only if certain Chakalov-Popoviciu quadrature formulae of Radau and Lobatto type exist corresponding to measures depending on f.

A spline function of degree $n \geq 2$ with the distinct nodes $\{t_k\}_{k=1}^m$, $t_k \in (0,1)$ and defects $z = (z_1, z_2, \ldots, z_m)$ can be written

$$s(t) = p_n(t) + \sum_{k=1}^{m} \sum_{i=0}^{z_k-1} c_{ki}(t_k - t)_+^{n-i} \tag{7}$$

where c_{ik} are real numbers and $p_n(t)$ is a polynomial with degree most equal n. Some interesting results about spline functions were obtain in [1], [2], [3], [4].

Choosing $z_k = 2s_k + 1$, $k = \overline{1,m}$, the spline function (7) can be written

$$s(t) = p_n(t) + \sum_{k=1}^{m} \sum_{i=0}^{2s_k} c_{ki}(t_k - t)_+^{n-i} \tag{8}$$

M.M. Spalević considers the problem:

Problem S1. Determine the spline function s, defined in (8) such that

$$\int_0^1 t^j s(t)dt = \int_0^1 t^j f(t)dt\,,\ j = 0, 1, \ldots, 2(\sum_{k=1}^{m} s_k + m) + n \tag{9}$$

Theorem 1 *Let $f \in C^{n+1}[0,1]$. There exist a unique spline function s, defined in (8), satisfying (9), if and only if the measure* $d\lambda(t) = \dfrac{(-1)^{n+1}}{n!} f^{(n+1)}(t)dt$

admits a Chakalov-Popoviciu quadrature formulae of Lobatto type

$$\int_0^1 f(t)d\lambda(t) = \sum_{k=0}^{n} \alpha_{k,1} f^{(k)}(0) + \sum_{k=0}^{n} \alpha_{k,2} f^{(k)}(1) + \sum_{k=1}^{m} \sum_{i=0}^{2s_k} A_{ik}^{L} f^{(i)}(x_k) + \mathcal{R}_{m,n}^{L}[f] \tag{10}$$

where

$$\mathcal{R}_{m,n}^{L}[f] = 0 \text{ for } f \in \mathcal{P}_{2(\sum_{k=1}^{m} s_k + m) + 2n + 1}$$

with distinct real zeros $x_k \in (0,1)$, $k = \overline{1,m}$.
The spline function (8) is given by

$$t_k = x_k \; , \; k = \overline{1,m}$$
$$c_{ki} = \frac{n!}{(n-1)!} A_{ik}^{L} \; , \; k = \overline{1,m} \; , \; i = \overline{0,2s_k}$$

where x_k *and* A_{ik}^{L} *are the nodes and coefficients, respective of Chakalov-Popoviciu quadrature formulae of Lobatto type, and the polynomial* $p_n(t)$ *is given by*

$$p_n(t) = \sum_{k=0}^{n} \frac{(t-1)^k}{k!} \left[(-1)^k n! \alpha_{n-k,2} + f^{(k)}(1) \right] .$$

Remark 1 *The case* $s_1 = s_2 = \cdots = s_m = 0$ *of Theorem 1 was obtain by M. Frontini, W. Gautschi, G.V. Milovanović in [5], and generalized, namely* $s_1 = \cdots = s_m = s$, $s \in \mathbb{N}$ *in [6] by M. Frontini and G.V. Milovanović.*

Theorem 2 *Let* $f \in C^{n+1}[0,1]$ *and* $r_x(t) = (t-x)_+^n$, $0 \le t \le 1$. *If the spline function s, defined in (8), satisfies the relations (9), then*

$$f(x) - s(x) = \mathcal{R}_{m,n}^{L}[r_x] \, , \; 0 < x < 1$$

where $\mathcal{R}_{m,n}^{L}[f]$ *is the remainder term in the Chakalov-Popoviciu quadrature formulae of Lobatto type (10).*

M.M. Spalević considers in [13] the problem:

PROBLEM S2. Determine the spline function s, defined in (8), such that

$$s^{(k)}(1) = p_n^{(k)}(1) = f^{(k)}(1)\ ,\ k = \overline{0,n} \tag{11}$$

$$\int_0^1 t^j s(t)dt = \int_0^1 t^j f(t)dt\ ,\ j = 0, 1, \ldots, 2(\sum_{k=1}^{m} s_k + m) - 1 \tag{12}$$

THEOREM 3 *Let $f \in C^{n+1}[0,1]$. There exist a unique spline function s, defined in (8), satisfying (11), (12), if and only if the measure*

$$d\lambda(t) = \frac{(-1)^{n+1}}{n!} f^{(n+1)}(t)dt$$

admits a Chakalov-Popoviciu quadrature formulae of Radau type

$$\int_0^1 f(t)d\lambda(t) = \sum_{k=0}^{n} \alpha_{k,1} f^{(k)}(0) + \sum_{k=1}^{m}\sum_{i=0}^{2s_k} A_{ik}^{\mathcal{R}} f^{(i)}(x_k) + \mathcal{R}_{m,n}^{\mathcal{R}}[f] \tag{13}$$

where

$$\mathcal{R}_{m,n}^{\mathcal{R}}[f] = 0 \text{ for } f \in \mathcal{P}_{2(\sum_{k=1}^{m} s_k + m)+n}$$

