Acta Universitatis Apulensis No. 31/2012
ISSN: 1582-5329 pp. 257-282

REGULARIZATION FOR A LAPLACE EQUATION WITH NONHOMOGENEOUS NEUMANN BOUNDARY CONDITION

Nguyen Huy Tuan and Ngo Van Hoa

Abstract. We consider the following problem

$$\begin{cases} u_{xx} + u_{yy} = 0, (x, y) \in (0, \pi) \times (0, 1) \\ u(0, y) = u(\pi, y) = 0, y \in (0, 1) \\ u_y(x, 0) = g(x), 0 < x < \pi \\ u(x, 0) = \varphi(x), 0 < x < \pi \end{cases}$$

The problem is shown to be ill-posed, as the solution exhibits unstable dependence on the given data functions. Using the new method, we regularize of problem and obtain some new results. Some numerical examples are given to illuminate the effect of our methods.

2000 *Mathematics Subject Classification*: 35K05, 35K99, 47J06, 47H10.

1. Introduction

We consider the Cauchy problem for the Laplace equation in a rectangle: determine the solution $u(x, y)$ for $0 < y \leq 1$ from the input data $\varphi(.) := u(., 0)$, when $u(x, y)$ satisfies

$$\begin{cases} u_{xx} + u_{yy} = 0, (x, y) \in (0, \pi) \times (0, 1) \\ u(0, y) = u(\pi, y) = 0, y \in (0, 1) \\ u_y(x, 0) = g(x), 0 < x < \pi \\ u(x, 0) = \varphi(x), 0 < x < \pi \end{cases} \tag{1}$$

where $\varphi(x), g(x) \in L^2(0, \pi)$ are noisy functions. If $g = 0$, the problem (1) becomes

$$\begin{cases} u_{xx} + u_{yy} = 0, (x, y) \in (0, \pi) \times (0, 1) \\ u(0, y) = u(\pi, y) = 0, y \in (0, 1) \\ u_y(x, 0) = 0, 0 < x < \pi \\ u(x, 0) = \varphi(x), 0 < x < \pi \end{cases} \tag{2}$$

which can scarcely be found in a series of articles analyzing the stability and convergence(see e.g [1, 2, 3, 4, 6, 8, 13]).

Many physical and engineering problems in areas like geophysics and seismology require the solution of a Cauchy problem for the Laplace equation. For example, certain problems related to the search for mineral resources, which involve interpretation of the earth's gravitational and magnetic fields, are equivalent to the Cauchy problem for the Laplace equation. The continuation of the gravitational potential observed on the surface of the earth in a direction away from the sources of the field is again such a problem.

The Cauchy problem for the Laplace equation and for other elliptic equations is in general ill-posed in the sense that the solution, if it exists, does not depend continuously on the initial data. This is because the Cauchy problem is an initial value problem which represents a transient phenomenon in a time-like variable while elliptic equations describe steady-state processes in physical fields. A small perturbation in the Cauchy data, therefore, affects the solution largely.

Very recently, Chu Li-Fu et al [11] approximated the problem (2) by the fourth-order method

$$\begin{cases} v^{\delta}_{xx} + v^{\delta}_{yy} - \beta^2 v^{\delta}_{xxyy} = 0, (x,y) \in (0,\pi)\times(0,1) \\ v^{\delta}(0,y) = v^{\delta}(\pi,y) = 0, y \in (0,1) \\ v^{\delta}_{y}(x,0) = 0, 0 < x < \pi \\ v^{\delta}(x,0) = \varphi^{\delta}(x), 0 < x < \pi \end{cases} \tag{3}$$

The solution of (3) is given by

$$v^{\delta}(x,y) = \sum_{n=1}^{\infty}\left[\left(\frac{\exp\{\sqrt{\frac{n^2}{1+\beta n^2}}y\} + \exp\{-\sqrt{\frac{n^2}{1+\beta n^2}}y\}}{2}\right) < \varphi^{\delta}, \sin nx > \right]\sin nx \tag{4}$$

where

$$< \varphi^{\delta}, \sin nx > = \frac{2}{\pi}\int_0^{\pi}\varphi^{\delta}(x)\sin nx dx.$$

Informally, by the method of separation of variables, the solution of problem (1) is as follows

$$u(x,y) = \sum_{n=1}^{\infty}\left[\left(\frac{e^{ny}+e^{-ny}}{2}\right)\varphi_n + \left(\frac{e^{ny}-e^{-ny}}{2n}\right)g_n\right]\sin nx \tag{5}$$

where

$$g(x) = \sum_{n=1}^{\infty} g_n \sin nx, \varphi(x) = \sum_{n=1}^{\infty}\varphi_n \sin nx. \tag{6}$$

The term e^{ny} in (5) increase rather quickly when n become large, so it is the unstability cause. Since the exact solution in (5), Chu-Li Fu and his coauthors replaced e^{ny} and e^{-ny} by two better terms $\exp\{\sqrt{\frac{n^2}{1+\beta n^2}}y\}$ and $\exp\{-\sqrt{\frac{n^2}{1+\beta n^2}}y\}$ respectively(when $g=0$). However, these terms contain complicated square, so it is difficult to estimate the error. In our opinion, it is complicated to consider the problem (1) using the method in (3).

Although (2) is considered in some papers, but there are not many results of (1). In the present paper, we shall introduce two new regularization methods to solve the Cauchy problem for the Laplace equation which is an extension of paper [11]. In the first method, we replace only e^{ny} by the better term $e^{ny-\beta n^2 y}$. The term e^{-ny} is bounded by 1, so it is not need to replaced. Based on the inequality $n-\beta n^2 \le \frac{1}{4\beta}$, the pertubing term becomes stability. In second method, we replace e^{ny} by the different term $\frac{e^{ny}}{1+\alpha e^n}$. The regularization methods are effective and convenient for dealing with some ill-posed problems. Moreover, we establish some new error estimates including the order of Hölder type. Especially, the convergence of the approximate solution at $y=1$ is also proved.
The paper is organized as follows. In Section 2, we introduce the first regularization method and obtain the convergence estimates. In Section 3, we use a second method to construct a stable approximation solution and give the convergence estimates. Finally, In Section 4, a numerical example is given to test the effectiveness of the proposed methods.

2. The first regularization method

Throught out this paper, we assume that the functions $\varphi, g \in L^2(0,\pi)$. Physically, φ, g can only be measured, there will be measurement errors, and we would actually have as data some function $\varphi^\epsilon = \sum\limits_{n=1}^{\infty} \varphi_n^\epsilon \sin nx \in L^2(0,\pi)$, $g^\epsilon = \sum\limits_{n=1}^{\infty} g_n^\epsilon \sin nx \in L^2(0,\pi)$ for which

$$\begin{aligned} \|\varphi^\epsilon - \varphi\| &= \|\varphi^\epsilon - u(.,0)\| \le \epsilon, \\ \|g^\epsilon - g\| &\le \epsilon, \end{aligned}$$

where the constant $\epsilon > 0$ represents a bound on the measurement error, $\|.\|$denotes the L^2-norm.
We modify the exact solution u as follows

$$u^\epsilon(x,y) = \sum_{n=1}^{\infty} \left[\left(\frac{e^{(n-\beta n^2)y} + e^{-ny}}{2} \right) \varphi_n + \left(\frac{e^{(n-\beta n^2)y} - e^{-ny}}{2n} \right) g_n \right] \sin nx \qquad (7)$$

where φ_n, g_n are defined by (6). And β is the regularization parameter which depend on ϵ.
Let the function v^ϵ be defined

$$v^\epsilon(x,y) = \sum_{n=1}^{\infty}\left[\left(\frac{e^{(n-\beta n^2)y}+e^{-ny}}{2}\right)\varphi_n^\epsilon + \left(\frac{e^{(n-\beta n^2)y}-e^{-ny}}{2n}\right)g_n^\epsilon\right]\sin nx. \qquad (8)$$

