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THE ELASTICITY OF THE THIN AND PERIODIC OBLIQUE BARS

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Abstract. The homogenization of linear elasticity problem for a two-dimensional domain of thin and periodic oblique bars which are periodic distributed is studied. These structures depend by the ε-period and δ-parameter. Then the elasticity coefficients are obtained.

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1. Introduction

The problem of the linear elasticity for oblique bars is an elliptical problem of the second order in a perforated domain. To homogenize this problem first we make ε tends to zero using L. Tartar's variational method. The result is a limit problem (named homogenized) which is set on a space domain without holes and has constant coefficients. These coefficients are integrals on the cell of periodicity from the Y-periodic correction functions, defined themselves on the periodicity cell. Because it is a reticulated structure, these coefficients depend on the thickness δ of the material from the periodicity cell.

In the literature was studied the homogenization of the linear elasticity for reticulated domain with different geometry (i.e. honeycomb, reinforced structure)

In this paper we make $\delta \to 0$. Our study is based on the dilatation method where the integrals on domains of thick δ becomes integrals on a fixed domain. Now, the parameter δ appears explicit in the integrals. The a-priori estimations for the correction functions make possible the transition to limit $\delta \to 0$ in the expression coefficients, finally obtaining homogenized coefficients which have the same symmetry and elipticity property as elasticity coefficients of the perforated material (named reinforced structure with oblique bars).

1.1. The periodic structure $\Omega_{\varepsilon,\delta}$ made from oblique bars.

Consider the open and bounded domain $\Omega = (0, l_1) \times (0, l_2)$, the reference cell $Y = \left(-\frac{1}{2}, \frac{1}{2}\right) \times \left(-\frac{1}{2}, \frac{1}{2}\right)$, the periodicity cell $Y_\delta = H_\delta \cup V_\delta \cup O_\delta$, where

$$H_\delta = \left\{ |y_1| < \frac{1}{2},\ |y_2| < \frac{\delta}{2} \right\}$$

$$V_\delta = \left\{ |y_1| < \frac{\delta}{2},\ |y_2| < \frac{1}{2} \right\}$$

and the oblique bar O_δ has the height $\sqrt{2}$ and the width δ. In the figure 1 is representing the periodicity cell Y_δ.

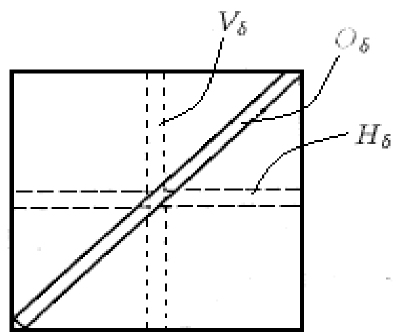

Fig.1. The periodicity cell Y_δ.

The periodicity cell Y_δ is distributed in the domain Ω with the period ε and along the OX_1 axe and OX_2, respectively. Thus it results the perforated domain $\Omega_{\varepsilon\delta}$, which represent the domain from Ω occupied by the distributed material along the bars with the width δ. In the figure 2 we represent the perforated domain $\Omega_{\varepsilon\delta}$ (named reticulated domain):

1.2. Setting the problem.

It is known that the problem of the linear elasticity for a domain $\Omega_{\varepsilon\delta}$ is

$$\begin{gathered} -\frac{\partial}{\partial x_j}\left(a_{ijkh}\frac{\partial u_k^{\varepsilon\delta}}{\partial x_k}\right) = f_i \text{ in } \Omega_{\varepsilon\delta},\ i = 1, 2,\ u^{\varepsilon\delta} = (u_1^{\varepsilon\delta}, u_2^{\varepsilon\delta}) \\ a_{ijkh}\frac{\partial u_k^{\varepsilon\delta}}{\partial x_k} \cdot n_j = 0 \text{ on } \partial T_{\varepsilon\delta} \\ u_k^{\varepsilon\delta} = 0 \text{ on } \partial\Omega, \end{gathered} \tag{1}$$

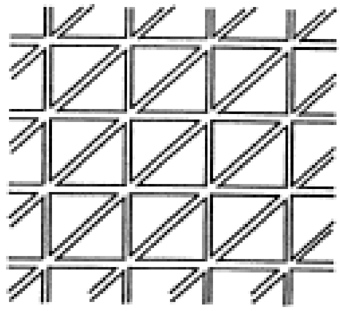

Fig.2. The perforated domain $\Omega_{\varepsilon\delta}$

where a_{ijkh} are the elasticity constants and satisfy the conditions of symmetry and elipticity
i) $a_{ijkh} = a_{ijhk} = a_{khij},\ \forall i, j, k, h \in \{1, 2\}$
ii) $\exists C_0 > 0$ such that $a_{ijkh} v_{ij} v_{kh} \geq C_0 v_{ij} v_{ij},\ \forall$ symmetric matrix v_{ij},
iii) $f = (f_1, f_2) \in \left[L^2(\Omega)\right]^2$.

