Acta Universitatis Apulensis No. 31/2012
ISSN: 1582-5329 pp. 191-197

ON CERTAIN SUBCLASS OF ANALYTIC FUNCTIONS DEFINED BY GENERALIZED DERIVATIVE OPERATOR

Serap BULUT

Abstract. In this paper, we define a general derivative operator and by means of this operator, introduce a new class $\mathcal{B}_{p,n}^{m}(\alpha,\delta,\lambda,l,\mu)$ of functions and obtain its relations with some well-known subclasses of analytic multivalent functions. Furthermore, we provide the sufficient conditions for functions to be in the class $\mathcal{B}_{p,n}^{m}(\alpha,\delta,\lambda,l,\mu)$.

2000 *Mathematics Subject Classification*: 30C45.

Keywords and phrases: Analytic function, Multivalent function, Starlike function, Convex function, Derivative operator.

1. Introduction and definitions

Let $\mathcal{H}$ be the subclass of analytic functions in the open unit disc

$$\mathbb{U}=\{z\in\mathbb{C}:|z|<1\}$$

and $\mathcal{H}[a,n]$ be the subclass of $\mathcal{H}$ consisting of the functions of the form

$$f(z)=a+a_nz^n+a_{n+1}z^{n+1}+\cdots.$$

Let $\mathcal{A}(p,n)$ denote the class of all functions of the form

$$f(z)=z^p+\sum_{k=p+n}^{\infty}a_kz^k \qquad (p,n\in\mathbb{N}=\{1,2,\ldots\}) \tag{1}$$

which are analytic in the open unit disc $\mathbb{U}$.

In particular, we set

$$\mathcal{A}(p,1):=\mathcal{A}_p \qquad \text{and} \qquad \mathcal{A}(1,1)=\mathcal{A}_1:=\mathcal{A}.$$

A function $f\in\mathcal{A}(p,n)$ is said to be p-valently starlike of order α $(0\leq\alpha<p)$ if it satisfies

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\}>\alpha \tag{2}$$

for all $z \in \mathbb{U}$. We say that f is in the class $\mathcal{S}_n^*(p,\alpha)$ for such functions. In particular, we set $\mathcal{S}_1^*(1,\alpha) = \mathcal{S}^*(\alpha)$.

A function $f \in \mathcal{A}(p,n)$ is said to be p-valently convex of order α $(0 \le \alpha < p)$ if it satisfies

$$\mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha \tag{3}$$

for all $z \in \mathbb{U}$. We say that f is in the class $\mathcal{K}_n(p,\alpha)$ for such functions. In particular, we set $\mathcal{K}_1(1,\alpha) = \mathcal{K}(\alpha)$.

We denote by $\mathcal{R}_n(p,\alpha)$ the class of functions in $\mathcal{A}(p,n)$ which satisfy

$$\mathrm{Re}\left\{\frac{f'(z)}{z^{p-1}}\right\} > \alpha \tag{4}$$

for all $z \in \mathbb{U}$. In particular, we set $\mathcal{R}_1(1,\alpha) = \mathcal{R}(\alpha)$.

For a function $f \in \mathcal{A}(p,n)$, we define the general differential operator $D_{\lambda,l,p}^{m,\delta}$ as follows:

$$\begin{aligned}
D_{\lambda,l,p}^{0,\delta} f(z) &= f(z), \\
D_{\lambda,l,p}^{1,\delta} f(z) &= \left(\delta - \frac{\lambda p}{p+l}\right) f(z) + \left(\frac{\lambda}{p+l} - \frac{\delta - 1}{p}\right) z f'(z) \\
&= D_{\lambda,l,p}^{\delta} f(z), \qquad \delta, \lambda, l \ge 0, \qquad (5) \\
D_{\lambda,l,p}^{2,\delta} f(z) &= D_{\lambda,l,p}^{\delta}\left(D_{\lambda,l,p}^{1,\delta} f(z)\right), \\
&\vdots \\
D_{\lambda,l,p}^{m,\delta} f(z) &= D_{\lambda,l,p}^{\delta}\left(D_{\lambda,l,p}^{m-1,\delta} f(z)\right), \qquad m \in \mathbb{N}. \qquad (6)
\end{aligned}$$

If f is given by (1), then by (5) and (6), we see that

$$D_{\lambda,l,p}^{m,\delta} f(z) = z^p + \sum_{k=p+n}^{\infty} \left[\frac{k}{p} + (k-p)\left(\frac{\lambda}{p+l} - \frac{\delta}{p}\right)\right]^m a_k z^k, \quad m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}. \tag{7}$$

Remark 1. If we set $n = 1$ in (7), then we have following operators.