Regarding the stability of the regularized solution we have the following result.
Theorem 1 *Let u^ϵ, v^ϵ be defined by* (7) *and* (8) *respectively. Then one has*

$$\|v^\epsilon(.,y) - u^\epsilon(.,y)\| \le \epsilon\sqrt{2e^{\frac{1}{2\beta}}+2}.$$

Proof.
It follows from (7) and (8) that

$$\begin{aligned}\|u^\epsilon(.,y) - v^\epsilon(.,y)\|^2 &\le 2\frac{\pi}{2}\sum_{n=1}^{\infty}\left[\left(\frac{e^{(n-\beta n^2)y}+e^{-ny}}{2}\right)(\varphi_n^\epsilon-\varphi_n)\right]^2 + \\ &\qquad 2\frac{\pi}{2}\sum_{n=1}^{\infty}\left[\left(\frac{e^{(n-\beta n^2)y}-e^{-ny}}{2n}\right)(g_n^\epsilon-g_n)\right]^2 \\ &\le 2\frac{\pi}{2}\sum_{n=1}^{\infty}\frac{e^{2(n-\beta n^2)y}+1}{2}\left[(\varphi_n^\epsilon-\varphi_n)^2+(g_n^\epsilon-g_n)^2\right].\end{aligned}$$

Using the inequality $n-\beta n^2 \le \frac{1}{4\beta}$, we get

$$\begin{aligned}\|u^\epsilon(.,y) - v^\epsilon(.,y)\|^2 &\le 2(e^{\frac{1}{2\beta}}+1)\|\varphi^\epsilon-\varphi\|^2 \\ &\le 2(e^{\frac{1}{2\beta}}+1)\epsilon^2.\end{aligned}$$

Theorem 2. *Let E be positive numbers such that $\|u(.,1)\|^2 + \|u_y(.,1)\|^2 \le E^2$. If we select $\beta = \frac{1}{2m\ln(\frac{1}{\epsilon})}$ $(0 < m < 2)$, then one has*

$$\|v^\epsilon(.,y) - u(.,y)\| \le \frac{1}{2m\ln(\frac{1}{\epsilon})}\frac{\sqrt{2}E}{(1-y)^2} + \sqrt{2\epsilon^2+2\epsilon^{2-m}}, \qquad (9)$$

for every $y \in [0,1)$.

Proof.
Step 1. First, we estimate the following error

$$\|u(.,y) - u^\epsilon(.,y)\| \le \frac{\sqrt{2}E\beta}{(1-y)^2}.$$

Infact, we have

$$u(x,y) - u^\epsilon(x,y) = \sum_{n=1}^{\infty} \left(\frac{e^{ny} - e^{(n-\beta n^2)y}}{2} \right) \left(\varphi_n + \frac{g_n}{n} \right) \sin nx. \tag{10}$$

From (5) and

$$u_y(x,y) = \sum_{n=1}^{\infty} n \left[\left(\frac{e^{ny} - e^{-ny}}{2} \right) \varphi_n + \left(\frac{e^{ny} + e^{-ny}}{2n} \right) g_n \right] \sin nx$$

we get

$$\begin{aligned} < u(x,1), \sin nx > &= \left(\frac{e^{n} + e^{-n}}{2} \right) \varphi_n + \left(\frac{e^{n} - e^{-n}}{2n} \right) g_n. \\ \frac{1}{n} < u_y(x,1), \sin nx > &= \left(\frac{e^{n} - e^{-n}}{2} \right) \varphi_n + \left(\frac{e^{n} + e^{-n}}{2n} \right) g_n. \end{aligned}$$

It implies that

$$< u(x,1), \sin nx > + \frac{1}{n} < u_y(x,1), \sin nx > = e^{n} (\varphi_n + \frac{g_n}{n}). \tag{11}$$

Combining (10) and (11) we get

$$u(x,y) - u^\epsilon(x,y) =$$
$$\sum_{n=1}^{\infty} \left(\frac{e^{ny} - e^{(n-\beta n^2)y}}{2e^{n}} \right) \left(< u(x,1), \sin nx > + \frac{1}{n} < u_y(x,1), \sin nx > \right) \sin nx.$$

Using the inequalities $(a+b)^2 \le 2(a^2+b^2)$ and $1 - e^{-x} \le x, \quad x > 0$, we get

$$\begin{aligned} &| < u(x,y) - u^\epsilon(x,y), \sin nx > |^2 = \\ &\quad = \left(\frac{e^{ny} - e^{(n-\beta n^2)y}}{2e^{n}} \right)^2 \left(< u(x,1), \sin nx > + \frac{1}{n} < u_y(x,1), \sin nx > \right)^2 \\ &\quad \le \frac{1}{2} e^{2(y-1)n} (1 - e^{-\beta n^2 y})^2 \left(| < u(x,1), \sin nx > |^2 + \frac{1}{n^2} | < u_y(x,1), \sin nx > |^2 \right) \\ &\quad \le e^{2(y-1)n} \beta^2 n^4 y^2 \left(| < u(x,1), \sin nx > |^2 + | < u_y(x,1), \sin nx > |^2 \right). \end{aligned} \tag{12}$$

It is easy to prove the inequality for $k, n > 0$

$$\frac{n^4}{e^{2kn}} \le \frac{4}{k^4}.$$

Thus, for $y < 1$

$$e^{2(y-1)n}\beta^2 n^4 \leq \frac{4\beta^2}{(1-y)^4}. \tag{13}$$

This follows from (13) and (14) that

$$|<u(x,y)-u^\epsilon(x,y),\sin nx>|^2 \leq \frac{2\beta^2}{(1-y)^4}\left(|<u(x,1),\sin nx>|^2+|<u_y(x,1),\sin nx>|^2\right).$$

Therefore, we obtain

$$\begin{aligned}
\|u(.,y)-u^\epsilon(.,y)\|^2 &= \frac{\pi}{2}\sum_{n=1}^{\infty}|<u(x,y)-u^\epsilon(x,y),\sin nx>|^2 \\
&\leq \frac{\pi}{2}\frac{2\beta^2}{(1-y)^4}\sum_{n=1}^{\infty}\left(|<u(x,1),\sin nx>|^2+|<u_y(x,1),\sin nx>|^2\right) \\
&\leq \frac{2\beta^2}{(1-y)^4}(\|u(.,1)\|^2+\|u_y(.,1)\|^2).
\end{aligned}$$

Hence

$$\|u(.,y)-u^\epsilon(.,y)\| \leq \frac{\sqrt{2}E\beta}{(1-y)^2}.$$

Step 2.
Using Theorem 2, we obtain

$$\|v^\epsilon(.,y)-u^\epsilon(.,y)\| \leq \sqrt{2e^{\frac{1}{2\beta}}+2\epsilon} = \sqrt{2\epsilon^2+2\epsilon^{2-m}}. \tag{14}$$

Since Step 1 and Step 2 give

$$\begin{aligned}
\|v^\epsilon(.,y)-u(.,y)\| &\leq \|v^\epsilon(.,y)-u^\epsilon(.,y)\|+\|u^\epsilon(.,y)-u(.,y)\| \\
&\leq \frac{1}{2m\ln(\frac{1}{\epsilon})}\frac{\sqrt{2}E}{(1-y)^2}+\sqrt{2\epsilon^2+2\epsilon^{2-m}}.
\end{aligned}$$

Remark 1.
1. If $g = 0$, the estimate (9) becomes

$$\|v^\epsilon(.,y)-u(.,y)\| \leq \frac{1}{\sqrt{2}m\ln(\frac{1}{\epsilon})}\frac{E}{(1-y)^2}+\sqrt{2\epsilon^2+2\epsilon^{2-m}}. \tag{15}$$

The order of error (15) is same in Theorem 2.3 in paper [11](page 483).
2. It follows from (9) that the error in $y = 1$ is not considered. This is disadvantage

point of this estimate. To improve this, we introduce next Theorem which the error in $y = 1$ is proved.

