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A PARTICULAR CLASS OF LINEAR AND POSITIVE STANCU - TYPE OPERATORS

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Abstract. The object of this paper is to introduce a particular class of Stancu - type operators, such that the test functions e_0 and e_1 are reproduced like in the classical case of Berstein type operator. Also, in our approach we give two theorems of error approximation and two Voronovskaja type theorems for this operators. Finally, we plot on the same graph the images generated for exponential function by the particular operator and by the classical Stancu operator.

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1.Preliminaries

Let $\mathbb{N}$ be the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.
In this section, we recall some notions and results which we will use in this article (see [8]).

We consider $I \subset R$, I an interval and we shall use the function sets: $E(I)$,$F(I)$ which are subsets of the set of real functions defined on I, $B(I) = \{f|f : I \to R,\ f$ bounded on $I\}$, $C(I) = \{f|f : I \to R,\ f$ continuous on $I\}$ and $C_B(I) = B(I) \cap C(I)$. For $x \in I$, consider the function $\psi_x : I \to R$, $\psi_x(t) = t - x$, for any $t \in R$.

Let a, b, a', b' be real numbers, $I \subset R$ interval, $a < b, a' < b'$, $[a, b] \subset I$, $[a', b'] \subset I$, and $[a, b] \cap [a', b'] \neq \phi$. For any $m \in \mathbb{N}$, consider the functions $\varphi_{m,k} : I \to R$ with the property that $\varphi_{m,k}(x) \geq 0$ for any $x \in [a', b']$, for any $k \in \{0, 1, 2, ..., m\}$ and the linear positive functionals $A_{m,k} : E([a, b]) \to \mathbb{R}$, for any $k \in \{0, 1, 2, ..., m\}$. For $m \in \mathbb{N}$, define the operator: $L_m : E([a, b]) \to F(I)$ by

$$(L_m f)(x) = \sum_{k=0}^{m} \varphi_{m,k}(x) A_{m,k}(f), \tag{1}$$

for any $f \in E([a, b])$, for any $x \in I$ and for $i \in \mathbb{N}_0$, define $T^*_{m,i}$ by

$$(T^*_{m,i} L_m)(x) = m^i (L_m \psi_x^i)(x) = m^i \sum_{k=0}^{m} \varphi_{m,k}(x) A_{m,k}(\psi_x^i), \tag{2}$$

for any $x \in [a, b] \cap [a', b']$.

In the following, let s be a fixed natural number, s even and we suppose that the operators $(L_m)_{m\geq 0}$ verify the condition: there exists the smallest $\alpha_s, \alpha_{s+1} \in [0;\infty)$ so that

$$\lim_{m\to\infty} \frac{(T^*_{m,j}L_m)(x)}{m^{\alpha_j}} = B_j(x) \in R, \tag{3}$$

for any $x \in [a,b] \cap [a',b'], j \in \{s, s+2\}$ and

$$\alpha_{s+2} < \alpha_s + 2. \tag{4}$$

If $I \subset \mathbb{R}$ is a given interval and $f \in C_B(I)$, then, the first order modulus of smoothness of f is the function $\omega(f;.) : [0;\infty) \to \mathbb{R}$ defined for any $\delta \geq 0$ by

$$\omega(f;\delta) = sup\{\left|f(x') - f(x'')\right| : x', x'' \in I; \left|x' - x''\right| \leq \delta\}.$$

In [8] were obtained the following results.

Proposition 0.1 *For $m \in \mathbb{N}$ the L_m operator is linear and positive.*

Theorem 0.1 *Let $f : [a,b] \to R$ be a function. If $x \in [a,b] \cap [a',b']$ and f is a s times derivable function in x, the function $f^{(s)}$ is continuous in x, then*

$$\lim_{m\to\infty} m^{s-\alpha_s} \left[(L_m f)(x) - \sum_{i=0}^{s} \frac{f^{(i)}(x)}{m^i i!}(T^*_{m,i}L_m)(x)\right] = 0. \tag{5}$$

If f is a s times differentiabile function on $[a,b]$, the function $f^{(s)}$ is continuous on $[a,b]$ and there exists $m(s) \in \mathbb{N}$ and $k_j \in \mathbb{R}$ so that for any natural number $m, m \geq m(s)$ and for any $x \in [a,b] \cap [a',b']$ we have

$$\frac{(T^*_{m,j}L_m)(x)}{m^{\alpha_j}} \leq k_j, \tag{6}$$

where $j \in \{s, s+2\}$, then the convergence given in (5) is uniform on $[a,b] \cap [a',b']$ and

$$m^{s-\alpha_s} \left|(L_m f)(x) - \sum_{i=0}^{s} \frac{f^{(i)}(x)}{m^i i!}(T^*_{m,i}L_m)(x)\right| \leq \tag{7}$$

$$\leq \frac{1}{s!}(k_s + k_{s+2})\omega\left(f^{(s)}; \frac{1}{\sqrt{m^{2+\alpha_s-\alpha_{s+2}}}}\right),$$

for any $x \in [a,b] \cap [a',b']$, for any natural number $m, m \geq m(s)$.

