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COEFFICIENTS BOUNDS FOR THE TRANSFORMATIONS, OF SOME SUBCLASSES OF UNIFORMLY TYPE FUNCTIONS, BY USING AN INTEGRAL OPERATOR

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Abstract. In this paper we define an integral operator and study the coefficients bounds for the subclasses of k-uniformly convex and starlike functions.

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1. Introduction

Let $\mathcal{H}(U)$ be the set of functions which are regular in the unit disc U, $A = \{f \in \mathcal{H}(U) : f(0) = f'(0) - 1 = 0\}$, $\mathcal{H}_u(U) = \{f \in \mathcal{H}(U) : f$ is univalent in $U\}$ and $S = \{f \in A : f$ is univalent in $U\}$.

Let D^n be the Sălăgean differential operator (see [4]) defined as:

$$D^n : A \to A \ , \quad n \in \mathbb{N} \text{ and } D^0 f(z) = f(z)$$
$$D^1 f(z) = Df(z) = zf'(z) \ , \quad D^n f(z) = D(D^{n-1} f(z)).$$

Remark 0.1. *If $f \in S$, $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$, $z \in U$ then $D^n f(z) = z + \sum_{j=2}^{\infty} j^n a_j z^j$.*

We recall here the analytically definitions of the well - known classes of starlike and convex functions

$$S^* = \left\{ f \in A : Re \frac{zf'(z)}{f(z)} > 0 \ , \ z \in U \right\}.$$

$$S^c = \left\{ f \in A : \left\{ \frac{zf''(z)}{f'(z)} + 1 \right\} > 0 \ , \ z \in U \right\}.$$

2. Preliminary results

Definition 0.1. *A function $f \in S$ is called uniformly convex of type α, $\alpha \geq 0$ if:*

$$Re\left\{1+\frac{zf''(z)}{f'(z)}\right\} \geq \alpha\left|\frac{zf''(z)}{f'(z)}\right|,\ z \in U. \tag{1}$$

We denote by $US^c(\alpha)$ the class of all this functions.

Remark 0.2. *The class $US^c(\alpha)$ it was defined by Kanas and Wisniowska in [1], by using the following geometrical interpretation:*

Let $0 \leq k < \infty$. A function $f \in S$ is called k-uniformly convex in U if the image of any circle arc γ contained in U, with the center ζ, where $|\zeta| \leq k$, is convex.

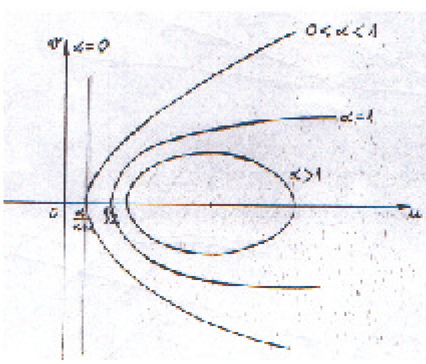
Geometrical interpretation: $f \in US^c(\alpha)$ if and only if $1 + zf''(z)/f'(z)$ take all values in D_α, where D_α is:

i) a elliptic region: $\dfrac{\left(u-\frac{\alpha^2}{\alpha^2-1}\right)^2}{\left(\frac{\alpha}{\alpha^2-1}\right)^2}+\dfrac{v^2}{\left(\frac{1}{\sqrt{\alpha^2-1}}\right)^2}<1$, for $\alpha > 1$

ii) a parabolic region: $v^2 < 2u - 1$, for $\alpha = 1$

iii) a hyperbolic region: $\dfrac{\left(u+\frac{\alpha^2}{1-\alpha^2}\right)^2}{\left(\frac{\alpha}{1-\alpha^2}\right)^2}-\dfrac{v^2}{\left(\frac{1}{\sqrt{1-\alpha^2}}\right)^2}>1$, and $u > 0$, for $0 < \alpha < 1$

iv) the half plane $u > 0$, for $\alpha = 0$.