Theorem 3. *Suppose that there is a positive constant E_1 such that*

$$\sum_{n=1}^{\infty} n^4 \left(| < u(x,1), \sin nx > |^2 + | < u_y(x,1), \sin nx > |^2 \right) < E_1^2.$$

Let us select $\beta = \frac{1}{2m \ln(\frac{1}{\epsilon})}$ $(0 < m < 2)$, then one has

$$\|v^\epsilon(.,y) - u(.,y)\| \leq \sqrt{\frac{\pi}{2}} \frac{E_1}{2m \ln(\frac{1}{\epsilon})} + \sqrt{\frac{\epsilon^2 + \epsilon^{2-m}}{2}}.$$

for every $y \in [0,1]$.

Proof.

First, we prove that

$$\|u(.,y) - u^\epsilon(.,y)\| \leq \beta \sqrt{\frac{\pi}{2}} E_1. \tag{16}$$

Since (12), we get

$$\begin{aligned} | < u(x,y) - u^\epsilon(x,y), \sin nx > |^2 &\leq e^{2(y-1)n} \beta^2 n^4 y^2 \\ &\quad \times \left(| < u(x,1), \sin nx > |^2 + | < u_y(x,1), \sin nx > |^2 \right). \end{aligned}$$

Then

$$\begin{aligned} \|u(x,y) - u^\epsilon(x,y)\|^2 &= \frac{\pi}{2} \sum_{n=1}^{\infty} | < u(x,y) - u^\epsilon(x,y), \sin nx > |^2 \\ &\leq \frac{\pi}{2} \beta^2 \sum_{n=1}^{\infty} \left(n^4 | < u(x,1), \sin nx > |^2 + n^4 | < u_y(x,1), \sin nx > |^2 \right) \\ &\leq \frac{\pi}{2} \beta^2 E_1^2. \end{aligned}$$

Hence, (16) is proved. Since (14) and (16), we obtain

$$\begin{aligned} \|v^\epsilon(.,y) - u(.,y)\| &\leq \|v^\epsilon(.,y) - u^\epsilon(.,y)\| + \|u^\epsilon(.,y) - u(.,y)\| \\ &\leq \sqrt{\frac{\pi}{2}} E_1 \beta + \sqrt{2\epsilon^2 + 2\epsilon^{2-m}} \\ &= \sqrt{\frac{\pi}{2}} \frac{E_1}{2m \ln(\frac{1}{\epsilon})} + \sqrt{2\epsilon^2 + 2\epsilon^{2-m}}. \end{aligned}$$

2. In Theorem 3, the logarithmic stability estimate is investigated. This often occurs in the boundary error estimate for ill-posed problems. In next section, we shall give the different regularization method which the order error of Hölder type is established.

3. The second regularization method

For $a \geq 1$ is a positive constant and α is the parameter regularization, we have the second approximated problem as follows

$$w^{\epsilon}(x,y) = \sum_{n=1}^{\infty} \left[\left(\frac{\frac{e^{ny}}{1+\alpha e^{na}} + e^{-ny}}{2} \right) \varphi_n + \left(\frac{\frac{e^{ny}}{1+\alpha e^{na}} - e^{-ny}}{2n} \right) g_n \right] \sin nx. \tag{17}$$

$$W^{\epsilon}(x,y) = \sum_{n=1}^{\infty} \left[\left(\frac{\frac{e^{ny}}{1+\alpha e^{na}} + e^{-ny}}{2} \right) \varphi_n^{\epsilon} + \left(\frac{\frac{e^{ny}}{1+\alpha e^{na}} - e^{-ny}}{2n} \right) g_n^{\epsilon} \right] \sin nx. \tag{18}$$

Theorem 4
Let $\varphi(x), g(x) \in L^2(0,\pi)$. Then, we have

$$\|W^{\epsilon}(.,y) - w^{\epsilon}(.,y)\| \leq \alpha^{-\frac{y}{a}} \epsilon.$$

Proof. We have

$$\begin{aligned} W^{\epsilon}(x,y) - w^{\epsilon}(x,y) &= \sum_{n=1}^{\infty} \left[\left(\frac{\frac{e^{ny}}{1+\alpha e^{na}} + e^{-ny}}{2} \right) (\varphi_n^{\epsilon} - \varphi^{\epsilon}) \right] \sin nx \\ &\quad + \sum_{n=1}^{\infty} \left[\left(\frac{\frac{e^{ny}}{1+\alpha e^{na}} - e^{-ny}}{2n} \right) (g_n^{\epsilon} - g_n) \right] \sin nx. \end{aligned}$$

For $n, x, \alpha, 0 \leq a \leq b$, it is not difficult to prove the inequality

$$\frac{e^{na}}{1+\alpha e^{nb}} \leq \alpha^{-\frac{a}{b}}. \tag{19}$$

Thus, we have

$$\frac{e^{na}}{1+\alpha e^{nb}} = \frac{e^{na}}{(1+\alpha e^{nb})^{\frac{a}{b}} (1+\alpha e^{nb})^{1-\frac{a}{b}}} \leq \frac{e^{na}}{(1+\alpha e^{nb})^{\frac{a}{b}}} \leq \alpha^{-\frac{a}{b}}.$$

Using the inequality

$$\begin{aligned}
\|W^\epsilon(.,y) - w^\epsilon(.,y)\|^2 &\leq 2\frac{\pi}{2}\sum_{n=1}^{\infty}\left[\left(\frac{\frac{e^{ny}}{1+\alpha e^{na}} + e^{-ny}}{2}\right)(\varphi_n^\epsilon - \varphi_n)\right]^2 \\
&\quad + 2\frac{\pi}{2}\sum_{n=1}^{\infty}\left[\left(\frac{\frac{e^{ny}}{1+\alpha e^{na}} - e^{-ny}}{2n}\right)(g_n^\epsilon - g_n)\right]^2 \\
&\leq 2\frac{\pi}{2}\sum_{n=1}^{\infty}\left[\left(\frac{e^{ny}}{1+\alpha e^{na}}\right)(\varphi_n^\epsilon - \varphi_n)\right]^2 \\
&\quad + 2\frac{\pi}{2}\sum_{n=1}^{\infty}\left[\left(\frac{e^{ny}}{1+\alpha e^{na}}\right)(g_n^\epsilon - g_n)\right]^2 \\
&\leq 2\alpha^{-2\frac{y}{a}}\left(\|\varphi^\epsilon - \varphi\|^2 + \|g^\epsilon - g\|^2\right) \\
&\leq 4\alpha^{-2\frac{y}{a}}\epsilon^2.
\end{aligned}$$

Hence

$$\|W^\epsilon(.,y) - w^\epsilon(.,y)\| \leq 2\alpha^{-\frac{y}{a}}\epsilon.$$

Theorem 5. *Let E be positive numbers such that $\|u(.,1)\|^2 + \|u_y(.,1)\|^2 \leq E^2$. If we select $\alpha = \epsilon^a$, then one has*

$$\|W^\epsilon(.,y) - u(.,y)\| \leq (\sqrt{\frac{E^2}{2}} + 2)\epsilon^{1-y} \tag{20}$$

for every $y \in [0,1]$.