with distinct real zeros $x_k \in (0,1)\ ,\ k = \overline{1,m}$.
The spline function (8) is given by

$$\begin{aligned} & t_k = x_k\ ,\ k = \overline{1,m} \\ & c_{ki} = \frac{n!}{(n-i)!} A_{ik}^{\mathcal{R}}\ ,\ k = \overline{1,m}\ ,\ i = \overline{0,2s_k} \end{aligned}$$

where x_k and $A_{ik}^{\mathcal{R}}$ are the nodes and coefficients, respective of Chakalov-Popoviciu quadrature formulae of Radau type, and the polynomial $p_n(t)$ is given by

$$p_n(t) = \sum_{k=0}^{n} \frac{f^{(k)}(1)}{k!}(t-1)^k\,.$$

THEOREM 4 *Let $f \in C^{n+1}[0,1]$ and $r_x(t) = (t-x)_+^n\,,\ 0 \le t \le 1$. If the spline function s, defined in (8), satisfies the relations (11), (12), then*

$$f(x) - s(x) = \mathcal{R}_{m,n}^{\mathcal{R}}[r_x]\,,\ 0 < x < 1$$

where $\mathcal{R}_{m,n}^{\mathcal{R}}[f]$ is the remainder term in the Chakalov-Popoviciu quadrature formulae of Radau type(13).

2. Main results

Similar as in [13] we shall consider next problem:

PROBLEM S3. Determine the spline function s, defined in (8) such that

$$\int_0^1 t^j s(t)dt = \int_0^1 t^j f(t)dt\,,\ j = 0,1,\ldots,2(\sum_{k=1}^{m} s_k + m) - 1 \qquad (14)$$

THEOREM 5 *Let $f \in C^{n+1}[0,1]$. There exists a unique spline function s, defined in (8), satisfying (14), if and only if the measure $d\lambda(t) = \frac{(-1)^{n+1}}{n!} f^{(n+1)}(t)dt$ admits a Chakalov-Popoviciu quadrature formulae of Radau type*

$$\int_0^1 f(t)d\lambda(t) = \sum_{k=0}^{n} \tilde{\alpha}_{k,1} f^{(k)}(1) + \sum_{k=1}^{m}\sum_{i=0}^{2s_k} \tilde{A}_{ik}^{\mathcal{R}} f^{(i)}(\tilde{x}_k) + \tilde{\mathcal{R}}_{m,n}^{\mathcal{R}}[f]\,, \qquad (15)$$

where

$$\tilde{\mathcal{R}}_{m,n}^{\mathcal{R}}[f] = 0 \text{ for } f \in \mathcal{P}_{2(\sum_{k=1}^{m} s_k + m)+n}$$

with distinct real zero $\tilde{x}_k \in (0,1)\ ,\ k = \overline{1,m}$.
The spline function (8) is given by

$$t_k = \tilde{x}_k\ ,\ k = \overline{1,m}$$
$$c_{ki} = \frac{n!}{(n-i)!}\tilde{A}_{ik}^{\mathcal{R}}\ ,\ k = \overline{1,m}\ ,\ i = \overline{0,2s_k}$$

where $\tilde{x}_k$ *and* $\tilde{A}_{ik}^{\mathcal{R}}$ *are the nodes and coefficients, respective of Chakalov-Popoviciu quadrature formulae of Radau type, and the polynomial* $p_n(t)$ *is given by*

$$p_n(t) = \sum_{k=0}^{n} \frac{(t-1)^k}{k!} \left[(-1)^k n! \tilde{\alpha}_{n-k,1} + f^{(k)}(1)\right] .$$

Proof. The relations (14) can be written

$$\int_0^1 t^j p_n(t)dt + \sum_{k=1}^{m} \sum_{i=0}^{2s_k} c_{ki} \int_0^1 t^j (t_k - t)_+^{n-i} dt = \int_0^1 t^j f(t)dt$$

Applying (n+1) integration by parts, we obtain

$$\sum_{k=0}^{n} (-1)^k \frac{j!}{(j+1+k)!} p_n^{(k)}(1) + \sum_{k=1}^{m} \sum_{i=0}^{2s_k} c_{ki} t_k^{n-i+j+1} \frac{j!(n-i)!}{(j+n-i+1)!} =$$

$$\sum_{k=0}^{n} (-1)^k \frac{j!}{(j+1+k)!} f^{(k)}(1) + (-1)^{n+1} \frac{j!}{(j+n+1)!} \int_0^1 t^{j+n+1} f^{(n+1)}(t)dt \, .$$

Above relation can be written such that

$$\sum_{k=0}^{n} (-1)^k \left[t^{n+j+1}\right]_{t=1}^{(n-k)} p_n^{(k)}(1) + \sum_{k=1}^{m} \sum_{i=0}^{2s_k} c_{ki}(n-i)! \left[t^{n+j+1}\right]_{t=t_k}^{(i)} =$$

$$\sum_{k=0}^{n} (-1)^k \left[t^{j+n+1}\right]_{t=1}^{(n-k)} f^{(k)}(1) + (-1)^{n+1} \int_0^1 t^{j+n+1} f^{(n+1)}(t)dt$$