For any $f \in C([0,1]), \alpha, \beta \in \mathbb{R}$ fixed, with $0 \leq \alpha < \beta$, the Stancu operators are defined by:

$$\left(P_m^{(\alpha,\beta)} f\right)(x) = \sum_{k=0}^{m} \binom{m}{k} x^k (1-x)^{m-k} f\left(\frac{k+\alpha}{m+\beta}\right) \tag{8}$$

for any $m \in \mathbb{N}$ and any $x \in [0,1]$.

Remark 0.1 *For $\alpha = \beta = 0$ in (8), we obtain Berstein's operators.*

In [5], the operator:

$$S_{n,\alpha,\beta}(f,x) = \left(\frac{n+\beta_2}{n}\right)^n \sum_{r=0}^{n} f\left(\frac{r+\alpha_1}{n+\beta_1}\right) \binom{n}{r} \left(x - \frac{\alpha_2}{n+\beta_2}\right)^r \left(\frac{n+\alpha_2}{n+\beta_2} - x\right)^{n-r} \tag{9}$$

is defined, where $\frac{\alpha_2}{n+\beta_2} \leq x \leq \frac{n+\alpha_2}{n+\beta_2}$, and $\alpha_k, \beta_k, k \in \{1,2\}$ are positive real numbers, provided by $0 \leq \alpha_2 \leq \alpha_1 \leq \beta_1 \leq \beta_2$.

Remark 0.2 *Note that for $n \in \mathbb{N}$, the variable x from (9) depends on n, being situated in an interval depending on n.*

We will consider the fixed real numbers α, β, with the property that
$0 \leq \alpha < \beta$. The following result is immediate.

Lemma 0.1 *If $m_1, m_2 \in \mathbb{N}, m_1 < m_2$, then $\frac{\alpha}{m_1+\beta} > \frac{\alpha}{m_2+\beta}$ and*

$$\left[\frac{\alpha}{m_1+\beta}; \frac{m_1+\alpha}{m_1+\beta}\right] \subset \left[\frac{\alpha}{m_2+\beta}; \frac{m_2+\alpha}{m_2+\beta}\right] \tag{10}$$

In the following, let $m_0 \in \mathbb{N}$ be fixed.

Lemma 0.2 *We have that*

$$\left[\frac{\alpha}{m_0+\beta}, \frac{m_0+\alpha}{m_0+\beta}\right] \subset \left[\frac{\alpha}{m+\beta}, \frac{m+\alpha}{m+\beta}\right] \tag{11}$$

for any $m \in \mathbb{N}$ and $m \geq m_0$.

Proof. It follows from (10).

Lemma 0.3 *The following inequalities:*

$$(m+\beta)x - \alpha \geq 0 \tag{12}$$

$$m + \alpha - (m+\beta)x \geq 0 \tag{13}$$

hold for any $m \in \mathbb{N}, m \geq m_0$ and any $x \in \left[\frac{\alpha}{m_0+\beta}; \frac{m_0+\alpha}{m_0+\beta}\right]$.

Proof. One applies Lemma 0.2.

Definition 0.1 *For the function* $f : [0,1] \longrightarrow \mathbb{R}$ *and* $m \in \mathbb{N}, m \geq m_0$, $\alpha, \beta \in \mathbb{R}$, *with* $0 \leq \alpha \leq \beta$, *we define the operator* $Q_m^{(\alpha,\beta)}$ *by:*

$$\left(Q_m^{(\alpha,\beta)} f\right)(x) = \frac{1}{m^m} \sum_{k=0}^{m} \binom{m}{k} ((m+\beta)x - \alpha)^k (m + \alpha - (m+\beta)x)^{m-k} f\left(\frac{k+\alpha}{m+\beta}\right) \tag{14}$$

for any $x \in \left[\frac{\alpha}{m_0+\beta}; \frac{m_0+\alpha}{m_0+\beta}\right]$

These operators are studied by Braica P.I., Pop O.T., Bărbosu D. and Pişcoran L. in [4].