Proof.

Since (7) and (17) give

$$\begin{aligned}
u(x,y) - w^\epsilon(x,y) &= \sum_{n=1}^{\infty}\left[\left(\frac{e^{ny} - \frac{e^{ny}}{1+\alpha e^{na}}}{2}\right)\varphi_n + \left(\frac{e^{ny} - \frac{e^{ny}}{1+\alpha e^{na}}}{2n}\right)g_n\right]\sin nx. \\
&= \sum_{n=1}^{\infty}\left(\frac{e^{ny} - \frac{e^{ny}}{1+\alpha e^{na}}}{2}\right)\left[\varphi_n + \frac{g_n}{n}\right]\sin nx \\
&= \frac{\alpha e^{n(a+y)}}{2(1+\alpha e^{na})}\left[\varphi_n + \frac{g_n}{n}\right]\sin nx.
\end{aligned} \tag{21}$$

Using (11), we get

$$\varphi_n + \frac{g_n}{n} = e^{-n}\left[< u(x,1), \sin nx > + \frac{1}{n} < u_y(x,1), \sin nx >\right]. \tag{22}$$

It follows from (21) and (22) that

$$< u(x,y) - w^{\epsilon}(x,y), \sin nx > \;=\; \frac{\alpha e^{n(a+y-1)}}{2(1+\alpha e^{na})}\left[< u(x,1), \sin nx > + \frac{1}{n} < u_y(x,1), \sin nx >\right]. \tag{23}$$

Thus

$$\begin{aligned} &| < u(x,y) - w^{\epsilon}(x,y), \sin nx > | \le \\ &\qquad \frac{\alpha e^{n(a+y-1)}}{2(1+\alpha e^{na})}\left[| < u(x,1), \sin nx > | + \frac{1}{n}| < u_y(x,1), \sin nx > |\right]. \end{aligned} \tag{24}$$

Using the inequalities $(a+b)^2 \le 2a^2 + 2b^2$ and (20), we obtain

$$\|u(.,y) - w^{\epsilon}(.,y)\|^2 = \frac{\pi}{2}\sum_{n=1}^{\infty} | < u(x,y) - w^{\epsilon}(x,y), \sin nx > |^2$$

$$\le 2\frac{\pi}{2}\sum_{n=1}^{\infty} \frac{\alpha^2 e^{2n(a+y-1)}}{4(1+\alpha e^{na})^2}\left[| < u(x,1), \sin nx > |^2 + \frac{1}{n^2}| < u_y(x,1), \sin nx > |^2\right]$$

$$\le \frac{1}{2}\alpha^{2\frac{1-y}{a}}\left[\|u(.,1)\|^2 + \|u_y(.,1)\|^2\right].$$

Applying the triangle inequality and Theorem 4, we obtain

$$\begin{aligned} \|u(.,y) - W^{\epsilon}(.,y)\| &\le \|u(.,y) - w^{\epsilon}(.,y)\| + \|w^{\epsilon}(.,y) - W^{\epsilon}(.,y)\| \\ &\le \sqrt{\frac{1}{2}\alpha^{2\frac{1-y}{a}}E^2} + 2\alpha^{-\frac{y}{a}}\epsilon \\ &\le \epsilon^{1-y}\left(\sqrt{\frac{E^2}{2}} + 2\right). \end{aligned}$$

Remark 2 The approximation error depends continuously on the measurement error for fixed $0 < y < 1$. However, as $y \to 1$, the accuracy of regularized solution becomes progressively lower. This is a common thing in the theory of ill-posed problems, if we do not have additional conditions on the smoothness of the solution. To retain the continuous dependence of the solution at $y = 1$, we introduce a stronger a priori assumption. We have the next Theorem

Theorem 6. *Suppose that there are positive real numbers k, E_2 such that*

$$\frac{\pi}{4}\sum_{n=1}^{\infty} e^{2kn}\left(| < u(x,1), \sin nx > |^2 + | < u_y(x,1), \sin nx > |^2\right) < E_2^2. \tag{25}$$

Let us select $\alpha = \epsilon^{\frac{a}{1+k}}$, $b = \min\{1+k, a\}$, *then one has*

$$\|W^\epsilon(.,y) - u(.,y)\| \quad \leq \quad (E_2+2)\epsilon^{\frac{b-y}{1+k}}. \tag{26}$$

for every $y \in [0,1]$.

Proof.
For the first term on the right-hand side of (23), we have

$$\begin{aligned} < u(x,y) - w^\epsilon(x,y), \sin nx > &= \frac{\alpha e^{n(a+y-1)}}{2(1+\alpha e^{na})}\left[< u(x,1), \sin nx > + \frac{1}{n} < u_y(x,1), \sin nx >\right] \\ &= \frac{\alpha e^{n(a+y-1-k)}}{2(1+\alpha e^{na})} e^{kn}\left[< u(x,1), \sin nx > + \frac{1}{n} < u_y(x,1), \sin nx >\right]. \end{aligned}$$

Using the inequality $(a+b)^2 \leq 2a^2 + 2b^2$, we obtain

$$\begin{aligned} \|u(.,y) - w^\epsilon(.,y)\|^2 &= \frac{\pi}{2}\sum_{n=1}^{\infty} | < u(x,y) - w^\epsilon(x,y), \sin nx > |^2 \\ &\leq \frac{1}{2}\alpha^{2\frac{1+k-y}{a}} \frac{\pi}{2}\sum_{n=1}^{\infty} e^{2kn}\left(| < u(x,1), \sin nx > |^2 + \frac{1}{n^2}| < u_y(x,1), \sin nx > |^2\right) \\ &\leq \frac{1}{2}\alpha^{2\frac{1+k-y}{a}} \frac{\pi}{2}\sum_{n=1}^{\infty} e^{2kn}\left(| < u(x,1), \sin nx > |^2 + | < u_y(x,1), \sin nx > |^2\right) \\ &\leq \alpha^{2\frac{1+k-y}{a}} E_2^2. \end{aligned}$$

Apply the triangle inequality

$$\begin{aligned} \|u(.,y) - W^\epsilon(.,y)\| &\leq \|u(.,y) - w^\epsilon(.,y)\| + \|w^\epsilon(.,y) - W^\epsilon(.,y)\| \\ &\leq \sqrt{\alpha^{2\frac{1+k-y}{a}} E_2^2} + 2\alpha^{-\frac{y}{a}}\epsilon \\ &\leq \sqrt{\epsilon^{2\frac{1+k-y}{1+k}} E_2^2} + 2\epsilon^{\frac{1+k-y}{1+k}}. \\ &\leq \epsilon^{\frac{b-y}{1+k}}(E_2+2). \end{aligned}$$

Remark 3. 1. We separately consider the case $0 \leq y < 1$ and the case $y = 1$ in order to emphasize the following facts. For the case $0 \leq y < 1$, the a priori bound $\|u(.,1)\| \leq E$ is sufficient. However, for the case $y = 1$, the stronger a priori bound in (25) must be imposed.