Denoting $d\lambda(t) = \dfrac{(-1)^{n+1}}{n!} f^{(n+1)}(t)dt$, we have

$$\int_0^1 t^{j+n+1} d\lambda(t) = \sum_{k=0}^{n} \frac{(-1)^{n-k}}{n!} \left[p_n^{(n-k)}(1) - f^{(n-k)}(1)\right] \cdot \left[t^{n+j+1}\right]_{t=1}^{(k)} + \quad (16)$$

$$\sum_{k=1}^{m} \sum_{i=0}^{2s_k} c_{ki} \frac{(n-i)!}{n!} \left[t^{n+j+1}\right]_{t=t_k}^{(i)} \ , \ \ j = 0, 1, \ldots, 2(\sum_{k=1}^{m} s_k + m) - 1$$

Putting $f(t) = t^{n+1}q(t)$, $q \in \mathcal{P}_{2(\sum_{k=1}^{m} s_k+m)-1}$ in the Chakalov-Popoviciu quadrature formulae of Radau type (15), we have

$$\int_0^1 t^{n+1}q(t)d\lambda(t) = \sum_{k=0}^{n} \tilde{\alpha}_{k,1} \left[t^{n+1}q(t)\right]_{t=1}^{(k)} + \sum_{k=1}^{m}\sum_{i=0}^{2s_k} \tilde{A}_{ik}^{\mathcal{R}} \left[t^{n+1}q(t)\right]_{t=\tilde{x}_k}^{(i)} \quad (17)$$

From relations (16) and (17) we obtain:

$$c_{ki} = \frac{n!}{(n-i)!}\tilde{A}_{ik}^{\mathcal{R}} \ , \ k = \overline{1,m} \ , \ i = \overline{0,2s_k}$$
$$p_n^{(k)}(1) = n!(-1)^k\tilde{\alpha}_{n-k,1} + f^{(k)}(1) \ , \ k = \overline{0,n}$$

THEOREM 6 *Let* $f \in C^{n+1}[0,1]$ *and* $r_x(t) = (t-x)_+^n$, $0 \le t \le 1$. *If the spline function s, defined in (8), satisfies the relations (14), then*

$$f(x) - s(x) = \tilde{\mathcal{R}}_{m,n}^{\mathcal{R}}[r_x] \, , \ 0 < x < 1$$

where $\tilde{\mathcal{R}}_{m,n}^{\mathcal{R}}[f]$ *is the remainder term in the Chakalov-Popoviciu quadrature formulae of Radau type (15).*

Proof. Using Taylor′ s formula, the function f can be written such that

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(1)}{k!}(x-1)^k + \frac{(-1)^{n+1}}{n!}\int_0^1 (t-x)_+^n f^{(n+1)}(t)dt$$

but

$$s(x) = \sum_{k=0}^{n} \frac{(x-1)^k}{k!}\left[(-1)^k n!\tilde{\alpha}_{n-k,1} + f^{(k)}(1)\right] + \sum_{k=1}^{m}\sum_{i=0}^{2s_k} \frac{n!}{(n-i)!}\tilde{A}_{ik}^{\mathcal{R}}(t_k-x)_+^{n-i} =$$
$$\sum_{k=0}^{n} \tilde{\alpha}_{k,1}\left[r_x(t)\right]_{t=1}^{(k)} + \sum_{k=0}^{n} \frac{f^{(k)}(1)}{k!}(x-1)^k + \sum_{k=1}^{m}\sum_{i=0}^{2s_k} \tilde{A}_{ik}^{\mathcal{R}}\left[r_x(t)\right]_{t=t_k}^{(i)}$$

From above relations we obtain

$$f(x)-s(x) = \int_0^1 r_x(t)d\lambda(t) - \sum_{k=0}^{n} \tilde{\alpha}_{k,1}\left[r_x(t)\right]_{t=1}^{(k)} - \sum_{k=1}^{m}\sum_{i=0}^{2s_k} \tilde{A}_{ik}^{\mathcal{R}}\left[r_x(t)\right]_{t=t_k}^{(i)} = \tilde{\mathcal{R}}_{m,n}^{\mathcal{R}}[r_x] \, .$$

In a similar way as in [13], we shall discuss the approximation of a function $f = f(t)$ on the interval $[0, 1]$ by a monospline function of degree $n \geq 2$ with the nodes $\{t_k\}_{k=\overline{1,m}}$ $t_k \in (0, 1)$ and defects $z = (z_1, z_2, \dots, z_m)$, $1 \leq z_k \leq n$, $k = \overline{1, m}$.

We shall show that the problem has a unique solution if and only if certain Chakalov-Popoviciu quadrature formulae of Radau and Lobatto type exist coresponding to measures depending on f.

Let

$$\begin{aligned} M(t) = \frac{(1-t)^n}{n!} - \sum_{k=0}^{n-1} B_k \frac{(n-1)!}{(n-k-1)!}(1-t)^{n-k-1} - \\ \sum_{k=1}^{m} \sum_{i=0}^{z_k - 1} a_{ik} \frac{(n-1)!}{(n-i-1)!}(t_k - t)_+^{n-i-1} \end{aligned} \tag{18}$$

be the monospline function, where B_k and a_{ik} are real numbers.

