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ON TRIVARIATE PSEUDO WEIBULL DISTRIBUTION

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Abstract. The trivarite Pseudo-Weibull distribution has been proposed. Some distributional properties of the distribution has been studied. The distribution of two concomitants has been obtained. The conditional distribution of one concomitant given the information of other has also been obtained. Moments of the resulting distributions has been computed.

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1. Introduction

The ordered random variables have found wide spread applications in many areas of statistics. Order and record statistics has been extensively used in modeling the extreem values in a sequence of random variables. The record statistics provide the information about the maximum (minimum) value among all the previously recorded observations. Chandler [4] introduced the basic theory of record values and record statistics. Since the time of its emergence, the record statistics has been a very popular area of study. The record statistics has been extensively used to characterize several probability distributions. Several characterizations of exponential distribution has been done by Ahsanullah [2]. Kirmani and Beg [9] has characterized several probability distributions by using expected records. Ahsanullah [3] has provided the comprehensive theory of record values and record statistics. The distribution of upper record statistics for a random sample from a distribution $F(x)$ is given by [3] as:

$$f_{k:n}(x_k) = \frac{1}{\Gamma(k)} f(x_k) \left[R(x_k)\right]^{k-1}; \tag{1}$$

where $R(x) = -\ln\left[1 - F(x)\right]$.

The joint distribution of k-th and m-th records is given by [3] as:

$$f_{k,m:n}(x_k, x_m) = \frac{r(x_k) f(x_m)}{\Gamma(k)\Gamma(m-k)} \left[R(x_k)\right]^{k-1} \left[R(x_m) - R(x_k)\right]^{m-k-1}, \tag{2}$$

where $r(x) = R'(x)$ and $-\infty < x_m < x_k < \infty$.

The concomitants of ordered random variables are also used in many areas of statistics. David and Nagaraja [5] has defined the concomitants of order statistics. The concomitants of record statistics when a random sample is available from a bivariate distribution are defined by [3] as:

$$f\left(y_k\right) = \int_{-\infty}^{\infty} f\left(y_k|x_k\right) f_{k:n}\left(x_k\right) dx_k. \tag{3}$$

The joint distribution of two concomitants of records is given as:

$$f\left(y_k, y_m\right) = \int_{-\infty}^{\infty} \int_{-\infty}^{x_m} f\left(y_k|x_k\right) f\left(y_m|x_m\right) f_{k,m:n}\left(x_k\right) dx_k dx_m. \tag{4}$$

The distribution of concomitants of records has been studied by many authors. The distribution of the pair of concomitants of records is defined by Shahbaz et al [13] as:

$$f\left(y_k, z_k\right) = \int_{-\infty}^{\infty} f\left(y_k, z_k|x_k\right) f_{k:n}\left(x_k\right) dx_k, \tag{5}$$

where $f_{k:n}\left(x_k\right)$ is defined in (1).

The Pseudo distributions are relatively new class of probability distributions in the subject of statistics. The pseudo distributions are effective models for modeling in stochastic processes, actuarial sciences and many other fields where standard probability distributions do not provide better fit. The pseudo distributions are introduced by Filus and Files [6], [7], [8]. Filus and Filus [6] introduced the pseudo Gaussian distribution as a linear combination of normal random variables. Shahbaz et al [11] has introduced the pseudo exponential distribution as the compound distribution of two random variables. Shahbaz and Shahbaz [12] has studied the distribution of concomitants of record statistics for bivariate pseudo Rayleigh distribution. Mohsin et al [10] studied properties of lower records for bivariate pseudo inverse Rayleigh distribution.

In the following section we have introduced trivariate pseudo Weibull distribution with some of its basic properties. The distribution of bivariate concomitants for the said distribution has been found in section 3.