2. The best possible worst case error $w(\epsilon)$ for identifying $u(x;y)$ from noisy data φ^ϵ with $\|\varphi - \varphi^\epsilon\| \leq \epsilon$ under the *smoothness assumption* $\|u(,1)\| \leq E$ is

$$w(\epsilon) = \epsilon \cosh\left(y.arcosh(\frac{E}{\epsilon})\right) = E^y(\frac{\epsilon}{2})^{1-y}(1+0(1))$$

for $\epsilon \to 0$. And there can be no regularization methods that provide a smaller error in the worst case sense. There exists special optimal regularization methods that guarantee this op- timal error bound. These results and more general results under stronger smoothness assumptions that allow to treat also the case y = 1 may be found in the papers [12, 16].

3.The error (26) is the order of Holder type for all $y \in [0,1]$. As we know, the convergence rate of ϵ^p, $(0 < p)$ is more quickly than the logarithmic order $\left(\ln(\frac{1}{\epsilon})\right)^{-q}$ $(q > 0)$ when $\epsilon \to 0$. Note that this error is not investigated in [11]. Moreover, we compare the method in [12](Theorem 3.2) and [16](Theorem 3.6) with optimal methods to conclude that the first method seems to be not of optimal order, but the second method is an order optimal method.

4. Numerical examples

In this section, some simple examples are devised for verifying the validity of the proposed method.

Example 1. We consider

$$\begin{cases} u_{xx} + u_{yy} = 0, \ (x,y) \in (0,\pi) \times (0,1), \\ u(0,y) = u(\pi,y) = 0, \ 0 < y < 1 \\ u_y(x,0) = 0, \ 0 < x < \pi \\ u(x,0) = \varphi(x) = \sin nx, \ 0 < x < \pi. \end{cases} \tag{27}$$

Then the exact solution to this problem is

$$u(x,y) = \frac{e^y + e^{-y}}{2} \sin x.$$

For convenience of computation, we consider the measured data

$$\varphi^\epsilon(x) = \left(\sqrt{\frac{2}{\pi}}\epsilon + 1\right)\varphi(x),$$

we have

$$\varphi_n^\epsilon = \frac{2}{\pi}\int_0^\pi \varphi^\epsilon(x)\sin nx dx = \begin{cases} \sqrt{\dfrac{2}{\pi}}\epsilon + 1, & n = 1 \\ 0, & n > 1, \end{cases} \tag{28}$$

and

$$\|\varphi^\epsilon - \varphi\|_{L^2(0,\pi)} = \left(\int_0^\pi \frac{2}{\pi}\epsilon^2 \left(\varphi(x)\right)^2 dx\right)^{1/2} = \left(\frac{2}{\pi}\epsilon^2 \int_0^\pi \sin^2 x dx\right)^{1/2} = \epsilon.$$

From (28), and (8) with notice that $\beta = \dfrac{1}{2\ln(\dfrac{1}{\epsilon})}$, we have the regularized solution of the first method

$$v^{\epsilon}(x,y) = \frac{1}{2}\left(e^{\left(1-\frac{1}{2\ln\frac{1}{\epsilon}}\right)y} + e^{-y}\right)\left(\sqrt{\frac{2}{\pi}}\epsilon + 1\right)\sin x. \tag{29}$$

From (28), and (19) with notice that $\alpha = \epsilon^2$, $a = 2$ we have the regularized solution of the second method

$$W^{\epsilon}(x,y) = \frac{1}{2}\left(\frac{e^{y}}{1+\epsilon^2 e^2} + e^{-y}\right)\left(\sqrt{\frac{2}{\pi}}\epsilon + 1\right)\sin x. \tag{30}$$

By applying the method in [11] with notice the expression (28), we have the regularized solution

$$w^{\epsilon}(x,y) = \frac{1}{2}\left(\exp\left\{\sqrt{\frac{1}{1+\frac{1}{\ln^2\left(\frac{2E}{\epsilon}\right)}}}y\right\} + \exp\left\{-\sqrt{\frac{1}{1+\frac{1}{\ln^2\left(\frac{2E}{\epsilon}\right)}}}y\right\}\right)\left(\sqrt{\frac{2}{\pi}}\epsilon + 1\right)\sin x \tag{31}$$

where $E = \|u(\cdot,1)\| = \sqrt{\frac{\pi}{8}}\left(e + e^{-1}\right)$.

If we put

$$y = \{0,\ 0.1,\ 0.2,\ 0.3,\ 0.4,\ 0.5,\ 0.6,\ 0.7,\ 0.8,\ 0.9\}$$

we get the following Table 1, Table 2, Table 3 show the comparison between the three methods of error between the regularization solution and the exact solution in the case $0 < y < 1$

Table 1

	$\epsilon = 10^{-1}$		
	The first method	The second method	The method in [11]
y	$\|v^\epsilon - u\|$	$\|W^\epsilon - u\|$	$\|w^\epsilon - u\|$
0	0.1000	0.0534	0.1000
0.1	0.0844	0.0490	0.0990
0.2	0.0669	0.0451	0.0962
0.3	0.0469	0.0417	0.0913
0.4	0.0241	0.0387	0.0843
0.5	0.0020	0.0360	0.0751
0.6	0.0321	0.0337	0.0634
0.7	0.0666	0.0318	0.0490
0.8	0.1064	0.0301	0.0316
0.9	0.1521	0.0288	0.0109

Table 2

	$\epsilon = 10^{-5}$		
	The first method	The second method	The method in [11]
y	$\|v^\epsilon - u\|$	$\|W^\epsilon - u\|$	$\|w^\epsilon - u\|$
0	10^{-5}	9.9995×10^{-6}	10^{-5}
0.1	0.0030	1.0050×10^{-5}	4.4263×10^{-4}
0.2	0.0066	1.0200×10^{-5}	0.0018
0.3	0.0109	1.0453×10^{-5}	0.0041
0.4	0.0161	1.0810×10^{-5}	0.0074
0.5	0.0222	1.1275×10^{-5}	0.0117
0.6	0.0294	1.1854×10^{-5}	0.0172
0.7	0.0378	1.2551×10^{-5}	0.0239
0.8	0.0476	1.3373×10^{-5}	0.0320
0.9	0.0591	1.4330×10^{-5}	0.0415

Table 3

	$\epsilon = 10^{-10}$		
	The first method	The second method	The method in [11]
y	$\|v^\epsilon - u\|$	$\|W^\epsilon - u\|$	$\|w^\epsilon - u\|$
0	10^{-10}	10^{-10}	10^{-10}
0.1	0.0015	1.0050×10^{-10}	2.4733×10^{-4}
0.2	0.0033	1.0201×10^{-10}	9.9416×10^{-4}
0.3	0.0055	1.0453×10^{-10}	0.0023
0.4	0.0081	1.0811×10^{-10}	0.0041
0.5	0.0112	1.1276×10^{-10}	0.0064
0.6	0.0148	1.1855×10^{-10}	0.0094
0.7	0.0190	1.2552×10^{-10}	0.0131
0.8	0.0240	1.3374×10^{-10}	0.0175
0.9	0.0298	1.4331×10^{-10}	0.0228

In the case $y = 1$, from (29), (30) we get

$$\begin{aligned} v^\epsilon(x,1) &= \frac{1}{2}\left(e^{\left(1-\frac{1}{2\ln\frac{1}{\epsilon}}\right)} + e^{-1}\right)\left(\sqrt{\frac{2}{\pi}}\epsilon + 1\right)\sin x, \\ W^\epsilon(x,1) &= \frac{1}{2}\left(\frac{e}{1+\epsilon^2 e^2} + e^{-1}\right)\left(\sqrt{\frac{2}{\pi}}\epsilon + 1\right)\sin x. \end{aligned}$$

By applying the method in [11] with the expression (29), we have the regularized solution in case $y = 1$

$$\begin{aligned} W^\epsilon(x,1) &= \frac{1}{2}\left(\exp\left\{\sqrt{\frac{1}{1+\frac{1}{\ln^2\left(\frac{E}{\epsilon}\left(\ln\frac{E}{\epsilon}\right)^{-1}\right)}}}\right\} + \exp\left\{-\sqrt{\frac{1}{1+\frac{1}{\ln^2\left(\frac{E}{\epsilon}\left(\ln\frac{E}{\epsilon}\right)^{-1}\right)}}}\right\}\right) \times \\ &\quad \left(\sqrt{\frac{2}{\pi}}\epsilon + 1\right)\sin x \end{aligned}$$

where $E = \|u_y(\cdot,1)\| = \sqrt{\frac{\pi}{8}}\left(e - e^{-1}\right)$.