Choosing $z_k = 2s_k + 1$, $k = \overline{1, m}$, the monospline function (18) can be written

$$\begin{aligned} M(t) = \frac{(1-t)^n}{n!} - \sum_{k=0}^{n-1} B_k \frac{(n-1)!}{(n-k-1)!}(1-t)^{n-k-1} - \\ \sum_{k=1}^{m} \sum_{i=0}^{2s_k} a_{ik} \frac{(n-1)!}{(n-i-1)!}(t_k - t)_+^{n-i-1} \end{aligned} \tag{19}$$

We consider the problem:

Problem M1. Determine the monospline M, defined in (19) such that

$$\int_0^1 t^j M(t)dt = \int_0^1 t^j f(t)dt\,,\ j = 0, 1, \dots, 2(\sum_{k=1}^{m} s_k + m) + n - 1 \tag{20}$$

Theorem 7 *Let $f \in C^n[0, 1]$. There exist a unique monospline M, defined in (19), satisfying (20), if and only if the measure*

$$d\lambda(t) = \frac{1}{(n-1)!}\left[1 + (-1)^{n-1} f^{(n)}(t)\right] dt$$

admits a Chakalov-Popoviciu quadrature formulae of Lobatto type

$$\int_0^1 f(t)d\lambda(t) = \sum_{k=0}^{n-1} \alpha_{k,1} f^{(k)}(0) + \sum_{k=0}^{n-1} \alpha_{k,2} f^{(k)}(1) + \sum_{k=1}^{m} \sum_{i=0}^{2s_k} A_{ik}^L f^{(i)}(x_k) + \mathcal{R}_{m,n}^L[f] \tag{21}$$

where

$$\mathcal{R}_{m,n}^L[f] = 0 \text{ for } f \in \mathcal{P}_{2(\sum_{k=1}^m s_k + m) + 2n - 1}$$

with distinct real zeros $x_k \in (0,1)$, $k = \overline{1,m}$.
The monospline (19) is given by

$$\begin{aligned} t_k &= x_k \ , \ k = \overline{1,m} \\ B_k &= \alpha_{k,2} + \frac{(-1)^{n-k}}{(n-1)!} f^{(n-1-k)}(1) \ , \ k = \overline{0, n-1} \\ a_{ik} &= A_{ik}^L \ , \ k = \overline{1,m} \ , \ i = \overline{0, 2s_k} \end{aligned}$$

where x_k *and* A_{ik}^L *are the nodes and coefficients, respective of Chakalov-Popoviciu quadrature formulae of Lobatto type.*

Proof. Using definition of the monospline (19), the relations (20) can be written

$$\begin{aligned} &\frac{1}{n!} \int_0^1 t^j (1-t)^n dt - \sum_{k=0}^{n-1} B_k \frac{(n-1)!}{(n-k-1)!} \int_0^1 t^j (1-t)^{n-k-1} dt - \\ &\sum_{k=1}^{m} \sum_{i=0}^{2s_k} a_{ik} \frac{(n-1)!}{(n-i-1)!} \int_0^1 t^j (t_k - t)_+^{n-i-1} dt = \int_0^1 t^j f(t) dt \end{aligned} \tag{22}$$

Applying n integration by parts we obtain

$$\int_0^1 t^j f(t) dt = \sum_{k=0}^{n-1} (-1)^k \frac{j!}{(j+1+k)!} f^{(k)}(1) + (-1)^n \frac{j!}{(j+n)!} \int_0^1 t^{j+n} f^{(n)}(t) dt \tag{23}$$

From relations (22) and (23) we have

$$\frac{j!}{(j+n+1)!} - \sum_{k=0}^{n-1} B_k \frac{(n-1)! j!}{(j+n-k)!} - \sum_{k=1}^{m} \sum_{i=0}^{2s_k} a_{ik} \frac{(n-1)! j!}{(j+n-i)!} t_k^{n+j-i} =$$

$$\sum_{k=0}^{n-1}(-1)^k\frac{j!}{(j+1+k)!}f^{(k)}(1)+(-1)^n\frac{j!}{(j+n)!}\int_0^1 t^{j+n}f^{(n)}(t)dt\,,$$

namely

$$\frac{1}{j+n+1}-\sum_{k=0}^{n-1}B_k(n-1)!\frac{(j+n)!}{(j+n-k)!}-\sum_{k=1}^{m}\sum_{i=0}^{2s_k}a_{ik}(n-1)!\frac{(j+n)!}{(j+n-i)!}t_k^{n+j-i}=$$

$$\sum_{k=0}^{n-1}(-1)^k\frac{(j+n)!}{(j+k+1)!}f^{(k)}(1)+(-1)^n\int_0^1 t^{j+n}f^{(n)}(t)dt$$

$$\frac{1}{(n-1)!}\int_0^1 t^{j+n}dt-\sum_{k=0}^{n-1}B_k\left[t^{j+n}\right]_{t=1}^{(k)}-\sum_{k=1}^{m}\sum_{i=0}^{2s_k}a_{ik}\left[t^{j+n}\right]_{t=t_k}^{(i)}=$$

$$\sum_{k=0}^{n-1}\frac{(-1)^k}{(n-1)!}\left[t^{j+n}\right]_{t=1}^{(n-1-k)}f^{(k)}(1)+\frac{(-1)^n}{(n-1)!}\int_0^1 t^{j+n}f^{(n)}(t)dt$$