2. The Trivariate Pseudo-Weibull Distribution

We introduced the trivariate Pseudo-Weibull distribution as a compound distribution of three random variables. The distribution is defined in the following:

Let the random variable X has the Weibull distribution with shape parameter β_1. The density function of X is:

$$f(x;\beta_1) = \beta_1 x^{\beta_1 - 1} \exp\left(-x^{\beta_1}\right),\ x > 0, \beta_1 > 0. \tag{6}$$

Further, let the random variable Y has the Weibull distribution with shape parameter β_2 and scale parameter $\phi_1(x)$, where $\phi_1(x)$ is some function of random variable X. The density function of Y is:

$$f(y|x) = \beta_2 \phi_1(x) y^{\beta_2 - 1} \exp\left\{-\phi_1(x) y^{\beta_2}\right\},\ y, \phi_1(x), \beta_2 > 0. \tag{7}$$

Finally, let the random variable Z also has the Weibull distribution with parameter shape parameter β_3 and scale parameter $\phi_2(x,y)$, where $\phi_2(x,y)$ is some function of random variables X and Y. The density function of Z is therefore:

$$f(z|x,y) = \beta_3 \phi_2(x,y) z^{\beta_3 - 1} \exp\left\{-\phi_2(x,y) z^{\beta_3}\right\},\ z, \phi_2(x,y), \beta_3 > 0. \tag{8}$$

The trivariate pseudo Weibull distribution is defined as the compound distribution of (6), (7) and (8) as:

$$\begin{aligned} f(x,y,z) = {} & \beta_1\beta_2\beta_3\phi_1(x)\phi_2(x,y) x^{\beta_1-1} y^{\beta_2-1} z^{\beta_3-1} \\ & \cdot \exp\left[-\left\{x^{\beta_1} + \phi_1(x) y^{\beta_2} + \phi_2(x,y) z^{\beta_3}\right\}\right], \end{aligned} \tag{9}$$

where $\phi_1(x) > 0, \phi_2(x,y) > 0, x, y, z, \beta_1, \beta_2, \beta_3 > 0$.

Using various choices of $\phi_1(x)$ and $\phi_2(x,y)$ we can define several pseudo Weibull distributions. Using $\phi_1(x) = x^{\beta_1}$ and $\phi_2(x,y) = x^{\beta_1} y^{\beta_2}$ we obtained following trivariate Pseudo-Weibull distribution:

$$f(x,y,z) = \beta_1\beta_2\beta_3 x^{3\beta_1-1} y^{2\beta_2-1} z^{\beta_3-1} \exp\left[-x^{\beta_1}\left\{1 + y^{\beta_2} + y^{\beta_2} z^{\beta_3}\right\}\right], \tag{10}$$

where $x, y, z, \beta_1, \beta_2, \beta_3 > 0$.

The product moments for distribution (10) are given as:

$$\begin{aligned} \mu'_{r,s,t} &= \int_0^\infty \int_0^\infty \int_0^\infty x^r y^s z^t f(x,y,z)\, dxdydz \\ &= \int_0^\infty \int_0^\infty \int_0^\infty x^r y^s z^t \beta_1\beta_2\beta_3 x^{3\beta_1-1} y^{2\beta_2-1} z^{\beta_3-1} \\ &\quad \cdot \exp\left[-x^{\beta_1}\left\{1 + y^{\beta_2} + y^{\beta_2} z^{\beta_3}\right\}\right] dxdydz. \end{aligned}$$

Making the transformation $w = x^{\beta_1}\left\{1 + y^{\beta_2} + y^{\beta_2} z^{\beta_3}\right\}$ and after some calculus, the product moments turned out to be:

$$\mu'_{r,s,t} = \Gamma\left(1 + \frac{r}{\beta_1} - \frac{s}{\beta_2}\right) \Gamma\left(1 + \frac{s}{\beta_2} - \frac{t}{\beta_3}\right) \Gamma\left(1 + \frac{t}{\beta_3}\right). \tag{11}$$

The product moments (11) exist if $s < \frac{\beta_2(\beta_1 + r)}{\beta_1}$ and $t < \frac{\beta_3(\beta_2 + s)}{\beta_2}$. Using (11), we can easily obtain the marginal moments. The conditional distribution of X given Y and Z is obtained as:

$$f(x|y,z) = \frac{f(x,y,z)}{f(y,z)}. \tag{12}$$

The joint distribution $f(y,z)$ is:

$$\begin{aligned} f(y,z) &= \int_0^\infty \beta_1\beta_2\beta_3 x^{3\beta_1-1} y^{2\beta_2-1} z^{\beta_3-1} \exp\left[-x^{\beta_1}\left\{1 + y^{\beta_2} + y^{\beta_2} z^{\beta_3}\right\}\right] dx \\ &= \frac{2\beta_2\beta_3 y^{2\beta_2-1} z^{\beta_3-1}}{(1 + y^{\beta_2} + y^{2\beta_2} z^{\beta 3})^3}, \text{ where } y, z, \beta_2, \beta_3 > 0. \end{aligned} \tag{13}$$