Let be

$$V_{\varepsilon\delta} = \{v \in H^1(\Omega_{\varepsilon\delta}) \mid v = 0 \text{ on } \partial\Omega\},$$

with the induced norm by the $H^1(\Omega_{\varepsilon\delta})$.

According to Lax-Milgram theorem, (1) has unique solution $u_k^{\varepsilon\delta} \in V_{\varepsilon\delta}$.

1.3. Basic results in the homogenization of the linear elasticity problem from periodic domain.

By the L. Tartar's variational method, if $\varepsilon \to 0$, we find

Theorem 1.1.[6] *There exists an extension operator* $P^{\varepsilon\delta} \in \mathcal{L}\left([V_{\varepsilon\delta}]^2; [H_0^1(\Omega)]^2\right)$ *such that*

$$P^{\varepsilon\delta} u^{\varepsilon\delta} \to u^\delta \textit{ weakly in } [H_0^1(\Omega)]^2,$$

where u^δ *is the solution of the problem*

$$\begin{aligned} -q_{ijkh}^\delta \frac{\partial^2 u_k^\delta}{\partial x_j \partial x_h} &= \frac{meas\ Y_\delta}{meas\ Y} f_i \textit{ in } \Omega \\ u_k^\delta &= 0 \textit{ on } \partial\Omega. \end{aligned} \tag{2}$$

The homogenization coefficients are defined by

$$q^{\delta}_{ijkh} = \int\limits_{Y_\gamma} \left(a_{ijkh} - a_{ijpr} \frac{\partial \chi^{kh}_{\delta,p}}{\partial y_r} \right) dy \tag{3}$$

The periodic corrector functions $\chi^{kh}_{\delta} = (\chi^{kh}_{\delta,1}, \chi^{kh}_{\delta,2})$ *are given by*

$$\begin{aligned} -\frac{\partial}{\partial y_j} \left(a_{ijlm} \frac{\partial(\chi^{kh}_{\delta,l} - y_k \delta_{hl})}{\partial y_m} \right) &= 0 \ in \ Y_\delta \\ a_{ijlm} \frac{\partial(\chi^{kh}_{\delta,l} - y_k \delta_{hl})}{y_m} \cdot n_j &= 0 \ on \ \partial T_\delta \\ \chi^{kh}_{\delta,l} \ are \ Y &- periodic. \end{aligned} \tag{4}$$

2.Main results

In this paper we consider the isotropic material

$$a_{ijkh} = \lambda \delta_{ij} \delta_{kh} + \mu(\delta_{ik}\delta_{jh} + \delta_{ih}\delta_{jk}), \tag{5}$$

where λ and μ are Lamé constants.

In the following we consider $\delta \to 0$ and we homogenize the problem (2) using the method from [4].

Theorem 2.1. *For the reticulated structure* $\Omega_{\varepsilon\delta}$*, for* $\delta \to 0$*, the following convergence holds:*

$$u^{\delta} \underset{\delta \to 0}{\rightharpoonup} u^* \ weakly \ in \ [H^1_0(\Omega)]^2, \tag{6}$$

where u^* *is the solution of the limit problem*

$$\begin{aligned} -q^*_{ijkh} \frac{\partial^2 u^*_k}{\partial x_j \partial x_h} &= (2+\sqrt{2}) f_i \ in \ \Omega \\ u^*_k &= 0 \ on \ \partial\Omega. \end{aligned} \tag{7}$$

The homogenized coefficients q^*_{ijkl} *are symmetric and elliptic and have the form:*

$$\begin{aligned} & q^*_{1111} = q^*_{2222} = 2\left(2 + \frac{\sqrt{2}}{2}\right) \mu \frac{\lambda+\mu}{\lambda+2\mu} \\ & q^*_{1122} = q^*_{2211} = \sqrt{2}\mu \frac{\lambda+\mu}{\lambda+2\mu} \\ & q^*_{1212} = q^*_{1221} = q^*_{2112} = q^*_{2121} = \sqrt{2}\mu \frac{\lambda+\mu}{\lambda+2\mu} \\ & q^*_{ijkh} = 0 \ in \ all \ the \ other \ cases \end{aligned} \tag{8}$$