Denoting $d\lambda(t)=\dfrac{1}{(n-1)!}\left[1+(-1)^{n-1}f^{(n)}(t)\right]dt$, above relations can be written:

$$\int_0^1 t^{j+n}d\lambda(t)=\sum_{k=0}^{n-1}\left[\frac{(-1)^{n-1-k}}{(n-1)!}f^{(n-1-k)}(1)+B_k\right]\cdot\left[t^{j+n}\right]_{t=1}^{(k)}+$$
$$\sum_{k=1}^{m}\sum_{i=0}^{2s_k}a_{ik}\left[t^{j+n}\right]_{t=t_k}^{(i)}\ ,\ \ j=0,1,\ldots,2(\sum_{k=1}^{m}s_k+m)+n-1 \tag{24}$$

Choosing in the Chakalov-Popoviciu quadrature formulae of Lobatto type (21), $f(t)=t^np(t)\,,\ p\in\mathcal{P}_{2(\sum_{k=1}^m s_k+m)+n-1}$, we obtain

$$\int_0^1 t^np(t)d\lambda(t)=\sum_{k=0}^{n-1}\alpha_{k,2}\left[t^np(t)\right]_{t=1}^{(k)}+\sum_{k=1}^{m}\sum_{i=0}^{2s_k}A_{ik}^L\left[t^np(t)\right]_{t=x_k}^{(i)} \tag{25}$$

From relations (24) and (25) we have

$$B_k=\alpha_{k,2}+\frac{(-1)^{n-k}}{(n-1)!}f^{(n-1-k)}(1)\ ,\ k=\overline{0,n-1}$$
$$a_{ik}=A_{ik}^L\ ,\ k=\overline{1,m}\ ,\ i=\overline{0,2s_k}$$

THEOREM 8 *Let $f \in C^n[0,1]$ and $r_x(t) = (t-x)_+^{n-1}$, $0 \le t \le 1$. If the monospline M, defined in (19), satisfies the relations (20), then*

$$M(x) - f(x) = \mathcal{R}^L_{m,n}[r_x]\,,\ 0 < x < 1$$

where $\mathcal{R}^L_{m,n}[f]$ is the remaider term in the Chakalov-Popoviciu quadrature formulae of Lobatto type (21).

Proof. Using Taylor′s formula, the function f can be written:

$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(1)}{k!}(x-1)^k + \frac{(-1)^n}{(n-1)!}\int_0^1 (t-x)_+^{n-1} f^{(n)}(t)dt\,,$$

but

$$M(x) = \frac{(1-x)^n}{n!} - \sum_{k=0}^{n-1}\left[\alpha_{k,2} + \frac{(-1)^{n-k}}{(n-1)!}f^{(n-1-k)}(1)\right]\frac{(n-1)!}{(n-k-1)!}(1-x)^{n-k-1} -$$

$$\sum_{k=1}^{m}\sum_{i=0}^{2s_k} A^L_{ik}\frac{(n-1)!}{(n-i-1)!}(x_k-x)_+^{n-i-1} =$$

$$\frac{1}{(n-1)!}\int_0^1 r_x(t)dt - \sum_{k=0}^{n-1}\alpha_{k,2}\left[r_x(t)\right]^{(k)}_{t=1} + \sum_{k=0}^{n-1}\frac{f^{(k)}(1)}{k!}(x-1)^k - \sum_{k=1}^{m}\sum_{i=0}^{2s_k} A^L_{ik}\left[r_x(t)\right]^{(i)}_{t=x_k}$$

From above relations we have

$$M(x) - f(x) = \int_0^1 r_x(t)d\lambda(t) - \sum_{k=0}^{n-1}\alpha_{k,2}\left[r_x(t)\right]^{(k)}_{t=1} - \sum_{k=1}^{m}\sum_{i=0}^{2s_k} A^L_{ik}\left[r_x(t)\right]^{(i)}_{t=x_k} = \mathcal{R}^L_{m,n}[r_x]\,.$$

PROBLEM M2. Determine the monospline M, defined in (19) such that

$$\int_0^1 t^j M(t)dt = \int_0^1 t^j f(t)dt\,,\ j = 0,1,\ldots,2(\sum_{k=1}^{m} s_k + m) - 1 \qquad (26)$$

THEOREM 9 *Let $f \in C^n[0,1]$. There exists a unique monospline M, defined in (19), satisfying (26), if and only if the measure*

$$d\lambda(t) = \frac{1}{(n-1)!}\left[1 + (-1)^{n-1} f^{(n)}(t)\right] dt$$

admits a Chakalov-Popoviciu quadrature formulae Radau type

$$\int_0^1 f(t)d\lambda(t) = \sum_{k=0}^{n-1} \tilde{\alpha}_{k,1} f^{(k)}(1) + \sum_{k=1}^{m}\sum_{i=0}^{2s_k} \tilde{A}_{ik}^{\mathcal{R}} f^{(i)}(\tilde{x}_k) + \tilde{\mathcal{R}}_{m,n}^{\mathcal{R}}[f]\,, \tag{27}$$

where

$$\tilde{\mathcal{R}}_{m,n}^{\mathcal{R}}[f] = 0 \text{ for } f \in \mathcal{P}_{2(\sum_{k=1}^{m} s_k+m)+n-1}$$