Remark 0.3. *From the geometrical interpretation it is easy to see that $US^c(\alpha) \subset S^c\left(\frac{\alpha}{\alpha+1}\right)$.*

Theorem 0.1. *[2] Let* $\alpha \geq 0$ *and* $f \in US^c(\alpha)$, $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. *Then* $|a_n| \leq \frac{(P_1)_{n-1}}{(1)_n}$, $n = 2, 3, \ldots$, *where* $(\lambda)_n$ *is the Pochhammer symbol, defined by* $(\lambda)_0 = 1$, $(\lambda)_n = \lambda(\lambda+1)\ldots(\lambda+n-1)$, $n \in \mathbb{N}$,

$$P_1 \equiv P_1(\alpha) = \begin{cases} \dfrac{8(arccos\,\alpha)^2}{\pi^2(1-\alpha^2)}\,, & 0 \leq \alpha < 1\,, \\ \dfrac{8}{\pi^2}\,, & \alpha = 1\,, \\ \dfrac{\pi^2}{4\sqrt{k}(\alpha^2-1)K^2(k)(1+k)}\,, & \alpha > 1\,. \end{cases}$$

and $K(k)$ *is the Legendre elliptic integral*

$$K(k) = \int_0^1 \frac{dt}{\sqrt{1-t^2}\sqrt{1-k^2t^2}}\,,\ k \in (0,1)$$

such that $\alpha = cosh[\pi K'(k)]/[4K(k)]$ *where* $K'(k) = K(\sqrt{1-k^2})$ *is the complementary integral of* $K(k)$.

Remark 0.4. *In connection with the class* $US^c(\alpha)$ *Kanas and Wisniowska define and study, in [3], the class* $\alpha - ST$ *by*

$$\alpha - ST := \left\{ f \in S\ :\ f(z) = zg'(z)\,,\ g \in US^c(\alpha) \right\}\,,\ \alpha \geq 0$$

.

Definition 0.2. *[5] A function* $f \in S$ *is said to be in the class* $SH(\alpha)$ *if it satisfies*

$$\left| \frac{zf'(z)}{f(z)} - 2\alpha\left(\sqrt{2}-1\right) \right| < Re\left\{ \sqrt{2}\,\frac{zf'(z)}{f(z)} \right\} + 2\alpha\left(\sqrt{2}-1\right),$$

for some α $(\alpha > 0)$ *and for all* $z \in U$.

Remark 0.5. *Geometric interpretation: Let* $\Omega(\alpha) = \left\{ \frac{zf'(z)}{f(z)}\ :\ z \in U\,,\ f \in SH(\alpha) \right\}$. *Then* $\Omega(\alpha) = \left\{ w = u + i \cdot v\ :\ v^2 < 4\alpha u + u^2\,,\ u > 0 \right\}$. *Note that* $\Omega(\alpha)$ *is the interior of a hyperbola in the right half-plane which is symmetric about the real axis and has vertex at the origin.*

Theorem 0.2. *[5] Let* $f(z) \in SH(\alpha)$ *and* $f(z) = z + a_2z^2 + a_3z^3 + \ldots$. *Then*

$$|a_2| \leq \frac{1+4\alpha}{1+2\alpha}\,,$$

$$|a_3| \leq \frac{(1+4\alpha)(3+16\alpha+24\alpha^2)}{4(1+2\alpha)^3}\,.$$

The estimations are sharp.

Remark 0.6. *For the extremal functions, of the inequalities from the above theorem, see [5].*

3. Main results

Definition 0.3. *Let $F(z) \in A$, $F(z) = z + b_2 z^2 + \cdots + b_n z^n + \ldots$, and $a \in \mathbb{R}^*$. We define the integral operator $L : A \to A$ by*

$$f(z) = L(F)(z) = \frac{1+a}{z^a} \int_0^z F(t) \left(t^{a-1} + t^{a+1} \right) dt \ . \tag{2}$$

Theorem 0.3. *Let $\alpha \geq 0$, $a \in \mathbb{R}^*$, and $F(z) \in US^c(\alpha)$. For $f(z) = L(F)(z)$, $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$, $z \in U$, where the integral operator L it is defined by (2), we have*

$$|a_2| \leq \left| \frac{a+1}{a+2} \right| \cdot \frac{P_1}{2} \ ,$$

$$|a_3| \leq \left| \frac{a+1}{a+3} \right| \cdot \left[\frac{P_1(P_1+1)}{6} + 1 \right]$$

and

$$|a_j| \leq \left| \frac{a+1}{a+j} \right| \cdot \frac{(P_1)_{j-3}}{(j-2)!} \cdot (P_1^*(j) + 1) \ \ , j = 4, 5, \ldots \ ,$$

where $P_1^*(j) = \dfrac{(P_1+j-2)(P_1+j-3)}{j(j-1)}$, $(\lambda)_n$ is the Pochhammer symbol and P_1 it is given in Theorem 0.1.