Proof.
We have

$$\text{meas } Y_\delta = (2+\sqrt{2})\delta(1-\delta)$$

and the a priori estimation [3]

$$\|grad\ \chi_\delta^{kh}\|_{[L^2(Y_\delta)]^{2\times 2}} \le C\delta^{\frac{1}{2}}, \tag{9}$$

where C is a positive constant, which is independent of δ.
From (3) and (9) we obtain

$$\delta^{-1} q_{ijkh}^{\delta} \underset{\delta\to 0}{\rightharpoonup} q_{ijkh}^{*}. \tag{10}$$

From (3) and the decomposition of Y_δ in $Y_\delta = H_\delta \cup V_\delta \cup O_\delta$, we obtain

$$\begin{aligned} \delta^{-1} q_{ijkh}^{\delta} &= (2+\sqrt{2})(1-\delta)a_{ijkh} - \\ &-\delta^{-1}\left(\int\limits_{H_\delta} + \int\limits_{V_\delta} + \int\limits_{O_\delta} - \int\limits_{K_\delta}\right)\left(a_{ijpr}\frac{\partial \chi_{\delta,p}^{kh}}{\partial y_r}\right) dy, \end{aligned} \tag{11}$$

where $K_\delta = H_\delta \cap V_\delta \cap O_\delta$.
Due to the estimation (9) and to the relation meas $K_\delta \le C\delta^2$, we have

$$\delta^{-1}\int\limits_{K_\delta} a_{ijpr}\frac{\partial \chi_{\delta,p}^{kh}}{\partial y_r} dy \underset{\delta\to 0}{\rightharpoonup} 0. \tag{12}$$

Now, we make a rotation of angle $-\dfrac{\pi}{4}$ and thus, the oblique bar O_δ becomes the horizontal bar $\widetilde{O}_\delta$ defined by

$$\widetilde{O}_\delta = \left\{(t_1,t_2) \mid |t_1| \le \frac{\sqrt{2}}{2},\ |t_2| \le \frac{\delta}{2}\right\}.$$

Now, we apply the dilatation method [3], which consists in the dilatation of the horizontal bar in the domain

$$Y_0 = \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) \times \left(-\frac{1}{2}, \frac{1}{2}\right)$$

with help of the transformation

$$\begin{aligned} z_1 &= \frac{\sqrt{2}}{2}y_1 + \frac{\sqrt{2}}{2}y_2 = t_1 \\ z_2 &= -\frac{\sqrt{2}}{2\delta}y_1 + \frac{\sqrt{2}}{2\delta}y_2 = \frac{t_2}{\delta} \end{aligned}$$

With the change of the function

$$\varphi_O\left(\frac{\sqrt{2}}{2}y_1 + \frac{\sqrt{2}}{2}y_2, -\frac{\sqrt{2}}{2\delta}y_1 + \frac{\sqrt{2}}{2\delta}y_2\right) = \varphi_O(z_1, z_2) = \varphi(y_1, y_2)$$

and using the estimation (9), we obtain the following weak convergence

$$\begin{aligned} \frac{\partial\left(\chi_\delta^{kh}\right)_O}{\partial z_1} &\underset{\delta\to 0}{\rightharpoonup} o_1^{kh} \quad \text{weakly in } \left[L^2(Y_0)\right]^2, \\ \delta^{-1}\frac{\partial\left(\chi_\delta^{kh}\right)_O}{\partial z_2} &\underset{\delta\to 0}{\rightharpoonup} o_2^{kh} \quad \text{weakly in } \left[L^2(Y_0)\right]^2. \end{aligned} \tag{13}$$

Applying, analogue, the dilatation method for the bars H_δ and V_δ, which, by the corresponding transformations pass into the domain $Y = \left(-\frac{1}{2}, \frac{1}{2}\right) \times \left(-\frac{1}{2}, \frac{1}{2}\right)$, with the help of the estimation (9), we found the weak convergences:

$$\begin{aligned} \frac{\partial\left(\chi_\delta^{kh}\right)_H}{\partial z_1} &\underset{\delta\to 0}{\rightharpoonup} w_1^{kh} \quad \text{weakly in } \left[L^2(Y)\right]^2, \\ \delta^{-1}\frac{\partial\left(\chi_\delta^{kh}\right)_H}{\partial z_2} &\underset{\delta\to 0}{\rightharpoonup} w_2^{kh} \quad \text{weakly in } \left[L^2(Y)\right]^2, \end{aligned} \tag{14}$$

and

$$\begin{aligned} \delta^{-1}\frac{\partial\left(\chi_\delta^{kh}\right)_V}{\partial z_1} &\underset{\delta\to 0}{\rightharpoonup} v_1^{kh} \quad \text{weakly in } \left[L^2(Y)\right]^2, \\ \frac{\partial\left(\chi_\delta^{kh}\right)_V}{\partial z_2} &\underset{\delta\to 0}{\rightharpoonup} v_2^{kh} \quad \text{weakly in } \left[L^2(Y)\right]^2, \end{aligned} \tag{15}$$

where $\varphi_H\left(y_1, \frac{y_2}{\delta}\right) = \varphi(y_1, y_2)$ and $\varphi_V\left(\frac{y_1}{\delta}, y_2\right) = \varphi(y_1, y_2)$ respectively.