with distinct real zeros $\tilde{x}_k \in (0,1)\ ,\ k = \overline{1,m}$.
The monospline defined in (19), is given by

$$\begin{aligned} t_k &= \tilde{x}_k\ ,\ k = \overline{1,m} \\ B_k &= \tilde{\alpha}_{k,1} + \frac{(-1)^{n-k}}{(n-1)!} f^{(n-1-k)}(1)\ ,\ k = \overline{0,n-1} \\ a_{ik} &= \tilde{A}_{ik}^{\mathcal{R}}\ ,\ k = \overline{1,m}\ ,\ i = \overline{0,2s_k} \end{aligned}$$

where $\tilde{x}_k$ *and* $\tilde{A}_{ik}^{\mathcal{R}}$ *are the nodes and coefficients, respective of Chakalov-Popoviciu quadrature formulae of Radau type.*

Proof. From relations (26) we obtain:

$$\begin{aligned} \int_0^1 t^{j+n} d\lambda(t) = \sum_{k=0}^{n-1} \left[\frac{(-1)^{n-1-k}}{(n-1)!} f^{(n-1-k)}(1) + B_k \right] \cdot \left[t^{j+n} \right]_{t=1}^{(k)} + \\ \sum_{k=1}^{m}\sum_{i=0}^{2s_k} a_{ik} \left[t^{j+n} \right]_{t=t_k}^{(i)}\ ,\ \ j = 0,1,\ldots,2(\sum_{k=1}^{m} s_k + m) - 1 \end{aligned} \tag{28}$$

where

$$d\lambda(t) = \frac{1}{(n-1)!} \left[1 + (-1)^{n-1} f^{(n)}(t) \right] dt$$

Choosing in Chakalov-Popoviciu quadrature formulae of Radau type (27), $f(t) = t^n p(t)\,,\ p \in \mathcal{P}_{2(\sum_{k=1}^{m} s_k+m)-1}$, we obtain

$$\int_0^1 t^n p(t) d\lambda(t) = \sum_{k=0}^{n-1} \tilde{\alpha}_{k,1} \left[t^n p(t) \right]_{t=1}^{(k)} + \sum_{k=1}^{m}\sum_{i=0}^{2s_k} \tilde{A}_{ik}^{\mathcal{R}} \left[t^n p(t) \right]_{t=\tilde{x}_k}^{(i)} \tag{29}$$

From relations (28) and (29) we obtain

$$\begin{aligned} B_k &= \tilde{\alpha}_{k,1} + \frac{(-1)^{n-k}}{(n-1)!} f^{(n-1-k)}(1)\ ,\ k = \overline{0, n-1} \\ a_{ik} &= \tilde{A}_{ik}^{\mathcal{R}}\ ,\ k = \overline{1, m}\ ,\ i = \overline{0, 2s_k} \end{aligned}$$

THEOREM 10 *Let* $f \in C^n[0,1]$ *and* $r_x(t) = (t-x)_+^{n-1}$, $0 \le t \le 1$. *If the monospline M, defined in (19), satisfies the relations (26), then*

$$M(x) - f(x) = \tilde{\mathcal{R}}_{m,n}^{\mathcal{R}}[r_x]\,,\ 0 < x < 1$$

where $\tilde{\mathcal{R}}_{m,n}^{\mathcal{R}}[f]$ *is the remainder term in the Chakalov-Popoviciu quadrature formulae of Radau type (27).*

Proof. Using Taylor′s formula, we can write

$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(1)}{k!}(x-1)^k + \frac{(-1)^n}{(n-1)!}\int_0^1 (t-x)_+^{n-1} f^{(n)}(t)dt\,,$$

but

$$M(x) = \frac{(1-x)^n}{n!} - \sum_{k=0}^{n-1}\left[\tilde{\alpha}_{k,1} + \frac{(-1)^{n-k}}{(n-1)!} f^{(n-1-k)}(1)\right]\frac{(n-1)!}{(n-k-1)!}(1-x)^{n-k-1} -$$

$$\sum_{k=1}^{m}\sum_{i=0}^{2s_k} \tilde{A}_{ik}^{\mathcal{R}} \frac{(n-1)!}{(n-i-1)!}(\tilde{x}_k - x)_+^{n-i-1} =$$

$$\frac{1}{(n-1)!}\int_0^1 r_x(t)dt - \sum_{k=0}^{n-1}\tilde{\alpha}_{k,1}\left[r_x(t)\right]_{t=1}^{(k)} + \sum_{k=0}^{n-1}\frac{f^{(k)}(1)}{k!}(x-1)^k - \sum_{k=1}^{m}\sum_{i=0}^{2s_k}\tilde{A}_{ik}^{\mathcal{R}}\left[r_x(t)\right]_{t=\tilde{x}_k}^{(i)}$$

From above relations we obtain:

$$M(x) - f(x) = \int_0^1 r_x(t)d\lambda(t) - \sum_{k=0}^{n-1}\tilde{\alpha}_{k,1}\left[r_x(t)\right]_{t=1}^{(k)} - \sum_{k=1}^{m}\sum_{i=0}^{2s_k}\tilde{A}_{ik}^{\mathcal{R}}\left[r_x(t)\right]_{t=\tilde{x}_k}^{(i)} = \tilde{\mathcal{R}}_{m,n}^{\mathcal{R}}[r_x]\,.$$