Proof. By differentiating in (2) we obtain

$$(1+a) \cdot F(z)(1+z^2) = a \cdot f(z) + z f'(z) \, .$$

From the above equation, for $F(z) = z + \sum_{j=2}^{\infty} b_j z^j$ and $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$, we have

$$a_2 = b_2 \cdot \frac{a+1}{a+2} \ ,$$

$$a_3 = (b_3 + 1) \cdot \frac{a+1}{a+3}$$

and

$$a_j = (b_j + b_{j-2}) \cdot \frac{a+1}{a+j} \ , j \geq 4 \, .$$

From Theorem 0.1 we have

$$|b_j| \leq \frac{(P_1)_{j-1}}{(1)_j} \ , \ j = 2, 3, \ldots$$

and thus we obtain $|a_2| \leq \left|\frac{a+1}{a+2}\right| \cdot \frac{P_1}{2}$, $|a_3| \leq \left|\frac{a+1}{a+3}\right| \cdot \left[\frac{P_1(P_1+1)}{6} + 1\right]$ and $|a_j| \leq \left|\frac{a+1}{a+j}\right| \cdot \frac{(P_1)_{j-3}}{(j-2)!} \cdot (P_1^*(j) + 1)$ $j = 4, 5, \ldots$ where $P_1^*(j) = \frac{(P_1 + j - 2)(P_1 + j - 3)}{j(j-1)}$.

In a similarly way with the proof of the above Theorem, by using the Remark 0.4 and the Theorem 0.1, we obtain:

Corollary 0.1. *Let $\alpha \geq 0$, $a \in \mathbb{R}^*$, and $F(z) \in \alpha - ST$. For $f(z) = L(F)(z)$, $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$, $z \in U$, where the integral operator L is defined by (2), we have:*

$$|a_2| \leq \left|\frac{a+1}{a+2}\right| \cdot P_1 \ ,$$

$$|a_3| \leq \left|\frac{a+1}{a+3}\right| \cdot \left[\frac{P_1(P_1+1)}{2} + 1\right]$$

and

$$|a_j| \leq \left|\frac{a+1}{a+j}\right| \cdot \frac{(P_1)_{j-3}}{(j-3)!} \cdot [P_1^{**}(j) + 1] \ , \ j = 4, 5, \ldots \ ,$$

where $P_1^{**}(j) = \frac{(P_1 + j - 2)(P_1 + j - 3)}{(j-1)(j-2)}$, $(\lambda)_j$ it is the Pochhammer symbol and P_1 it is given in the Theorem 0.1.

Theorem 0.4. *Let $\alpha \geq 0$, $a \in \mathbb{R}^*$, and $F(z) \in SH(\alpha)$. For $f(z) = L(F)(z)$, $f(z) = z + a_2 z^2 + a_3 z^3 + \ldots$, $z \in U$, where the integral operator L it is defined by (2), we have*

$$|a_2| \leq \left|\frac{a+1}{a+2}\right| \cdot \frac{1+4\alpha}{1+2\alpha} \ \textit{and} \ |a_3| \leq \left|\frac{a+1}{a+3}\right| \cdot \frac{7 + 52\alpha + 136\alpha^2 + 128\alpha^3}{4(1+2\alpha)^3} .$$

Proof. By using the Theorem 0.2 for $F(z) \in SH(\alpha)$, $F(z) = z + b_2 z^2 + b_3 z^3 + \ldots$, we have:

$$|b_2| \leq \frac{1+4\alpha}{1+2\alpha}, \ |b_3| \leq \frac{(1+4\alpha)(3+16\alpha+24\alpha^2)}{4(1+2\alpha)^3} .$$

In the proof of the Theorem 0.3 we obtain, for $f(z) = L(F)(z)$, $f(z) = z + a_2 z^2 + a_3 z^3 + \ldots$, the following relations between the coefficients:

$$a_2 = b_2 \cdot \frac{a+1}{a+2}, \ a_3 = (b_3 + 1) \cdot \frac{a+1}{a+3} .$$

By using the estimations for the coefficients b_2 and b_3, into the above relations, we complete the proof.

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