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UNIVALENCY OF ANALYTIC FUNCTIONS ASSOCIATED WITH SCHWARZIAN DERIVATIVE

The authors would like to dedicate this paper to the late Professor Shigeo Ozaki

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Abstract. Let $\mathcal{A}$ be the class of analytic functions $f(z)$ in the open unit disk U normalized with $f(0) = 0$ and $f'(0) = 1$. For $f(z) \in \mathcal{A}$, a new univalency of $f(z)$ associated with Schwarzian derivative of $f(z)$ is discussed.

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1. Introduction

Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disk $U = \{z \in C : |z| < 1\}$. For $f(z) \in \mathcal{A}$, the following differential operator

$$\{f(z), z\} = \left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2 \tag{1.2}$$

$$= \frac{f'''(z)}{f'(z)} - \frac{3}{2}\left(\frac{f''(z)}{f'(z)}\right)^2$$

is said to be the Schwarzian derivative of $f(z)$ or the Schwarzian differential operator of $f(z)$. For the Schwarzian derivative of $f(z) \in \mathcal{A}$, the following results by Nehari [2] are well-known.

Theorem A. *If $f(z) \in \mathcal{A}$ is univalent in U, then*

$$|\{f(z), z\}| \leq \frac{6}{(1 - |z|^2)^2} \qquad (z \in U). \tag{1.3}$$

The equality is attained by Koebe function $f(z)$ given by

$$f(z) = \frac{z}{(1-z)^2} \tag{1.4}$$

and its rotation.

Theorem B. *If $f(z) \in \mathcal{A}$ satisfies*

$$|\{f(z), z\}| \leq \frac{2}{(1-|z|^2)^2} \qquad (z \in U), \tag{1.5}$$

then $f(z)$ is univalent in U.

For Theorem B, Hille [1] has noticed that 2 in (1.5) is the best possible constant.

Let us define the function $g(z)$ by

$$\begin{aligned} g(z) &= \frac{f'(x)(1-|x|^2)}{f\left(\frac{z+x}{1+\overline{x}z}\right) - f(x)} \\ &= \frac{1}{z} + \overline{x} - \frac{1}{2}(1-|x|^2)\frac{f''(x)}{f'(x)} - \frac{1}{6}(1-|x|^2)^2 \left\{ \left(\frac{f''(x)}{f'(x)}\right)' - \frac{1}{2}\left(\frac{f''(x)}{f'(x)}\right)^2 \right\} z + \cdots \\ &= \frac{1}{z} + h(z,x) \end{aligned} \tag{1.6}$$

for $f(z) \in \mathcal{A}$ and some complex x such that $|x| < 1$, where

$$h(z,x) = \overline{x} - \frac{1}{2}(1-|x|^2)\frac{f''(x)}{f'(x)} - \frac{1}{6}(1-|x|^2)^2 \left\{ \left(\frac{f''(x)}{f'(x)}\right)' - \frac{1}{2}\left(\frac{f''(x)}{f'(x)}\right)^2 \right\} z + \cdots . \tag{1.7}$$

Then, it is easy to see that $g(z)$ is univalent in U if and only if $f(z)$ is univalent in U.

On the other hand, Ozaki and Nunokawa [3] have given the following result.

Theorem C. *If $f(z) \in \mathcal{A}$ is univalent in U, then*

$$|h'(0,x)| \leq \frac{(1-|x|^2)^2}{6}|\{f(x), x\}| \leq 1 \qquad (|x| < 1). \tag{1.8}$$

If $f(z) \in \mathcal{A}$ satisfies

$$|h'(0,x)| \leq \frac{1}{3} \qquad (|x| < 1), \tag{1.9}$$

then $f(z)$ *is univalent in* U.

To discuss the univalency for our problem, we have to recall here the following result which is called Darboux theorem.