PROBLEM M3. Determine the monospline M, defined in (19), such that

$$M^{(j)}(1) = f^{(j)}(1)\ ,\ j = \overline{0, n-1} \tag{30}$$

$$\int_0^1 t^j M(t)dt = \int_0^1 t^j f(t)dt\,,\ j = 0, 1, \ldots, 2(\sum_{k=1}^{m} s_k + m) - 1 \tag{31}$$

THEOREM 11 *Let $f \in C^n[0,1]$. There exists a unique monospline M, defined in (19),satisfying (30) and (31), if and only if the measure*

$$d\lambda(t) = \frac{1}{(n-1)!}\left[1 + (-1)^{n-1} f^{(n)}(t)\right] dt$$

admits a Chakalov-Popoviciu quadrature formulae of Radau type

$$\int_0^1 f(t)d\lambda(t) = \sum_{k=0}^{n-1} \alpha_{k,1} f^{(k)}(0) + \sum_{k=1}^{m}\sum_{i=0}^{2s_k} A_{ik}^{\mathcal{R}} f^{(i)}(x_k) + \mathcal{R}_{m,n}^{\mathcal{R}}[f]\,, \tag{32}$$

where

$$\mathcal{R}_{m,n}^{\mathcal{R}}[f] = 0 \text{ for } f \in \mathcal{P}_{2(\sum_{k=1}^{m} s_k + m) + n - 1}$$

with distinct real zeros $x_k \in (0,1)$, $k = \overline{1,m}$.
The monospline (19) is given by

$$\begin{aligned}
t_k &= x_k\ ,\ k = \overline{1,m}\\
B_k &= \frac{(-1)^{n-k}}{(n-1)!} f^{(n-1-k)}(1)\ ,\ k = \overline{0,n-1}\\
a_{ik} &= A_{ik}^{\mathcal{R}}\ ,\ k = \overline{1,m}\ ,\ i = \overline{0,2s_k}
\end{aligned}$$

where x_k and $A_{ik}^{\mathcal{R}}$ are the nodes and coefficients, respectively of Chakalov-Popoviciu quadrature formulae of Radau type.

Proof. From the definition (19) of monospline M, we obtain

$$M^{(j)}(1) = B_{n-j-1} \cdot (n-1)! \cdot (-1)^{j+1}\ ,\ \ j = \overline{0,n-1}$$

and using the relations (30) we have

$$B_j = \frac{(-1)^{n-j}}{(n-1)!} f^{(n-1-j)}(1)\ ,\ j = \overline{0,n-1} \tag{33}$$

From relations (31) we obtain:

$$\int_0^1 t^{j+n} d\lambda(t) = \sum_{k=0}^{n-1} \left[\frac{(-1)^{n-1-k}}{(n-1)!} f^{(n-1-k)}(1) + B_k \right] \cdot \left[t^{j+n} \right]_{t=1}^{(k)} + \sum_{k=1}^{m} \sum_{i=0}^{2s_k} a_{ik} \left[t^{j+n} \right]_{t=t_k}^{(i)} \; , \; j = 0, 1, \ldots, 2(\sum_{k=1}^{m} s_k + m) - 1 \tag{34}$$

where

$$d\lambda(t) = \frac{1}{(n-1)!} \left[1 + (-1)^{n-1} f^{(n)}(t) \right] dt$$

and using relation (33) we can write

$$\int_0^1 t^{n+j} d\lambda(t) = \sum_{k=1}^{m} \sum_{i=0}^{2s_k} a_{ik} \left[t^{j+n} \right]_{t=t_k}^{(i)} \; , \; j = 0, 1, \ldots, 2(\sum_{k=1}^{m} s_k + m) - 1 \tag{35}$$

Choosing in the Chakalov-Popoviciu quadrature formulae of Radau type (32), $f(t) = t^n p(t)\,,\ p \in \mathcal{P}_{2(\sum_{k=1}^{m} s_k + m) - 1}$, we obtain

$$\int_0^1 t^n p(t) d\lambda(t) = \sum_{k=1}^{m} \sum_{i=0}^{2s_k} A_{ik}^{\mathcal{R}} \left[t^n p(t) \right]_{t=x_k}^{(i)} \tag{36}$$

From relations (35) and (36) we have

$$a_{ik} = A_{ik}^{\mathcal{R}} \; , \; k = \overline{1, m} \; , \; i = \overline{0, 2s_k}$$

THEOREM 12 *Let* $f \in C^n[0,1]$ *and* $r_x(t) = (t-x)_+^{n-1}\,,\ 0 \le t \le 1$. *If the monospline M, defined in (19), satisfies the relations (30) and (31), then*

$$M(x) - f(x) = \mathcal{R}_{m,n}^{\mathcal{R}}[r_x]\,,\ 0 < x < 1$$

where $\mathcal{R}_{m,n}^{\mathcal{R}}[f]$ *is the remainder term in the Chakalov-Popoviciu quadrature formulae of Radau type (32).*