We have the following Table 4, Table 5, Table 6 show the comparison between the three methods of error between the regularization solution and the exact solution in the case $y = 1$

Table 4

$\epsilon = 10^{-1}$		
The first method	The second method	The method in [11]
$\|v^\epsilon - u\|$	$\|W^\epsilon - u\|$	$\|w^\epsilon - u\|$
0.2047	0.0277	0.0136

Table 5

$\epsilon = 10^{-5}$		
The first method	The second method	The method in [11]
$\|v^\epsilon - u\|$	$\|W^\epsilon - u\|$	$\|w^\epsilon - u\|$
0.0723	1.5×10^{-5}	0.0527

Table 6

$\epsilon = 10^{-10}$		
The first method	The second method	The method in [11]
$\|v^\epsilon - u\|$	$\|W^\epsilon - u\|$	$\|w^\epsilon - u\|$
0.0365	10^{-10}	0.0289

Example 2. In this example, we take $g(x) = 0$ and the exact data $\varphi(x) = \sum\limits_{n=1}^{\infty} \frac{2}{n\cosh n} \sin nx$. It is easy to verify that

$$u(x,y) = \sum_{n=1}^{\infty} \frac{2\cosh ny}{n\cosh n} \sin nx$$

is the exact solution of the problem (1). That is $u(x,y)$ satisfies

$$\begin{cases} u_{xx} + u_{yy} = 0, \ (x,y) \in (0,\pi) \times (0,1), \\ u(0,y) = u(\pi,y) = 0, \ 0 < y < 1 \\ u_y(x,0) = 0, \ 0 < x < \pi \\ u(x,0) = \sum\limits_{n=1}^{\infty} \dfrac{2}{n\cosh n} \sin nx, \ 0 < x < \pi. \end{cases} \tag{32}$$

For simplicity in computation, the measured data $\varphi^\epsilon(x)$ is given by

$$\varphi^\epsilon(x) = \varphi(x) + \epsilon(2 - x),$$

we have

$$\begin{aligned} \varphi_n^\epsilon &= \frac{2}{\pi} \int_0^\pi \varphi^\epsilon(x) \sin nx dx \\ &= \frac{2}{\pi} \left(\int_0^\pi \sum_{m=1}^{\infty} \frac{2}{m\cosh m} \sin mx \sin nx dx + \frac{\epsilon(\pi - 2)(-1)^n}{n} + \frac{2\epsilon}{n} \right). \end{aligned} \tag{27}$$

Note that

$$\int_0^\pi \sin mx \sin nx dx = \begin{cases} \dfrac{\pi}{2}, & \text{if } m = n \\ 0, & \text{if } m \neq n. \end{cases} \tag{34}$$

From (33), (8) with notice that $\beta = \dfrac{1}{2\ln(\dfrac{1}{\epsilon})}$ and (34), we have the regularized solution of the first method

$$v^\epsilon(x,y) = \sum_{n=1}^{\infty} \frac{1}{2}\left[e^{\left(n - \frac{n^2}{2\ln\frac{1}{\epsilon}}\right)y} + e^{-ny}\right]\left[\frac{2}{n\cosh n} + \frac{\epsilon(\pi-2)(-1)^n}{n} + \frac{2\epsilon}{n}\right]\sin nx. \tag{35}$$

From (33), (19) with notice that $\alpha = \epsilon^2$, $a = 2$ and (34), we have the regularized solution of the second method

$$W^\epsilon(x,y) = \sum_{n=1}^{\infty} \frac{1}{2}\left(\frac{e^{ny}}{1+\epsilon^2 e^{2n}} + e^{-ny}\right).\left[\frac{2}{n\cosh n} + \frac{\epsilon(\pi-2)(-1)^n}{n} + \frac{2\epsilon}{n}\right]\sin nx. \tag{36}$$

By applying the method in [11] with the expression (33) and notice that (34), we have the regularized solution

$$\begin{aligned} w^\epsilon(x,y) \quad = \quad & \sum_{n=1}^{\infty} \frac{1}{2}\left(\exp\left\{\sqrt{\frac{n^2}{1+\frac{n^2}{\ln^2\left(\frac{2E}{\epsilon}\right)}}}y\right\} + \exp\left\{-\sqrt{\frac{n^2}{1+\frac{n^2}{\ln^2\left(\frac{2E}{\epsilon}\right)}}}y\right\}\right) \times \\ & \left[\frac{2}{n\cosh n} + \frac{\epsilon(\pi-2)(-1)^n}{n} + \frac{2\epsilon}{n}\right]\sin nx. \end{aligned} \tag{28}$$

where $E = \dfrac{\pi\sqrt{\pi}}{\sqrt{3}}$.

If we put

$$y = \{0,\ 0.1,\ 0.2,\ 0.3,\ 0.4,\ 0.5,\ 0.6,\ 0.7,\ 0.8,\ 0.9\}$$

we get the following Table 7, Table 8, Table 9 show the comparison between the three methods of error between the regularization solution and the exact solution in the case $0 < y < 1$.

Table 7

	$\epsilon = 10^{-1}$		
	The first method	The second method	The method in [11]
y	$\|v^\epsilon - u\|$	$\|W^\epsilon - u\|$	$\|w^\epsilon - u\|$
0	0.2790	0.1367	0.2790
0.1	0.2190	0.1041	0.2858
0.2	0.1787	0.0937	0.3081
0.3	0.1392	0.0986	0.3514
0.4	0.1047	0.1202	0.4253
0.5	0.1039	0.1604	0.5437
0.6	0.1701	0.2238	0.7262
0.7	0.2977	0.3212	0.9987
0.8	0.4963	0.4752	1.3941
0.9	0.8144	0.7401	1.9345

Table 8

	$\epsilon = 10^{-5}$		
	The first method	The second method	The method in [11]
y	$\|v^\epsilon - u\|$	$\|W^\epsilon - u\|$	$\|w^\epsilon - u\|$
0	2.7904×10^{-5}	2.6725×10^{-5}	2.7904×10^{-5}
0.1	5.7872×10^{-3}	2.8003×10^{-5}	2.9285×10^{-4}
0.2	1.3876×10^{-2}	3.5756×10^{-5}	1.3119×10^{-3}
0.3	2.5333×10^{-2}	6.3696×10^{-5}	3.3247×10^{-3}
0.4	4.1937×10^{-2}	1.5910×10^{-4}	6.9744×10^{-3}
0.5	6.6725×10^{-2}	4.8226×10^{-4}	1.3676×10^{-2}
0.6	0.1051	1.6232×10^{-3}	2.6878×10^{-2}
0.7	0.1678	5.8859×10^{-3}	5.6691×10^{-2}
0.8	0.2774	2.2956×10^{-2}	0.1385
0.9	0.4933	9.9740×10^{-2}	0.4035