(**i**) $D_{\lambda,l,p}^{m,1} = I_p(m,\lambda,l)$ defined by Cătaş [2].

(**ii**) $D_{\lambda,0,1}^{m,\delta} = D_{\delta,\lambda}^m$ defined and studied by Darus and Ibrahim [4].

(**iii**) $D_{\lambda,0,1}^{m,1} = D_\lambda^m$ which is Al-Oboudi (generalized Sălăgean) differential operator [1].

(**iv**) $D_{1,0,1}^{m,1} = D^m$ which is Sălăgean differential operator [8].

Remark 2. It follows from the (7) that

$$\begin{aligned} p\,(p+l)\, D_{\lambda,l,p}^{m+1,\delta} f\,(z) &= p\,[\delta\,(p+l)-\lambda p]\, D_{\lambda,l,p}^{m,\delta} f\,(z) \\ &\quad + [\lambda p+(1-\delta)\,(p+l)]\, z \left(D_{\lambda,l,p}^{m,\delta} f\,(z)\right)' \end{aligned} \tag{8}$$

for $m \in \mathbb{N}_0$ and $z \in \mathbb{U}$.

In order to prove our main results, we shall require the following lemma.

Lemma 1.1. [6] Let h be analytic in $\mathbb{U}$ with $h\,(0)=1$ and suppose that

$$\operatorname{Re}\left\{1+\frac{zh'\,(z)}{h\,(z)}\right\} > \frac{3\alpha-1}{2\alpha} \quad (z \in \mathbb{U})\,.$$

Then

$$\operatorname{Re}\{h\,(z)\} > \alpha$$

for $z \in \mathbb{U}$ and $\frac{1}{2} \leq \alpha < 1$.

2. Main results

Definition 1. We say that a function $f \in \mathcal{A}\,(p,n)$ is in the class $\mathcal{B}_{p,n}^{m}\,(\alpha,\delta,\lambda,l,\mu)$ $(p,n \in \mathbb{N};\ m \in \mathbb{N}_0;\ \delta,\lambda,l \geq 0;\ \mu \geq 0;\ 0 \leq \alpha < 1)$ if

$$\left| \frac{D_{\lambda,l,p}^{m+1,\delta} f\,(z)}{z^p} \left(\frac{z^p}{D_{\lambda,l,p}^{m,\delta} f\,(z)} \right)^{\mu} - p \right| < p-\alpha \quad (z \in \mathbb{U})\,.$$

Remark 3. The family $\mathcal{B}_{p,n}^{m}\,(\alpha,\delta,\lambda,l,\mu)$ is a new comprehensive class of analytic functions which includes various new classes of analytic functions as well as some very well known ones. For example,

(**i**) For $m=0$ and $\mu=1$, we have the class

$$\mathcal{B}_{p,n}^{0}\,(\alpha,\delta,\lambda,l,1) \equiv \mathcal{S}_n^*(p,\alpha).$$

(**ii**) For $m=1$, $\delta=\lambda=0$ and $\mu=1$, we have the class

$$\mathcal{B}_{p,n}^{1}\,(\alpha,0,0,l,1) \equiv \mathcal{K}_n(p,\alpha).$$

(**iii**) For $m=0$ and $\mu=0$, we have the class

$$\mathcal{B}_{p,n}^{0}\,(\alpha,\delta,\lambda,l,0) \equiv \mathcal{R}_n(p,\alpha).$$

(**iv**) For $p=1$ and $\delta=1$, we have the class

$$\mathcal{B}_{1,n}^{m}\left(\alpha,1,\lambda,l,\mu\right)\equiv\mathcal{BI}\left(m,n,\mu,\alpha,\lambda,l\right)$$

introduced by Lupas [7].

(**v**) For $p=n=1$, $\delta=1$ and $\lambda=1$, we have the class

$$\mathcal{B}_{1,1}^{m}\left(\alpha,1,1,l,\mu\right)\equiv\mathcal{B}\left(m,\mu,\alpha,\lambda\right)$$

introduced and studied by Stanciu and Breaz [9].