2.Main results

Now, supose that $\alpha = 0$. Then the operators from (14) become

$$(Q_m^\beta f)(x) = \frac{1}{m^m} \sum_{k=0} m \binom{m}{k} ((m+\beta)x)^k (m - (m+\beta)x)^{m-k} f\left(\frac{k}{m+\beta}\right) \tag{15}$$

for any $m \in \mathbb{N}, m \geq m_0$ and any $x \in \left[0, \frac{m_0}{m_0+\beta}\right]$.

Proposition 0.2 *The operators* $(Q_m^\beta)_{m \geq m_0}$ *are linear and positive.*

Proof. It follows immediately from (15).

Remark 0.3 *For* $\beta = 0$ *in (15), we obtain Berstein's operators.*

Lemma 0.4 *For* $m \in \mathbb{N}, m \geq m_0$ *and* $x \in \left[0, \frac{m_0}{m_0+\beta}\right]$ *we have*

$$(Q_m^\beta e_0)(x) = 1 \tag{16}$$

$$(Q_m^\beta e_1)(x) = x \tag{17}$$

$$(Q_m^\beta e_2)(x) = \frac{m-1}{m} x^2 + \frac{1}{m+\beta} x \tag{18}$$

Lemma 0.5 *For* $m \in \mathbb{N}, m \geq m_0$ *and* $x \in \left[0, \frac{m_0}{m_0+\beta}\right]$, *the following identities*

$$(T_{m,0}\ Q_m^\beta)(x) = 1 \tag{19}$$

$$(T_{m,1}\ Q_m^\beta)(x) = 0 \tag{20}$$

$$(T_{m,2}\ Q_m^{\beta})(x) = -mx^2 + \frac{m^2}{m+\beta} \tag{21}$$

$$(T_{m,3}\ Q_m^{\beta})(x) = (-2m^3+2m)x^3 - \frac{3m^2}{m+\beta}x^2 + \frac{m^3}{m+\beta}x \tag{22}$$

$$(T_{m,4}\ Q_m^{\beta})(x) = (3m^2-6)x^4 + \frac{-16m^3+12m^2}{m+\beta}x^3 + \frac{3m^4-7m^3}{(m+\beta)^2}x^2 + \frac{m^4}{(m+\beta)^3}x \tag{23}$$

hold.

Proof. We take (2) and Lemma 0.4 into account.

Theorem 0.2 *Let $f:[0,1]\longrightarrow\mathbb{R}$ be a function continuous on $[0,1]$. Then, we have*

$$\lim_{m\longrightarrow\infty} Q_m^{\beta} f = f \tag{24}$$

uniformly on $\left[0,\frac{m_0}{m_0+\beta}\right]$ *and exists* $m^* = max(m_0, m(0))$ *so that*

$$|(Q_m^{\beta} f)(x) - f(x)| \le \frac{9}{4}\omega\left(f;\frac{1}{\sqrt{m}}\right) \tag{25}$$

for any $x\in\left[0,\frac{m_0}{m_0+\beta}\right]$*, any* $m\in\mathbb{N}, m\ge m^*$.

Proof. Using Theorem 0.1, for $s=0$, and Lemma 0.5 are obtained immediately the conclusions of the theorem.

Theorem 0.3 *Let $f:[0,1]\longrightarrow\mathbb{R}$ be a function s times differentiabile on $[0,1]$, having s order derivate continuous on $[0,1]$.*

Then, for $s=2$ we have

$$\lim_{m\longrightarrow\infty} m((Q_m^{\beta} f)(x) - f(x)) = \frac{x(1-x)}{2} f^{(2)}(x) \tag{26}$$

uniformly on $\left[0,\frac{m_0}{m_0+\beta}\right]$ *and exists* $m^* = max(m_0, m(2), m(0))$ *so that*

$$m\left|(Q_m^{\beta} f)(x) - f(x)\right| \le \frac{5}{8}M + \frac{39}{32}\omega\left(f^{(2)},\frac{1}{\sqrt{m}}\right) \tag{27}$$

for any $x\in\left[0,\frac{m_0}{m_0+\beta}\right]$*, any* $m\in\mathbb{N}, m\ge m^*$*, where* $M=\max_{x\in[0,1]}|f^{(2)}(x)|$.

For $s=4$ we have

$$\lim_{m\longrightarrow\infty} m^2\left((Q_m^{\beta}f)(x)-f(x)-\frac{1}{2m}\left(-x+\frac{m}{m+\beta}x\right)f^{(2)}(x)-\right.$$
$$\left.-\frac{1}{6m^3}f^{(2)}(x)(T_{m,3}\ Q_m^{\beta})\right)=$$
$$=\frac{3}{24}(x(1-x))^2f^{(4)}(x), \tag{28}$$

for any $x\in\left[0,\frac{m_0}{m_0+\beta}\right]$.

Proof. Is the same way using Theorem 0.1 and Lemma 0.5, we obtain immediately the conclusions.