Using (10) and (13) in (12), the conditional distribution of X given Y and Z is given as:

$$f(x|y,z) = \frac{1}{2}\beta_1 \left(1 + y^{\beta_2} + y^{\beta_2} z^{\beta_3}\right)^3 x^{3\beta_1-1} \exp\left[-x^{\beta_1}\left\{1 + y^{\beta_2} + y^{\beta_2} z^{\beta_3}\right\}\right].$$

The *hth* conditional moment of X is:

$$\begin{aligned} E\left(X^h|y,z\right) &= \int_0^\infty \frac{1}{2} x^h \beta_1 \left(1+y^{\beta_2}+y^{\beta_2} z^{\beta_3}\right)^3 x^{3\beta_1-1} \exp\left[-x^{\beta_1}\left\{1+y^{\beta_2}+y^{\beta_2} z^{\beta_3}\right\}\right] dx \\ &= \frac{1}{2\left(1 + y^{\beta_2} + y^{\beta_2} z^{\beta_3}\right)^{h/\beta_1}} \Gamma\left(3 + \frac{h}{\beta_1}\right). \end{aligned} \tag{14}$$

The conditional mean and variance can be readily obtained from (14). The conditional distribution of Y and Z given X is:

$$f(y,z|x) = \frac{f(x,y,z)}{f(x)}. \tag{15}$$

Using (6) and (10) in (15), the conditional distribution of Y and Z given X is:

$$f(y,z|x) = \beta_2\beta_3 x^{2\beta 1} y^{2\beta_2-1} z^{\beta_3-1} \exp\left\{-x^{\beta_1} y^{\beta_2}\left(1 + z^{\beta_3}\right)\right\}. \tag{16}$$

The joint conditional moments of Y and Z given X are obtained as:

$$\begin{aligned} E\left(Y^h Z^q|x\right) &= \int_0^\infty \int_0^\infty y^h z^q f(y,z|x)\, dydz \\ &= \int_0^\infty \int_0^\infty y^h z^q \beta_2\beta_3 x^{2\beta 1} y^{2\beta_2-1} z^{\beta_3-1} \exp\left\{-x^{\beta_1} y^{\beta_2}\left(1 + z^{\beta_3}\right)\right\} dydz \\ &= x^{-h\beta_1/\beta_2} \Gamma\left(1 + \frac{h}{\beta_2} - \frac{q}{\beta_3}\right) \Gamma\left(1 + \frac{q}{\beta_3}\right). \end{aligned} \tag{17}$$

The conditional means, variances and covariances can be readily obtained from (17). In the following section we have obtained the distribution of the concomitants of record statistics for (21).

3. Bivariate Concomitants of Upper Records

We have defined the trivariate pseudo-Weibull distribution in (9) with a special case in (10). The distribution of bivariate concomitant of records is given in (5). We now derive the distribution of bivariate concomitant of records. The joint conditional distribution of Y and Z given X is given in (16). The distribution of k-th upper record, $X_k = X$, is:

$$f_{k:n}(x_k) = \frac{\beta_1}{\Gamma(k)} x_k^{k\beta_1 - 1} \exp\left(-x_k^{\beta_1}\right), \; x_k, \beta_1 > 0. \tag{18}$$

