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CLASSES OF ANALYTIC FUNCTIONS AND APPLICATIONS

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Abstract. In this paper we introduce new classes of analytic functions and for the functions from these classes is studied the convexity of an integral operator.

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1. Introduction

Let $\mathcal{A}$ be the class of analytic functions f in the open unit disk $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$, $f(0) = f'(0) - 1 = 0$ and $\mathcal{S}$ be the subclass of univalent functions in the class $\mathcal{A}$.

We denote by $\mathcal{S}^*(\alpha)$ the class of starlike functions by the order α, $0 \leq \alpha < 1$. If $f \in \mathcal{S}^*(\alpha)$, then f verify the inequality

$$Re \frac{zf'(z)}{f(z)} > \alpha, \quad (z \in \mathcal{U}). \tag{1}$$

Also we denote with $\mathcal{K}(\alpha)$ the class of convex functions by the order α, $0 \leq \alpha < 1$. The function $f \in \mathcal{K}(\alpha)$ verify the inequality

$$Re \left(\frac{zf''(z)}{f'(z)} + 1 \right) > \alpha, \quad (z \in \mathcal{U}). \tag{2}$$

A function $f \in \mathcal{K}(\alpha)$ if and only if $zf' \in \mathcal{S}^*(\alpha)$.

Petru T. Mocanu [2] defines the class of α-convex functions, which is denoted M_α, α be a real number. If the function $f \in M_\alpha$, then $f(0) = f'(0) - 1 = 0$ and f verifies the inequlity

$$Re \left[(1-\alpha) \frac{zf'(z)}{f(z)} + \alpha \left(\frac{zf''(z)}{f'(z)} + 1 \right) \right] > 0, \tag{3}$$

for all $z \in \mathcal{U}$.

J. Stankiewicz and A. Wisniowska [4] had introduced the class of univalent functions $\mathcal{SH}(\beta)$, for some $\beta > 0$. If $f \in \mathcal{SH}(\beta)$, then f verifies the next inequality:

$$Re\left(\sqrt{2}\frac{zf'(z)}{f(z)}\right) + 2\beta(\sqrt{2}-1) > \left|\frac{zf'(z)}{f(z)} - 2\beta(\sqrt{2}-1)\right|, \tag{4}$$

for some $\beta > 0$, $f \in \mathcal{S}$ and for all $z \in \mathcal{U}$.

In the paper [3], F. Ronning introduced the class of univalent functions $\mathcal{SP}(\alpha, \beta)$, $\alpha > 0$, $\beta \in [0,1)$, the class of all functions $f \in \mathcal{S}$ which have the property

$$Re\frac{zf'(z)}{f(z)} + \alpha - \beta \geq \left|\frac{zf'(z)}{f(z)} - (\alpha + \beta)\right|, \quad (z \in \mathcal{U}). \tag{5}$$

B.A. Frasin and M. Darus [1] have defined the class $\mathcal{B}(\alpha)$, for the functions $f \in \mathcal{A}$, which verify the condition:

$$\left|\frac{z^2 f'(z)}{f^2(z)} - 1\right| < 1 - \alpha, \tag{6}$$

for all $\alpha, 0 \leq \alpha < 1$ and $z \in \mathcal{U}$.

2. New classes of analytic functions

Definition 1. *The function $f \in \mathcal{A}$ is the starlike function by the order $|\alpha|$, α be a complex number, if and only if $|\alpha| < 1$ and f verify the inequality*

$$Re\frac{zf'(z)}{f(z)} > |\alpha|, \quad (z \in \mathcal{U}). \tag{7}$$

We denote $\mathcal{PS}^*(|\alpha|)$ the class of starlike functions by the order $|\alpha|$, α be a complex number, $|\alpha| < 1$.

Remark 0.1. *For α be a real number, $0 \leq \alpha < 1$, we have $\mathcal{PS}^*(|\alpha|) = \mathcal{S}^*(\alpha)$.*

Definition 2. *The function $f \in \mathcal{A}$ is said to be the convex function by the order $|\alpha|$, α be a complex number, if and only if $|\alpha| < 1$ and f verify the inequality*

$$Re\left\{\frac{zf''(z)}{f'(z)} + 1\right\} > |\alpha|, \quad (z \in \mathcal{U}). \tag{8}$$

We denote by $\mathcal{PK}^*(|\alpha|)$ the class of convex functions by the order $|\alpha|$, α be a complex number, $|\alpha| < 1$.