Due to the Y-periodicity of the function χ_δ^{kh} we have:

$$\int\limits_Y w_{1,j}^{kh}dy = 0, \ \int\limits_Y v_{2,j}^{kh}dy = 0 \text{ and } \int\limits_{Y_0} o_{1,j}^{kh}dy = 0, \ j \in \{1, 2\}. \tag{16}$$

Now consider the relation (11) for $\delta \to 0$, and, due to the convergences (10), (13), (14), (15) and the relation (16), we find:

$$\begin{aligned} q^*_{ijkh} &= (2+\sqrt{2})a_{ijkh} - a_{ij2r}\int\limits_Y w^{kh}_{2,r}dy - a_{ij1r}\int\limits_Y v^{kh}_{1,r}dy \\ & -\frac{\sqrt{2}}{2}(-a_{ij1r}+a_{ij2r})\int\limits_{Y_0} o^{kh}_{2,r}dy. \end{aligned} \tag{17}$$

In the following we multiply the equation $(4)_1$ with the test functions $\delta^{-1}\left(\varphi(y_1,y_2),0\right)$, respectively $\delta^{-1}\left(0,\varphi(y_1,y_2)\right)$, where

$$\varphi(y_1,y_2) = \varphi_1(y_1)\cdot\varphi_2(y_2)\cdot\varphi_3\left(-\frac{\sqrt{2}}{2}y_1+\frac{\sqrt{2}}{2}y_2\right).$$

We consider the functions φ_1 and φ_2 periodic with the period equal 1, and the function φ_3 with the period $\frac{\sqrt{2}}{2}$, therefore the function φ is Y-periodic. We integrate by parts using the transformations of H_δ, V_δ, O_δ in Y, respectively Y_0, then we consider $\delta \to 0$. Thus, we obtain

$$\begin{aligned} &\left(I^1_H+\frac{\sqrt{2}}{2}I^2_H\right)\varphi_2(0)+\left(I^1_V-\frac{\sqrt{2}}{2}I^2_V\right)\varphi_1(0)+I^3_H\left(\frac{\partial\varphi_2}{\partial y_2}\right)(0)+ \\ &+I^3_V\left(\frac{\partial\varphi_1}{\partial y_1}\right)(0)+\frac{\sqrt{2}}{2}\left(I^1_O+I^2_O\right)\varphi_3(0)+\frac{1}{2}\left(I^3_O-I^4_O\right)\left(\frac{\partial\varphi_3}{\partial z}\right)(0)=0, \end{aligned} \tag{18}$$

where

$$I^1_H = \int\limits_Y [a_{ilp1}w^{kh}_{l,p}-a_{i1kh}]\frac{\partial}{\partial y_1}\left[\varphi_1(y_1)\varphi_3\left(-\frac{\sqrt{2}}{2}y_1\right)\right]dy,$$

$$I^2_H = \int\limits_Y [a_{ilp2}w^{kh}_{l,p}-a_{i2kh}]\varphi_1(z_1)\left(\frac{\partial\varphi_3}{\partial z}\right)\left(-\frac{\sqrt{2}}{2}z_1\right)dz,$$

$$I^3_H = \int\limits_Y [a_{ilp2}w^{kh}_{l,p}-a_{i2kh}]\varphi_1(y_1)\varphi_3\left(-\frac{\sqrt{2}}{2}y_1\right)dy,$$

$$I^1_V = \int\limits_Y [a_{ilp2}v^{kh}_{l,p}-a_{i2kh}]\left(\frac{\partial}{\partial y_2}\left[\varphi_2(y_2)\varphi_3\left(-\frac{\sqrt{2}}{2}y_1+\frac{\sqrt{2}}{2}y_2\right)\right]\right)(0,z_2)dz,$$

$$I_V^2 = -\int\limits_Y [a_{ilp1} v_{l,p}^{kh} - a_{i1kh}] \varphi_2(z_2) \left(\frac{\partial \varphi_3}{\partial z}\right) \left(\frac{\sqrt{2}}{2} z_2\right) dz,$$