Using (16) and (18) in (5), the joint distribution of the pair of concomitants, $Y_k = Y$ and $Z_k = Z$, is obtained below:

$$\begin{aligned} f(y_k, z_k) &= \int_0^\infty \beta_2\beta_3 x_k^{2\beta 1} y_k^{2\beta_2 - 1} z_k^{\beta_3 - 1} \exp\left\{-x_k^{\beta_1} y_k^{\beta_2}\left(1 + z_k^{\beta_3}\right)\right\} \\ &\cdot \frac{\beta_1}{\Gamma(k)} x_k^{k\beta_1 - 1} \exp\left(-x_k^{\beta_1}\right) dx_k \\ &= \frac{\beta_1\beta_2\beta_3}{\Gamma(k)} y_k^{2\beta_1 - 1} z_k^{\beta_3 - 1} \int_0^\infty x_k^{\beta_1(k+2)-1} \exp\left\{-x_k^{\beta_1}\left(1 + y_k^{\beta_2} + y_k^{\beta_2} z_k^{\beta_3}\right)\right\} dx_k. \end{aligned}$$

Making the transformation $x_k^{\beta_1}\left(1 + y_k^{\beta_2} + y_k^{\beta_2} z_k^{\beta_3}\right) = w$ and simplifying, the joint distribution of two concomitants, $Y_k = Y$ and $Z_k = Z$, is obtained as:

$$f(y_k, z_k) = \frac{\beta_2\beta_3 k(k+1) y_k^{2\beta_2 - 1} z_k^{\beta_3 - 1}}{\left(1 + y_k^{\beta_2} + y_k^{\beta_2} z_k^{\beta_3}\right)^{k+2}}, \; y_k, z_k > 0. \tag{19}$$

The joint moments of two concomitants are given by:

$$\mu'_{s,t} = E\left(Y_k^s, Z_k^t\right) = \int_0^\infty \int_0^\infty y_k^s z_k^t \frac{\beta_2\beta_3 k(k+1) y_k^{2\beta_2 - 1} z_k^{\beta_3 - 1}}{\left(1 + y_k^{\beta_2} + y_k^{\beta_2} z_k^{\beta_3}\right)^{k+2}} dy_k dz_k$$

Using [1] and some simplification, the product moments of two concomitants are given as:

$$\mu'_{s,t} = \frac{1}{\Gamma(k)} \Gamma\left(k - \frac{s}{\beta_2}\right) \Gamma\left(1 + \frac{s}{\beta_2} - \frac{t}{\beta_3}\right) \Gamma\left(1 + \frac{t}{\beta_3}\right) \cdot \tag{20}$$

The means, variances and covariance can be easily obtained by using (20). The marginal distribution of $Z_k = z$ is readily obtained from (19) as:

$$f\left(z_k\right)=\int_0^{\infty}\frac{\beta_2\beta_3k\left(k+1\right)y_k^{2\beta_2-1}z_k^{\beta_3-1}}{\left(1+y_k^{\beta_2}+y_k^{\beta_2}z_k^{\beta_3}\right)^{k+2}}dy_k=\frac{\beta_3z_k^{\beta_3-1}}{\left(1+z_k^{\beta_3}\right)^2},\ \text{where } z_k,\beta_3>0\cdot \tag{21}$$

Using (19) and (21), the conditional distribution of $Y_k = Y$ given $Z_k = Z$ is:

$$f\left(y_k|z_k\right)=\frac{\beta_2k\left(k+1\right)y_k^{2\beta_2-1}\left(1+z_k^{\beta_3}\right)^2}{\left(1+y_k^{\beta_2}+y_k^{\beta_2}z_k^{\beta_3}\right)^{k+2}},\ y_k,z_k>0 \tag{22}$$

The conditional moments of Y_k given Z_k are given as:

$$E\left(Y_k^s|z_k\right)=\int_0^{\infty}y_k^s\frac{\beta_2k\left(k+1\right)y_k^{2\beta_2-1}\left(1+z_k^{\beta_3}\right)^2}{\left(1+y_k^{\beta_2}+y_k^{\beta_2}z_k^{\beta_3}\right)^{k+2}}dy_k$$

Using [1], the conditional moments are given as:

$$E\left(Y_k^s|z_k\right)=\frac{\Gamma\left(k-\frac{s}{\beta_2}\right)\Gamma\left(2+\frac{s}{\beta_2}\right)}{\left(1+z^{\beta_3}\right)^{\frac{s}{\beta_2}}\Gamma\left(k\right)},\ s<k\beta_2. \tag{23}$$

The conditional mean and variance of Y_k given Z_k can be readily obtained from (23).

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