Remark 0.2. *For α be a real number, $0 \le \alpha < 1$, we have $\mathcal{PK}^*(|\alpha|) = \mathcal{K}^*(\alpha)$.*

Definition 3. *The class of univalent functions $\mathcal{PH}(|\beta|)$, β be a complex number, $\beta \neq 0$, is the class defined by*

$$\mathcal{PH}(|\beta|) = \left\{ f \in \mathcal{S} : \left| \frac{zf'(z)}{f(z)} - 2|\beta|(\sqrt{2}-1) \right| < Re\left[\sqrt{2}\frac{zf'(z)}{f(z)} + 2|\beta|(\sqrt{2}-1) \right] \right\} \quad (9)$$

for all $z \in \mathcal{U}$.

Remark 0.3. *We have $\mathcal{SH}(\beta) \subset \mathcal{PH}(|\beta|)$.*

Definition 4. *The class of univalent functions $\mathcal{PV}(|\alpha|, |\beta|)$, α, β be complex numbers, $\alpha \neq 0$, $|\beta| \le 1$, is the class defined by*

$$\mathcal{PV}(|\alpha|, |\beta|) = \left\{ f \in \mathcal{S} : \left| \frac{zf'(z)}{f(z)} - (|\alpha| + |\beta|) \right| \le Re\frac{zf'(z)}{f(z)} + |\alpha| - |\beta| \right\} \quad (10)$$

for all $z \in \mathcal{U}$.

Remark 0.4. *We observe that $\mathcal{SP}(\alpha, \beta) \subset \mathcal{PV}(|\alpha|, |\beta|)$, α, β be real numbers, $\alpha > 0$, $\beta \in [0, 1)$.*

Definition 5. *The class of univalent functions $\mathcal{PB}(|\alpha|)$, α be a complex number, $\alpha \neq 0$, $|\alpha| < 1$ is the class defined by*

$$\mathcal{PB}(|\alpha|) = \left\{ f \in \mathcal{A} : \left| \frac{z^2 f'(z)}{f^2(z)} - 1 \right| < 1 - |\alpha| \right\}, \quad (11)$$

for all $z \in \mathcal{U}$.

Remark 0.5. *We have $\mathcal{B}(\alpha) \subset \mathcal{PB}(|\alpha|)$, α be a real number, $\alpha \in (0, 1)$.*

Definition 6. *The class $M_{|\alpha|}$ of $|\alpha|$-convex functions, α be a complex number, is the class defined by*

$$Re\left[(1 - |\alpha|)\frac{zf'(z)}{f(z)} + |\alpha| \left(\frac{zf''(z)}{f'(z)} + 1 \right) \right] > 0, \quad (12)$$

for all $z \in \mathcal{U}$, $f \in \mathcal{A}$.

3. Convexity of an integral operator

We consider the integral operator $I_{|\alpha|,|\beta|}$, defined by

$$I_{|\alpha|,|\beta|}(z) = \int_0^z \left(\frac{f(u)}{u}\right)^{|\alpha|} (f'(u))^{|\beta|}\, du, \tag{13}$$

for some α, β be complex numbers, the function f from new classes of analytic functions and in this case, we determine the order of convexity for this integral operator.

Theorem 0.6. *Let α, β be complex numbers, $|\alpha| + |\beta| < 1$. If the function $f \in \mathcal{PS}^*(|\alpha|)$ and $z \cdot f' \in \mathcal{PS}^*(|\beta|)$, then the integral operator $I_{|\alpha|,|\beta|}$ is convex by the order $1 - |\alpha| + |\alpha|^2 - |\beta| + |\beta|^2$.*

Proof. We have

$$\frac{zI''_{|\alpha|,|\beta|}(z)}{I'_{|\alpha|,|\beta|}(z)} + 1 = |\alpha|\left(\frac{zf'(z)}{f(z)} - 1\right) + |\beta|\left(\frac{zf''(z)}{f'(z)} + 1\right) - |\beta| + 1, \tag{14}$$

for all $z \in \mathcal{U}$ and hence we get

$$\begin{aligned} Re\left(\frac{zI''_{|\alpha|,|\beta|}(z)}{I'_{|\alpha|,|\beta|}(z)} + 1\right) &= |\alpha| Re\frac{zf'(z)}{f(z)} - |\alpha| + \\ &+ |\beta| Re\left(\frac{zf''(z)}{f'(z)} + 1\right) - |\beta| + 1. \end{aligned} \tag{15}$$