Proof. Using Taylor's formula we can write

$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(1)}{k!}(x-1)^k + \frac{(-1)^n}{(n-1)!} \int_0^1 (t-x)_+^{n-1} f^{(n)}(t) dt\,,$$

but

$$M(x) = \frac{(1-x)^n}{n!} + \sum_{k=0}^{n-1} \frac{f^{(k)}(1)}{k!}(x-1)^k - \sum_{k=1}^{m}\sum_{i=0}^{2s_k} A_{ik}^{\mathcal{R}} \frac{(n-1)!}{(n-i-1)!}(x_k-x)_+^{n-i-1} =$$

$$\frac{1}{(n-1)!}\int_0^1 r_x(t)dt + \sum_{k=0}^{n-1} \frac{f^{(k)}(1)}{k!}(x-1)^k - \sum_{k=1}^{m}\sum_{i=0}^{2s_k} A_{ik}^{\mathcal{R}} \left[r_x(t)\right]_{t=x_k}^{(i)}$$

From above relations we can write

$$M(x) - f(x) = \int_0^1 r_x(t)d\lambda(t) - \sum_{k=1}^{m}\sum_{i=0}^{2s_k} A_{ik}^{\mathcal{R}} \left[r_x(t)\right]_{t=x_k}^{(i)} = \mathcal{R}_{m,n}^{\mathcal{R}}[r_x]\,.$$

References

[1] N. Breaz, D. Breaz, *A spline aproximation of the factors path in MDF*, Acta Universitatis Apulensis, No. 2/2001, 35-46.

[2] N. Breaz, *A cross validation method for estimating the relative weights in a spline model*, Seminar on Numerical and Statistical Calculus, Babes-Bolyai, University of Cluj-Napoca, Preprint, 2004, 57-64

[3] N. Breaz, *Numerical experiments with least squares spline estimators in a parametric regression model*, Acta Universitatis Apulensis, no 8/2004, Proceedings of Int. Conf. on Theory and Appl. of Math. and Inf., Thessaloniki, Greece, 2004, 50-59

[4] N. Breaz, D. Breaz, *On the spline functions in data analysis framework*, Acta Universitatis Apulensis, No. 11/2006, Proceedings of Int. Conf. on Theory and Appl. of Math. and Inf., Albac, 2005,197-210

[5] M. Frontini, W. Gautschi, G.V. Milovanović , *Moment-preserving spline approximation on finite intervals*, Numer. Math. 50(1987) 503-518.

[6] M. Frontini, G.V. Milovanović , *Moment-preserving spline approximation on finite intervals and Turán quadratures*, Facta Univ. Ser. Math. Inform. 4(1989) 45-56.

[7] W.Gautschi , G.V. Milovanović, *S-orthogonality and construction of Gauss-Turán-type quadrature formulae*, J.Comput.Appl.Math., 86(2)(1997), 205-218.

[8] G.V. Milovanović, *Construction of s-orthogonal polynomials and Turán quadrature formulae*, Numerical Methods and Approximation, Theory III (Niš,1987), (ed. G.V.Milovanović), (Unv. of Niš, Niš, 1988), 311-328.

[9] G.V. Milovanović, M.M Spalević, *A numerical procedure for coefficients in generalized Gauss-Turán quadratures*, Filomat 9(1995) 1-8.

[10] G.V. Milovanović, M.M Spalević, *Construction of Chakalov-Popoviciu type quadrature formulae*, Rend. Circ. Mat. Plermo (Suppl.) (2) II (52) (1998), 625-636.

[11] G.V. Milovanović, M.M Spalević, *Quadrature formulae conected to σ-orthogonal polinomials*, J. Comput. Appl. Math. 140 (2002), 619-637.

[12] M.M. Spalević , *Product of Turán cuadratures for cube, simplex, surface of the sphere, $\overline{E}_n^r$, $E_n^{r^2}$*, J. Comput. Appl. Math. 106 (1999) 99-115.

[13] M.M. Spalević , *Calculation of Chakalov-Popoviciu Quadratures of Radau and Lobatto type*, ANZIAM J. 43 (2002), 429-447.

[14] D.D. Stancu , *Generalizarea unor formule de interpolare pentru functiile de mai multe variabile si unele contributii asupra formulei de integrare numerica a lui Gauss*, Buletin St. Acad. Române 9(1957), 287-313.

[15] D.D. Stancu , *O metoda pentru construirea de formule de cuadratura de grad inalt de exactitate*, Comunic. Acad. R.P.Rom. 8(1958), 349-358.

[16] D.D. Stancu , *On the Gaussian quadrature formulae*, Studia Univ. "Babe's-Bolyai", Matematica I (1958), 71-84.

[17] D.D. Stancu , *Sur quelques formules générales de quadrature du type Gauss-Christoffel*, Mathematica 1(24) (1959), 167-182.

[18] D.D. Stancu, *Quadrature formulas with multiple Gaussian nodes*, SIAM Numer. Anal. Ser. B2 (1965), 129-143.

[19] D.D. Stancu, A.H. Stroud , *Quadrature formulas with simple Gaus-*

sian nodes and multiple fixed nodes, Math. Comp. 17 (1963), 384-394.

[20] D. D. Stancu , G. Coman , P. Blaga, *Analiza Numerica si Teoria Aproximarii*, Presa Universitara Clujeana , Cluj-Napoca 2002.

Author:

Ana Maria Acu
University "Lucian Blaga" of Sibiu
Department of Mathematics
Str. Dr. I. Rațiu, No. 5-7
550012 - Sibiu, Romania
e-mail: acuana77@yahoo.com