Table 9

	$\epsilon = 10^{-100}$		
	The first method	The second method	The method in [11]
y	$\|v^\epsilon - u\|$	$\|W^\epsilon - u\|$	$\|w^\epsilon - u\|$
0	2.7904×10^{-100}	9.3543×10^{-102}	2.7904×10^{-100}
0.1	2.9301×10^{-4}	1.7770×10^{-11}	1.1212×10^{-6}
0.2	7.0961×10^{-4}	1.3024×10^{-11}	4.8247×10^{-6}
0.3	1.3178×10^{-3}	1.5092×10^{-10}	1.2342×10^{-5}
0.4	2.2400×10^{-3}	1.7813×10^{-10}	2.6659×10^{-5}
0.5	3.7143×10^{-3}	1.4884×10^{-10}	5.5097×10^{-5}
0.6	6.2598×10^{-3}	2.7030×10^{-10}	1.1790×10^{-4}
0.7	1.1201×10^{-2}	2.8086×10^{-10}	2.8402×10^{-4}
0.8	2.2862×10^{-2}	1.8466×10^{-10}	8.8781×10^{-4}
0.9	6.2976×10^{-2}	1.2795×10^{-9}	5.2159×10^{-3}

and we have the graphic is displayed in Figures 2, 3, 4, 5, 6, 7, 8, 9, 10 on the rectangular domain $[0, \pi] \times [0, 0.9]$

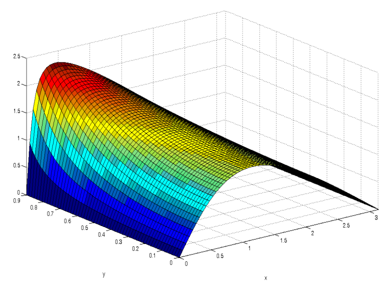

Figure 1: The exact solution.

In the case $y = 1$, we have the exact solution $u(x, 1) = \pi - x$. From (35), (36) we get the regularized solution of the first method and the second method in case $y = 1$

$$
\begin{aligned}
v^\epsilon(x,1) &= \sum_{n=1}^{\infty} \frac{1}{2}\left[e^{\left(n - \frac{n^2}{2\ln\frac{1}{\epsilon}}\right)} + e^{-n}\right]\left[\frac{2}{n\cosh n} + \frac{\epsilon(\pi-2)(-1)^n}{n} + \frac{2\epsilon}{n}\right]\sin nx,\\
W^\epsilon(x,1) &= \sum_{n=1}^{\infty} \frac{1}{2}\left(\frac{e^n}{1+\epsilon^2 e^{2n}} + e^{-n}\right)\left[\frac{2}{n\cosh n} + \frac{\epsilon(\pi-2)(-1)^n}{n} + \frac{2\epsilon}{n}\right]\sin nx.
\end{aligned}
$$

By applying the method in [11] with the expression (33) and notice that (34),

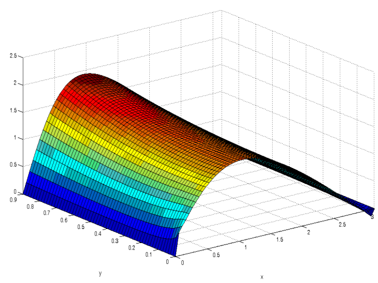

Figure 2: The regularized solution with $\epsilon = 10^{-1}$ by applying the first method.

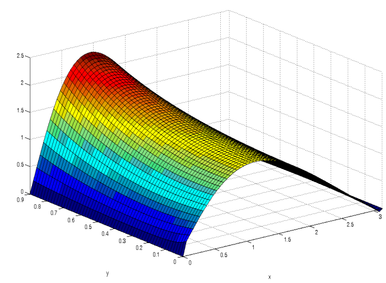

Figure 3: The regularized solution with $\epsilon = 10^{-1}$ by applying the second method.

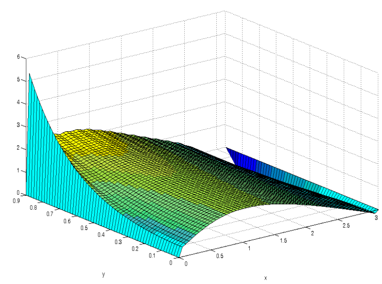

Figure 4: The regularized solution with $\epsilon = 10^{-1}$ by applying the method in [11].

we have the regularized solution in case $y = 1$

$$w^{\epsilon}(x,1) = \sum_{n=1}^{\infty} \frac{1}{2} \left(\exp\left\{ \sqrt{\frac{n^2}{1 + \frac{n^2}{\ln^2\left(\frac{120}{\epsilon}\left(\ln\frac{120}{\epsilon}\right)^{-1}\right)}}} \right\} + \exp\left\{ -\sqrt{\frac{n^2}{1 + \frac{n^2}{\ln^2\left(\frac{120}{\epsilon}\left(\ln\frac{120}{\epsilon}\right)^{-1}\right)}}} \right\} \right) \times \left[\frac{2}{n\cosh n} + \frac{\epsilon(\pi-2)(-1)^n}{n} + \frac{2\epsilon}{n} \right] \sin nx.$$

We have the following Table 10, Table 11, show the comparison between the three methods of error between the regularization solution and the exact solution in the case $y = 1$

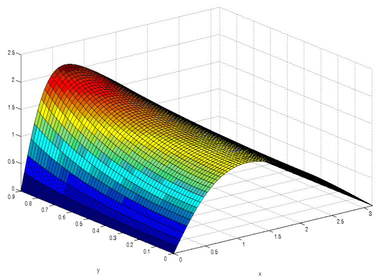

Figure 5: The regularized solution with $\epsilon = 10^{-5}$ by applying the first method.

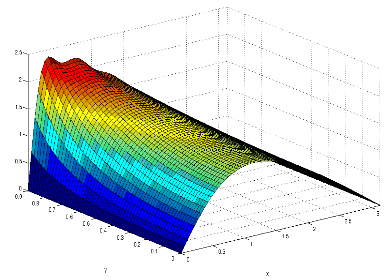

Figure 6: The regularized solution with $\epsilon = 10^{-5}$ by applying the second method.

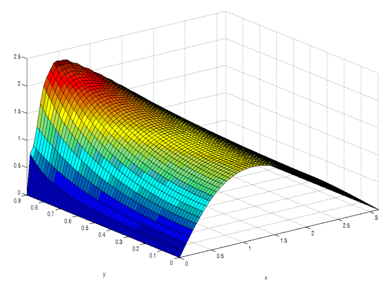

Figure 7: The regularized solution with $\epsilon = 10^{-5}$ by applying the method in [11].

Table 10

$\epsilon = 10^{-10}$		
The first method	The second method	The method in [11]
$\|v^\epsilon - u\|$	$\|W^\epsilon - u\|$	$\|w^\epsilon - u\|$
0.9734	0.4980	0.7239

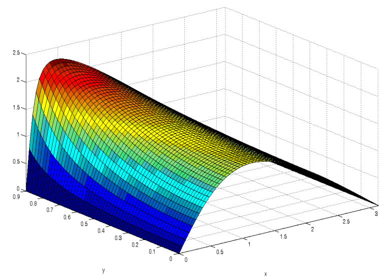

Figure 8: The regularized solution with $\epsilon = 10^{-100}$ by applying the first method.