Since $f \in \mathcal{PS}^*(|\alpha|)$ and $z \cdot f' \in \mathcal{PS}^*(|\beta|)$ we have

$$Re\left(\frac{zI''_{|\alpha|,|\beta|}(z)}{I'_{|\alpha|,|\beta|}(z)} + 1\right) \geq |\alpha|^2 - |\alpha| + |\beta|^2 - |\beta| + 1 \tag{16}$$

and hence, by hypothesis $|\alpha| + |\beta| < 1$, it results that the integral operator $I_{|\alpha|,|\beta|}$ is convex by the order $1 - (1 - |\alpha|)|\alpha| - (1 - |\beta|)|\beta|$. □

Theorem 0.7. *Let α, β, γ, δ be complex numbers $\alpha \neq 0$, $\beta \neq 0$, $\delta \neq 0$, $|\gamma| < 1$, $f \in \mathcal{PK}(|\gamma|)$ and $f \in \mathcal{PH}(|\delta|)$.*

If

$$0 < \sqrt{2}|\alpha||\delta| + |\beta||\gamma| + 1 - 2|\alpha||\delta| - |\alpha| - |\beta| < 1 \tag{17}$$

then the integral operator $I_{|\alpha|,|\beta|}$ *is convex by the order*

$$\sqrt{2}|\alpha||\delta| + |\beta||\gamma| + 1 - 2|\alpha||\delta| - |\alpha| - |\beta|.$$

Proof. From (14) we have

$$\sqrt{2}\left(\frac{zI''_{|\alpha|,|\beta|}(z)}{I'_{|\alpha|,|\beta|}(z)} + 1\right) = \sqrt{2}|\alpha|\frac{zf'(z)}{f(z)} - \sqrt{2}|\alpha| + \sqrt{2}|\beta|\left(\frac{zf''(z)}{f'(z)} + 1\right) - \sqrt{2}|\beta| + \sqrt{2},$$

hence, we obtain

$$\begin{aligned}\sqrt{2}Re\left(\frac{zI''_{\alpha,\beta}(z)}{I'_{\alpha,\beta}(z)} + 1\right) &= \sqrt{2}|\alpha|Re\frac{zf'(z)}{f(z)} - \sqrt{2}|\alpha| + \\ &+ \sqrt{2}|\beta|Re\left(\frac{zf''(z)}{f'(z)} + 1\right) - \sqrt{2}|\beta| + \sqrt{2}. \quad (18)\end{aligned}$$

From (18) we get

$$\begin{aligned}\sqrt{2}Re\left(\frac{zI''_{|\alpha|,|\beta|}(z)}{I'_{|\alpha|,|\beta|}(z)} + 1\right) &= |\alpha|\left\{Re\left[\sqrt{2}\frac{zf'(z)}{f(z)}\right] + 2|\delta|(\sqrt{2}-1)\right\} - 2|\alpha||\delta|(\sqrt{2}-1) - \\ &- \sqrt{2}|\alpha| + |\beta|\sqrt{2}Re\left(\frac{zf''(z)}{f'(z)} + 1\right) - \sqrt{2}|\beta| + \sqrt{2}. \quad (19)\end{aligned}$$

Since, $f \in \mathcal{K}(|\gamma|)$ and $f \in \mathcal{SH}(|\delta|)$, by (19) we have

$$\begin{aligned}\sqrt{2}Re\left(\frac{zI''_{|\alpha|,|\beta|}(z)}{I'_{|\alpha|,|\beta|}(z)} + 1\right) &> |\alpha|\left|\frac{zf'(z)}{f(z)} - 2|\delta|(\sqrt{2}-1)\right| - 2|\alpha||\delta|(\sqrt{2}-1) - \\ &- \sqrt{2}|\alpha| + |\beta||\gamma|\sqrt{2} - \sqrt{2}|\beta| + \sqrt{2}.\end{aligned}$$