(**vi**) For $p=1$, $\delta=1$, $\lambda=1$ and $l=0$, we have the class

$$\mathcal{B}_{1,n}^{m}\left(\alpha,1,1,0,\mu\right)\equiv\mathcal{BS}_{n}\left(m,\mu,\alpha\right)$$

introduced by Cătaş and Lupaş [3].

(**vii**) For $m=0$ and $p=n=1$, the class

$$\mathcal{B}\left(\mu,\alpha\right)=\left\{f\in\mathcal{A}:\left|f'(z)\left(\frac{z}{f(z)}\right)^{\mu}-1\right|<1-\alpha;\ \mu\geq 0,\ 0\leq\alpha<1,\ z\in\mathbb{U}\right\}$$

introduced by Frasin and Jahangiri [6].

(**viii**) For $m=0$, $p=n=1$ and $\mu=2$, the class

$$\mathcal{B}\left(\alpha\right)=\left\{f\in\mathcal{A}:\left|\frac{z^{2}f'(z)}{f^{2}(z)}-1\right|<1-\alpha;\ 0\leq\alpha<1,\ z\in\mathbb{U}\right\}$$

introduced by Frasin and Darus [5].

The object of the present paper is to investigate the sufficient condition for functions to be in the class $\mathcal{B}_{p,n}^{m}\left(\alpha,\delta,\lambda,l,\mu\right).$

Theorem 2.1. Let $f\in\mathcal{A}\left(p,n\right)$ be the function of the form (1), $\mu\geq 0$ and $\frac{1}{2}\leq\alpha<1$. If

$$\operatorname{Re}\left\{\frac{p\left(p+l\right)}{\lambda p+\left(1-\delta\right)\left(p+l\right)}\frac{D_{\lambda,l,p}^{m+2,\delta}f\left(z\right)}{D_{\lambda,l,p}^{m+1,\delta}f\left(z\right)}-\frac{\mu p\left(p+l\right)}{\lambda p+\left(1-\delta\right)\left(p+l\right)}\frac{D_{\lambda,l,p}^{m+1,\delta}f\left(z\right)}{D_{\lambda,l,p}^{m,\delta}f\left(z\right)}+\frac{p\left(p+l\right)\left(\mu-1\right)}{\lambda p+\left(1-\delta\right)\left(p+l\right)}+1\right\}>\frac{3\alpha-1}{2\alpha}, \tag{9}$$

then $f\in\mathcal{B}_{p,n}^{m}\left(\alpha,\delta,\lambda,l,\mu\right).$

Proof. Define the function $h(z)$ by

$$h(z)=\frac{D_{\lambda,l,p}^{m+1,\delta}f\left(z\right)}{z^{p}}\left(\frac{z^{p}}{D_{\lambda,l,p}^{m,\delta}f\left(z\right)}\right)^{\mu}. \tag{10}$$

Then the function $h(z)$ is analytic in $\mathbb{U}$ and $h(0)=1$. Therefore, differentiating (10) logarithmically and using (8), the simple computation yields

$$\begin{aligned}\frac{zh'(z)}{h(z)} &= \frac{p\left(p+l\right)}{\lambda p+\left(1-\delta\right)\left(p+l\right)}\frac{D_{\lambda,l,p}^{m+2,\delta}f\left(z\right)}{D_{\lambda,l,p}^{m+1,\delta}f\left(z\right)}-\frac{\mu p\left(p+l\right)}{\lambda p+\left(1-\delta\right)\left(p+l\right)}\frac{D_{\lambda,l,p}^{m+1,\delta}f\left(z\right)}{D_{\lambda,l,p}^{m,\delta}f\left(z\right)}\\ &\quad+\frac{p\left(p+l\right)\left(\mu-1\right)}{\lambda p+\left(1-\delta\right)\left(p+l\right)}.\end{aligned}$$

By the hypothesis of the theorem, we have

$$\operatorname{Re}\left\{1+\frac{zh'\left(z\right)}{h\left(z\right)}\right\}>\frac{3\alpha-1}{2\alpha}.$$

Hence, by Lemma 1.1, we have

$$\operatorname{Re}\left\{\frac{D_{\lambda,l,p}^{m+1,\delta}f\left(z\right)}{z^{p}}\left(\frac{z^{p}}{D_{\lambda,l,p}^{m,\delta}f\left(z\right)}\right)^{\mu}\right\}>\alpha.$$

Therefore, in view of Definition 1, $f\in\mathcal{B}_{p,n}^{m}\left(\alpha,\delta,\lambda,l,\mu\right)$. ♦

As consequences of the above theorem we have the following corollaries.