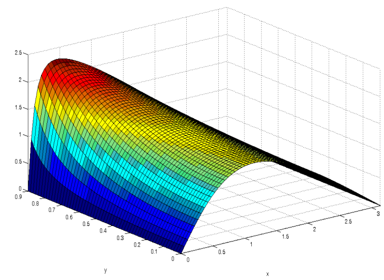

Figure 9: The regularized solution with $\epsilon = 10^{-100}$ by applying the second method.

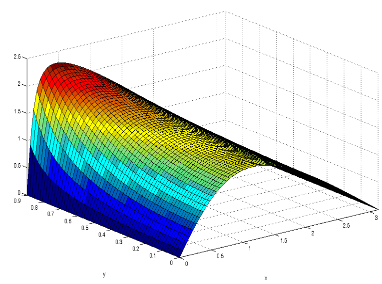

Figure 10: The regularized solution with $\epsilon = 10^{-100}$ by applying the method in [11].

Table 11

$\epsilon = 10^{-300}$		
The first method	The second method	The method in [11]
$\|v^\epsilon - u\|$	$\|W^\epsilon - u\|$	$\|w^\epsilon - u\|$
0.4036	9.3464×10^{-10}	0.2261

and we have the graphic is displayed in Figures 11, 12 in the case $y = 1$

Looking at from Table 1 to Table 11, a comparison between the three methods, the error in the second method converges to zero more quickly many times than the first method and the method in [11]. This shows that our approach has a nice regularizing effect and give a better approximation with comparison to the paper [11].

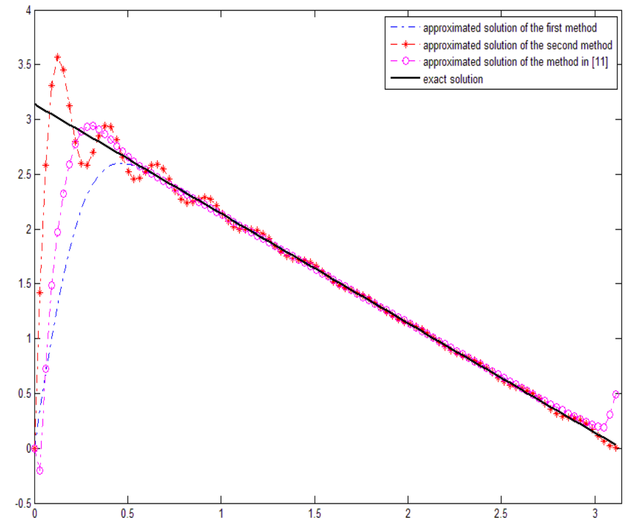

Figure 11: The exact solution and the regularized solution in three methods with $\epsilon = 10^{-10}$.

Acknowledgments

The authors would like to thank the anonymous referees for their valuable comments leading to the improvement of our manuscript.

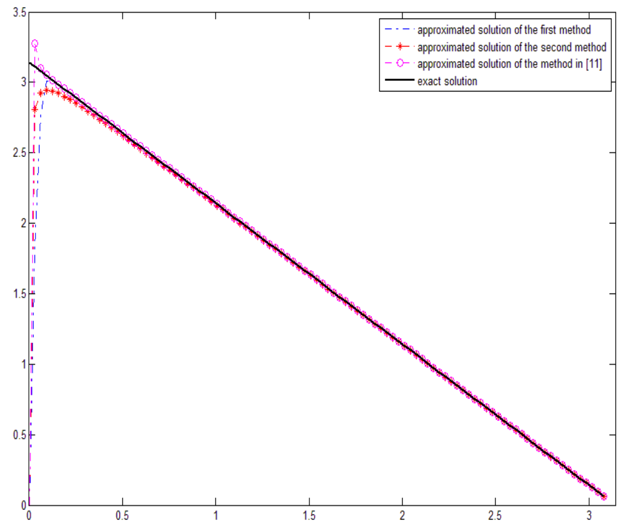

Figure 12: The exact solution and the regularized solution in three methods with $\epsilon = 10^{-300}$.

References

[1] J.R. Cannon, *Error estimates for some unstable continuation problems,* J. Soc. Industr. Appl. Math. 12 (1964) 27084.

[2] J.R. Cannon, K. Miller, *Some problems in numerical analytic continuation,* SIAM Numer. Anal. 2 (1965) 878.

[3] M. Charton, H.-J. Reinhardt, *Approximation of Cauchy problems for elliptic equations using the method of lines,* WSEAS Trans. Math. 4/2 (2005) 649.

[4] J. Douglas Jr.,*A numerical method for analytic continuation, in: Boundary Value Problems in Difference Equation,* University of Wisconsin Press, Madison, WI, 1960, pp. 17989.

[5] J. Hadamard, *Lectures on the Cauchy Problem in Linear Differential Equations,* Yale University Press, New Haven, CT, 1923.

[6] D.N. Hao, P.M. Hien, *Stability results for the Cauchy problem for the Laplace equation in a strip,* Inverse Problems 19 (2003) 83344.

[7] Lesnic D, Elliott L and Ingham D B 1998 *The boundary element solution of the Laplace and biharmonic equations subjected to noisy boundary data* Int. J. Numer. Methods Eng. 43 4792

[8] M. Kubo, *L2-conditional stability estimate for the Cauchy problem for the Laplace equation,* J. Inverse Ill-Posed Probl. 2 (1994) 25361.

[9] Payne L 1975 Improperly Posed Problems in Partial Differential Equations (Philadelphia: SIAM)

[10] Z. Qian, C.L.Fu and X.T. Xiong, *Fourth-order modified method for the Cauchy problem for the Laplace equation,* Journal of Computational and Applied Mathematics, 192: 205- 218(2006).

[11] Qian, Zhi; Fu, Chu-Li; Li, Zhen-Ping, *Two regularization methods for a Cauchy problem for the Laplace equation.* J. Math. Anal. Appl. 338 (2008), no. 1, 479–489.

[12] T. Reginska and U. Tautenhahn (2009),*Conditional stability estimates and regular- ization with applications to Cauchy problems for the Helmholtz equation,* Num. Funct. Anal. and Optimiz. 30, 1065–1097

[13] J.-J. Reinhardt, H. Han, D.N. Hao, *Stability and regularization of discrete approximation to the Cauchy problem for the Laplace equation,* SIAM Numer. Anal. 36 (1999) 89005.

[14] Dang Duc Trong and Nguyen Huy Tuan *Regularization and error estimates for nonhomogeneous backward heat problems*, Electron. J. Diff. Eqns., Vol. 2006 , No. 04, 2006, pp. 1-10.

[15] Dang Duc Trong and Nguyen Huy Tuan, *Regularization and error estimate for the nonlinear backward heat problem using a method of integral equation.*, Nonlinear Analysis, Volume 71, Issue 9, 1 November 2009, Pages 4167-4176

[16] U. Tautenhahn (1996), *Optimal stable solution of Cauchy problems for elliptic equa- tions,* Z. Anal. Anw. 15, 961–984

[17] U.Tautenhahn,*On the method of Lavrentiev regularization for nonlinear ill-posed problems,* Inverse Probl. 18:191-207(2002).

[18] Tautenhahn U 1998 *Optimality for ill-posed problems under general source conditions* Numer. Funct. Anal. Optim. 19 3778

Nguyen Huy Tuan
Division of Applied Mathematics,
Ton Duc Thang University
Nguyen Huu Tho Street, District 7,
Hochiminh City, Vietnam.
email:*tuannh@tdt.edu.vn*

Ngo Van Hoa
Division of Applied Mathematics,
Ton Duc Thang University
Nguyen Huu Tho Street, District 7,
Hochiminh City, Vietnam.
email:*ngovanhoa@tdt.edu.vn*