Because $|\alpha|\left|\frac{zf'(z)}{f(z)} - 2|\delta|(\sqrt{2}-1)\right| > 0$, obtain that

$$\sqrt{2}Re\left(\frac{zI''_{|\alpha|,|\beta|}(z)}{I'_{|\alpha|,|\beta|}(z)} + 1\right) > \sqrt{2}(|\beta||\gamma| - 2|\alpha||\delta| - |\alpha| - |\beta| + 1) + 2|\alpha||\delta|$$

and we get

$$Re\left(\frac{zI''_{|\alpha|,|\beta|}(z)}{I'_{|\alpha|,|\beta|}(z)}+1\right) > |\beta||\gamma| - 2|\alpha||\delta| - |\alpha| - |\beta| + 1 + \sqrt{2}|\alpha||\delta|. \qquad (20)$$

By (17) and (20), we obtain that the integral operator $I_{|\alpha|,|\beta|}$ is convex by the order

$$\sqrt{2}|\alpha||\delta| + |\beta||\gamma| + 1 - 2|\alpha||\delta| - |\alpha| - |\beta|.$$

□

Corollary 0.1. *Let α, β be complex numbers, $|\alpha| < 1$, $|\beta| \neq 0$, $f \in \mathcal{PK}(|\alpha|)$ and $f \in \mathcal{PH}(|\alpha|)$.*

If

$$|\beta| < \frac{1 + |\alpha|^2(\sqrt{2} - 2) - |\alpha|}{1 - |\alpha|}, \qquad (21)$$

then the integral operator $I_{|\alpha|,|\beta|}$ is convex, by the order

$$1 + |\beta|(|\alpha| - 1) + |\alpha|^2(\sqrt{2} - 2) - |\alpha|.$$

Proof. We take $\gamma = \delta = \alpha$, $|\alpha| < 1$, $\alpha \neq 0$ in Theorem 0.7. □

Corollary 0.2. *Let α, β be complex numbers, $\alpha \neq 0$, $\beta \neq 0$, $|\beta| < 1$ $f \in \mathcal{PK}(|\beta|)$ and $f \in \mathcal{PH}(|\beta|)$.*

If

$$|\alpha| < \frac{|\beta|^2 - |\beta| + 1}{|\beta|(2 - \sqrt{2}) + 1}, \qquad (22)$$

then the integral operator $I_{|\alpha|,|\beta|}$ is convex, by the order

$$1 + |\beta|^2 - |\beta| - |\alpha|(2|\beta| + 1 - \sqrt{2}|\beta|).$$

Proof. For $\gamma = \delta = \beta$, $|\beta| < 1$, $\beta \neq 0$, in Theorem 0.7, we obtain Corollary 0.2. □

Corollary 0.3. *Let α, β be complex numbers, $\alpha \neq 0$, $\beta \neq 0$, $|\beta| < 1$, $f \in \mathcal{PK}(|\beta|)$ and $f \in \mathcal{PH}(|\alpha|)$.*

If

$$1 + |\beta|^2 - |\beta| - (2 - \sqrt{2})|\alpha|^2 - |\alpha| > 0, \tag{23}$$

then the integral operator $I_{|\alpha|,|\beta|}$ *is convex, by the order*

$$1 + |\beta|^2 - |\beta| - (2 - \sqrt{2})|\alpha|^2 - |\alpha|.$$

Proof. We take $\gamma = \beta$, $\delta = \alpha$, $|\beta| < 1$, $\beta \neq 0$, $\alpha \neq 0$ in Theorem 0.7 we obtain the Corollary 0.3. □

Corollary 0.4. *Let* α, β *be complex numbers,* $\beta \neq 0$*,* $\alpha \in (\frac{1}{2\sqrt{2}-1}, 1)$*,* $f \in \mathcal{PK}(|\alpha|)$ *and* $f \in \mathcal{PH}(|\beta|)$*.*

If

$$|\beta| < \frac{|\alpha|}{(2\sqrt{2} - 1)|\alpha| - 1}, \tag{24}$$

then the integral operator $I_{|\alpha|,|\beta|}$ *is convex, by the order*

$$|\beta|[(2\sqrt{2} - 1)|\alpha| - 1] - |\alpha| + 1.$$

Proof. We take $\gamma = \alpha$, $\delta = \beta$, $|\alpha| < 1$, $\alpha \neq 0$, $\beta \neq 0$ in Theorem 0.7. □