Choosing $m=1$, $\mu=1$, $\alpha=\frac{1}{2}$, $p=n=1$, $\delta=\lambda=1$ and $l=0$, we have

Corollary 2.2. If $f\in\mathcal{A}$ and

$$\operatorname{Re}\left\{\frac{z^{2}f'''(z)+2zf''(z)}{zf''(z)+f'(z)}-\frac{zf''(z)}{f'(z)}\right\}>-\frac{1}{2},$$

then

$$\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}>\frac{1}{2}.$$

That is, $f\in\mathcal{K}\left(\frac{1}{2}\right)$.

Choosing $m=1$, $\mu=0$, $\alpha=\frac{1}{2}$, $p=n=1$, $\delta=\lambda=1$ and $l=0$, we have

Corollary 2.3. If $f \in \mathcal{A}$ and

$$\mathrm{Re}\left\{\frac{z^2 f'''(z) + 2zf''(z)}{zf''(z) + f'(z)}\right\} > -\frac{1}{2},$$

then

$$\mathrm{Re}\left\{f'(z) + zf''(z)\right\} > \frac{1}{2}.$$

Choosing $m = 0$, $\mu = 1$, $\alpha = \frac{1}{2}$, $p = n = 1$, $\delta = \lambda = 1$ and $l = 0$, we have

Corollary 2.4. If $f \in \mathcal{A}$ and

$$\mathrm{Re}\left\{\frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right\} > -\frac{3}{2},$$

then

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \frac{1}{2}.$$

That is, $f \in \mathcal{S}^*\left(\frac{1}{2}\right)$.

Choosing $m = 0$, $\mu = 0$, $\alpha = \frac{1}{2}$, $p = n = 1$, $\delta = \lambda = 1$ and $l = 0$, we have

Corollary 2.5. If $f \in \mathcal{A}$ and

$$\mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \frac{1}{2},$$

then

$$\mathrm{Re}\left\{f'(z)\right\} > \frac{1}{2}.$$

That is, $f \in \mathcal{R}\left(\frac{1}{2}\right)$.

References

[1] F.M. Al-Oboudi, *On univalent functions defined by a generalized Sălăgean operator*, Int. J. Math. Math. Sci. **2004**, no. 25-28, 1429-1436.

[2] A. Cătaş, *On certain classes of p-valent functions defined by multiplier transformations*, in: Proceedings of the International Symposium on Geometric Function Theory and Applications, İstanbul, Turkey, August 2007.

[3] A. Cătaş and A.A. Lupaş, *On sufficient conditions for certain subclass of analytic functions defined by differential Sălăgean operator*, Int. J. Open Problems Complex Analysis **1** (2009), no. 2, 14-18.

[4] M. Darus and R.W. Ibrahim, *On new subclasses of analytic functions involving generalized differential and integral operators*, Eur. J. Pure Appl. Math. **4** (2011), no. 1, 59-66.

[5] B.A. Frasin and M. Darus, *On certain analytic univalent functions*, Int. J. Math. Math. Sci. **25** (2001), no. 5, 305-310.

[6] B.A. Frasin and Jay M. Jahangiri, *A new and comprehensive class of analytic functions*, An. Univ. Oradea Fasc. Mat. **15** (2008), 59-62.

[7] A.A. Lupaş, *A note on a subclass of analytic functions defined by multiplier transformations*, Int. J. Open Problems Complex Analysis **2** (2010), no. 2, 154-159.

[8] G.S. Sălăgean, *Subclasses of univalent functions*, in: Lecture Notes in Math., vol. 1013, Springer-Verlag, Berlin, Heidelberg and New York, 1983, pp. 362-372.

[9] L. Stanciu and D. Breaz, *A subclass of analytic functions defined by multiplier transformation*, Acta Univ. Apulensis **27** (2011), 225-228.

Serap BULUT
Kocaeli University
Civil Aviation College
Arslanbey Campus
Kartepe, Kocaeli / Turkey
email: *serap.bulut@kocaeli.edu.tr*