Lemma 1. *Let* E *be a domain covered by Jordan curve* C *and let* $w = f(z)$ *be analytic in* E. *If a point* z *moves on* C *in the positive direction, then* w *also moves on the Jordan curve* $\Gamma = f(C)$ *in the positive direction. Let* Δ *be the inside of the curve* Γ. *Then* $w = f(z)$ *is univalent in* E *and maps* E *onto* Δ *conformally.*

Proof. Let $w_0 \in \Delta$ and $\phi(z) = w - w_0 = f(z) - w_0$. Then $\phi(z)$ is analytic in $E, \phi(z) \neq 0$ on C, and

$$\frac{1}{2\pi}\int_C d\arg\phi(z) = \frac{1}{2}\int_\Gamma d\arg(w - w_0). \tag{1.10}$$

¿From the argument theorem, the left hand side of (1.10) shows that the number of zeros of $\phi(z)$ in E and the right hand side of (1.10) shows the argument momentum when w moves on Γ in the positive direction. Therefore, the right hand side of (1.10) should be just one. This shows us that $\phi(z) = f(z) - w_0$ has one zero in E.

Let us put $w_0 = f(z_0)$. Then there exists only one point $z_0 \in E$ for an arbitrary $w_0 \in \Delta$. This means that $f(z)$ is univalent in E.

For the case of $w_0 \notin \Delta$, we obtain that

$$\int_C d\arg(w - w_0) = 0, \tag{1.11}$$

which gives us that $\phi(z) = f(z) - w_0$ has no zero in E. This completes the proof of the lemma.

We note that we owe the proof of Lemma 1 by Tsuji [4].

2. Univalency of functions associated with Schwarzian derivative

An application for Lemma 1 derives

Theorem 1. *If* $f(z) \in \mathcal{A}$ *satisfies*

$$\mathrm{Re} h'(z, x) > \alpha \qquad (z \in U) \tag{2.1}$$

for some real α $(\alpha > 1)$ *and for all* $|x| < 1$, *then* $f(z)$ *is univalent in* U, *where* $h(z, x)$ *is given by* (1.7).

Proof. Let us put $0 < |z| < 1$ and $|x| < 1$. Then, using $g(z)$ and $h(z,x)$ given by (1.7), we have that

$$g(z) - \frac{1}{z} = h(z,x) \tag{2.2}$$

is analytic in U. Note that $f(z)$ is univalent in U if and only if $g(z)$ is univalent in U. We know that

$$\left(g(z_2) - \frac{1}{z_2}\right) - \left(g(z_1) - \frac{1}{z_1}\right) = h(z_2,x) - h(z_1,x) = \int_{z_1}^{z_2} \left(\frac{dh(z,x)}{dz}\right) dz, \tag{2.3}$$

where the integral is taken on the line segment z_1z_2 such that $z_1 \neq z_2$ and $0 < |z_1| = |z_2| = r < 1$. Letting

$$z = z_1 + (z_2 - z_1)t \qquad (0 \leq t \leq 1),$$

we have that

$$\int_{z_1}^{z_2} \left(\frac{dh(z,x)}{dz}\right) dz = (z_2 - z_1) \int_0^1 \left(\frac{dh(z,x)}{dz}\right) dz. \tag{2.4}$$

Therefore, we obtain that

$$g(z_2) - g(z_1) + \frac{z_2 - z_1}{z_1 z_2} = (z_2 - z_1) \int_0^1 h'(z,x)dt.$$

This gives us that

$$\begin{aligned} \frac{g(z_2) - g(z_1)}{z_2 - z_1} &= \int_0^1 h'(z,x)dt - \frac{1}{z_1 z_2} \\ &= \int_0^1 \left(h'(z,x) - \frac{1}{z_1 z_2}\right) dt. \end{aligned} \tag{2.5}$$

If there exist two points z_1 and z_2 such that $z_1 \neq z_2$ and $|z_1| = |z_2| = r < 1$ for which $g(z_1) = g(z_2)$, then we have that

$$0 = \int_0^1 \operatorname{Re}\left(h'(z,x) - \frac{1}{z_1 z_2}\right) dt > \int_0^1 \left(\alpha - \frac{1}{|z_1 z_2|}\right) dt = \frac{\alpha r^2 - 1}{r^2}.$$

Therefore, letting $r \to 1^-$, we see that

$$\int_0^1 \operatorname{Re}\left(h'(z,x) - \frac{1}{z_1 z_2}\right) dt > 0.$$

This is the contradiction and shows that there exist no points z_1 and z_2 such that $z_1 \neq z_2$ and $g(z_1) = g(z_2)$ in U. Since $g(z)$ is univalent in U, using Lemma 1, we conclude that $f(z)$ is univalent in U.

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