Theorem 0.8. *Let* $\alpha, \beta, \gamma, \delta, \eta$ *be complex numbers,* $\alpha \neq 0$*,* $\beta \neq 0$*,* $\gamma \neq 0$*,* $|\delta| < 1$*,* $|\eta| < 1$*,* $\eta \neq 0$*,* $f \in \mathcal{K}(|\eta|)$ *and* $f \in \mathcal{PV}(|\gamma|, |\delta|)$*.*

If

$$0 < 1 - |\alpha|(|\gamma| - |\delta| + 1) + |\beta|(|\eta| - 1) < 1, \tag{25}$$

then the integral operator $I_{|\alpha|,|\beta|}$ *is convex, by the order*

$$1 - |\alpha|(|\gamma| - |\delta| + 1) + |\beta|(|\eta| - 1).$$

Proof. From (15) we obtain

$$\begin{aligned} Re\left(\frac{zI''_{|\alpha|,|\beta|}(z)}{I'_{|\alpha|,|\beta|}(z)} + 1\right) &= |\alpha|\left[Re\frac{zf'(z)}{f(z)} + |\gamma| - |\delta|\right] - |\alpha|(|\gamma| - |\delta|) - |\alpha| + \\ &+ |\beta| Re\left(\frac{zf''(z)}{f'(z)} + 1\right) - |\beta| + 1 \end{aligned} \tag{26}$$

Since $f \in \mathcal{PV}(|\gamma|, |\delta|)$ and $f \in \mathcal{K}(|\eta|)$, we have

$$Re\left(\frac{zI''_{|\alpha|,|\beta|}(z)}{I'_{|\alpha|,|\beta|}(z)}+1\right) \geq |\alpha|\left|\frac{zf'(z)}{f(z)}-(|\gamma|+|\delta|)\right| - |\alpha|(|\gamma|-|\delta|)-|\alpha| + \\ + |\beta||\eta|-|\beta|+1 \quad (27)$$

Because $|\alpha|\left|\frac{zf'(z)}{f(z)}-(|\gamma|+|\delta|)\right| > 0$, we obtain that

$$Re\left(\frac{zI''_{|\alpha|,|\beta|}(z)}{I'_{|\alpha|,|\beta|}(z)}+1\right) > 1-|\alpha|(|\gamma|-|\delta|+1)+|\beta|(|\eta|-1)$$

and from (25), it results that the integral operator $I_{|\alpha|,|\beta|}$ is convex by the order

$$1-|\alpha|(|\gamma|-|\delta|+1)+|\beta|(|\eta|-1).$$

□

Corollary 0.5. *Let α, β be complex numbers, $|\alpha| > 0$, $|\beta| < 1$, $\beta \neq 0$, $f \in \mathcal{K}(|\beta|)$ and $f \in \mathcal{PV}(|\alpha|, |\beta|)$.*

If

$$0 < 1-|\alpha|(|\alpha|-|\beta|+1)+|\beta|(|\beta|-1) < 1, \quad (28)$$

then the integral operator $I_{|\alpha|,|\beta|}$ is convex, by the order

$$1-|\alpha|(|\alpha|-|\beta|+1)+|\beta|(|\beta|-1).$$

Proof. From Theorem 0.8, for $\gamma = \alpha$, $\delta = \eta = \beta$, $\alpha \neq 0$, $\beta \neq 0$, $|\beta| < 1$, we have the Corollary 0.5. □

Theorem 0.9. *Let α, β be complex numbers, $\alpha \neq 0$, $|\beta| < 1$, $|\alpha| = 1-|\beta|$, $f \in M_{|\beta|}$, then the integral operator $I_{|\alpha|,|\beta|}$ is convex.*

Proof. From (15), for $|\alpha| = 1-|\beta|$, we have

$$Re\left(\frac{zI''_{|\alpha|,|\beta|}(z)}{I'_{|\alpha|,|\beta|}(z)}+1\right) = (1-|\beta|)Re\left(\frac{zf'(z)}{f(z)}\right)+|\beta|Re\left(\frac{zf''(z)}{f'(z)}+1\right),$$

and since, $f \in M_{|\beta|}$, we have

$$Re\left(\frac{zI''_{|\alpha|,|\beta|}(z)}{I'_{|\alpha|,|\beta|}(z)}+1\right) > 0, \quad (z \in \mathcal{U}),$$

hence, it results that the integral operator $I_{|\alpha|,|\beta|}$ is convex. □

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