$$I_V^3 = \int\limits_Y [a_{ilp1} v_{l,p}^{kh} - a_{i1kh}] \varphi_2(z_2) \varphi_3 \left(\frac{\sqrt{2}}{2} z_2\right) dz,$$

$$I_O^1 = \int\limits_{Y_0} \left[(a_{i1p1} + a_{i2p1}) o_{1,p}^{kh} + (-a_{i1p1} + a_{i2p1}) o_{2,p}^{kh} - \frac{2}{\sqrt{2}} a_{i1kh}\right] \cdot$$

$$\cdot \left(\varphi_2 \frac{\partial \varphi_1}{\partial y_1}\right) \left(\frac{\sqrt{2}}{2} z_1, \frac{\sqrt{2}}{2} z_1\right) dz,$$

$$I_O^2 = \int\limits_{Y_0} \left[(a_{i1p2} + a_{i2p2}) o_{1,p}^{kh} + (-a_{i1p2} + a_{i2p2}) o_{2,p}^{kh} - \frac{2}{\sqrt{2}} a_{i2kh}\right] \cdot$$

$$\cdot \left(\varphi_1 \frac{\partial \varphi_2}{\partial y_2}\right) \left(\frac{\sqrt{2}}{2} z_1, \frac{\sqrt{2}}{2} z_1\right) dz,$$

$$I_O^3 = \int\limits_{Y_0} \left[(a_{i1p1} + a_{i2p1}) o_{1,p}^{kh} + (-a_{i1p1} + a_{i2p1}) o_{2,p}^{kh} - \frac{2}{\sqrt{2}} a_{i1kh}\right] \cdot$$

$$\cdot \varphi_1 \left(\frac{\sqrt{2}}{2} z_1\right) \varphi_2 \left(\frac{\sqrt{2}}{2} z_1\right) dz,$$

$$I_O^4 = \int\limits_{Y_0} \left[(a_{i1p2} + a_{i2p2}) o_{1,p}^{kh} + (-a_{i1p2} + a_{i2p2}) o_{2,p}^{kh} - \frac{2}{\sqrt{2}} a_{i2kh}\right] \cdot$$

$$\cdot \varphi_1 \left(\frac{\sqrt{2}}{2} z_1\right) \varphi_2 \left(\frac{\sqrt{2}}{2} z_1\right) dz,$$

Let choose in relation (18) the functions $\varphi_1,\ \varphi_2,\ \varphi_3$ such that $\varphi_1(0) = \varphi_2(0) = \varphi_3(0) = 0$ and $\dfrac{\partial \varphi_1}{\partial y_1} \neq 0,\ \dfrac{\partial \varphi_2}{\partial y_2} \neq 0,\ \dfrac{\partial \varphi_3}{\partial z} \neq 0$. Therefore, we obtain $I_H^3 = 0,\ I_V^3 = 0,\ I_O^3 - I_O^4 = 0$, and, thus,

$$a_{ilp2} \int\limits_{-\frac{1}{2}}^{\frac{1}{2}} w_{l,p}^{kh} dy_2 = a_{i2kh}, \quad a_{ilp1} \int\limits_{-\frac{1}{2}}^{\frac{1}{2}} v_{l,p}^{kh} dy_1 = a_{i1kh},$$

$$(-a_{i1p1} - a_{i2p1} + a_{i1p2} + a_{i2p2}) \int_{-\frac{1}{2}}^{\frac{1}{2}} o_{1,p}^{kh} dz_2 +$$

$$+(a_{i1p1} - a_{i1p2} - a_{i2p1} + a_{i2p2}) \int_{-\frac{1}{2}}^{\frac{1}{2}} o_{2,p}^{kh} dz_2 = \frac{2}{\sqrt{2}}(-a_{i1kh} + a_{i2kh}).$$

From the Y-periodicity we obtain:

$$\int_Y w_{1,p}^{kh} dy = \int_Y v_{2,p}^{kh} dy = \int_{Y_0} o_{1,p}^{kh} dy = 0, \ p \in \{1,2\} \tag{19}$$

$$a_{i2p2} \int_Y w_{2,p}^{kh} dy = a_{i2kh}, \quad a_{i1p1} \int_Y v_{1,p}^{kh} dy = a_{i1kh}, \tag{20}$$

$$(a_{i1p1} - a_{i1p2} - a_{i2p1} + a_{i2p2}) \int_Y o_{2,p}^{kh} dy = 2(-a_{i1kh} + a_{i2kh}). \tag{21}$$

Replacing the relations (19), (20) and (21) in (17), we obtain the homogenization coefficients (8).

From the dilatation method [3] we obtain the weak convergence (6).

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