Remark 0.4 *The relation (26), (28) are Voronovskaja type theorems.*

3.Application

Next, using graphical representation, we will plot some graphs for this type of polynomials.

We choose $0=\alpha<3=\beta$ and we compare the following polynomials:

$$(P_m^{0,3}f)(x)=\sum_{k=0}^{m}\binom{m}{k}x^k(1-x)^{m-k}f\left(\frac{k}{m+3}\right) \tag{29}$$

the classical Stancu polynomial, with:

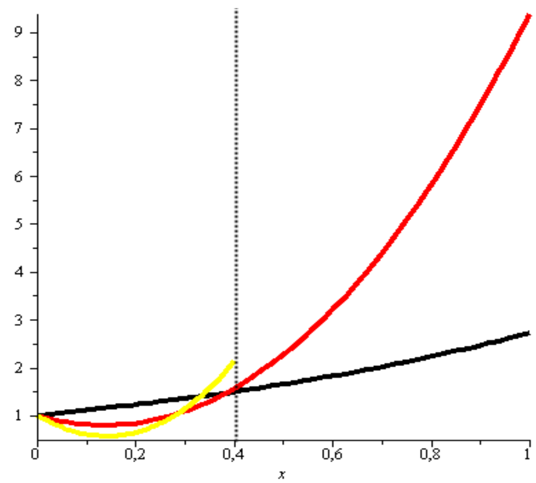
$$(Q_m^3f)(x)=\frac{1}{m^m}\sum_{k=0}^{m}\binom{m}{k}((m+3)x)^k(m-(m+3)x)^{m-k}f\left(\frac{k}{m+3}\right) \tag{30}$$

the Stancu polynomial of Berstein type. We fix $m_0=2$. For $m=3$, we plot

- with black $f:[0,1]\longrightarrow\mathbb{R}, f(x)=exp(x)$

- with red $(P_3^{(0,3)}exp(\cdot))(x)=(1-x)^3+3x(1-x)^2exp\left(\frac{1}{6}\right)+3x^2(1-x)\cdot$ $\cdot exp\left(\frac{2}{6}\right)+x^3exp\left(\frac{3}{6}\right)$, for $x\in[0,1]$

- with yellow $(Q_3^3exp(\cdot))(x)=(1-2x)^3+2x(1-2x)^2exp\left(\frac{1}{6}\right)+(2x)^2\cdot$ $\cdot(1-2x)exp\left(\frac{2}{6}\right)+(2x)^3\left(\frac{3}{6}\right)$, for $x\in\left[0,\frac{2}{5}\right]$

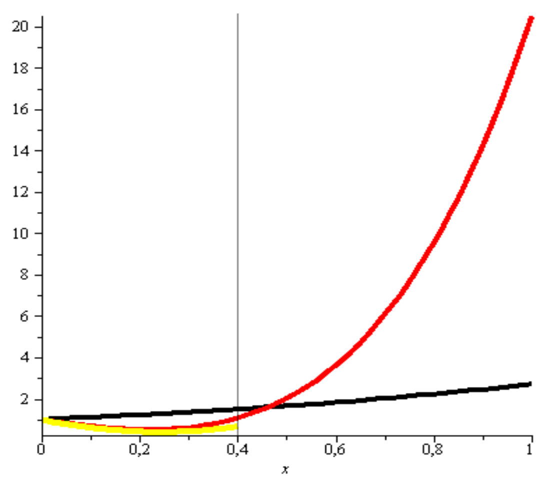


We fix $m_0 = 2$. For $m = 4$ we plot

- with black $f : [0,1] \longrightarrow \mathbb{R}, f(x) = exp(x)$

- with red $(P_4^{(0,3)} exp(\cdot))(x) = (1-x)^4 + 4x(1-x)^3 exp\left(\frac{1}{7}\right) + 6x^2 \cdot$
$\cdot (1-x)^2 exp\left(\frac{2}{7}\right) + 4x^3(1-x) exp\left(\frac{3}{7}\right) + x^4 exp\left(\frac{4}{7}\right)$, for $x \in [0,1]$

- with yellow $(Q_4^3 exp(\cdot))(x) = \frac{1}{256}(4-7x)^4 + \frac{1}{256}(7x)(4-7x)^3 exp\left(\frac{1}{7}\right) +$
$+ \frac{1}{256}(7x)^2(4-7x)^2 exp\left(\frac{2}{7}\right) + \frac{1}{256}(7x)^3(4-7x) exp\left(\frac{3}{7}\right) + \frac{1}{256}(7x)^4 exp\left(\frac{4}{7}\right)$, for $x \in \left[0, \frac{2}{5}\right]$



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