

## A NEW CLASS $\Gamma^2(M, A, \Delta)$ OF THE DOUBLE DIFFERENCE SEQUENCES OF FUZZY NUMBERS

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ABSTRACT. The aim of this paper is to introduce and study a new concept of strong double  $\Gamma^2(M, A, \Delta)$  of fuzzy numbers and also some properties of the resulting sequence spaces of fuzzy numbers were examined.

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### 1. INTRODUCTION

The concept of fuzzy sets and fuzzy set operations were first introduced by Zadeh [30] and subsequently several authors have discussed various aspects of the theory and applications of fuzzy sets such as fuzzy topological spaces, similarity relations of fuzzy orderings, fuzzy measures of fuzzy events, fuzzy mathematical programming.

Let  $(x_{mn})$  be a double sequence of real or complex numbers. Then the series  $\sum_{m,n=1}^{\infty} x_{mn}$  is called a double series. The double series  $\sum_{m,n=1}^{\infty} x_{mn}$  is said to be convergent if and only if the double sequence  $(S_{mn})$  is convergent, where

$$S_{mn} = \sum_{i,j=1}^{m,n} x_{ij} (m, n = 1, 2, 3, \dots) \text{ (see[1])}.$$

We denote  $w^2$  as the class of all complex double sequences  $(x_{mn})$ . A sequence  $x = (x_{mn})$  is said to be double analytic if

$$\sup_{mn} |x_{mn}|^{1/m+n} < \infty.$$

The vector space of all prime sense double analytic sequences are usually denoted by  $\Lambda^2$ . A sequence  $x = (x_{mn})$  is called double entire sequence if

$$|x_{mn}|^{1/m+n} \rightarrow 0 \text{ as } m, n \rightarrow \infty.$$

The vector space of all prime sense double entire sequences are usually denoted by  $\Gamma^2$ . The space  $\Lambda^2$  and  $\Gamma^2$  is a metric space with the metric

$$d(x, y) = \sup_{mn} \left\{ |x_{mn} - y_{mn}|^{1/m+n} : m, n : 1, 2, 3, \dots \right\}, \quad (1)$$

for all  $x = \{x_{mn}\}$  and  $y = \{y_{mn}\}$  in  $\Gamma^2$ .

Let  $(x_{mn})$  be a double sequence of real or complex numbers. Then the series  $\sum_{m,n=1}^{\infty} x_{mn}$  is called a double series. The double series  $\sum_{m,n=1}^{\infty} x_{mn}$  is called convergent if and only if the double sequence  $(s_{mn})$  is convergent, where  $s_{mn} = \sum_{i,j=1}^{m,n} x_{ij}$  ( $m, n = 1, 2, 3, \dots$ ) (see[1]). A sequence  $x = (x_{mn})$  is said to be double analytic if

$\sup_{m,n} |x_{mn}|^{1/m+n} < \infty$ . The vector space of all double analytic sequences is usually denoted by  $\Lambda^2$ . A sequence  $x = (x_{mn})$  is called double entire sequence if  $|x_{mn}|^{1/m+n} \rightarrow 0$  as  $m, n \rightarrow \infty$ . The vector space of double entire sequences is usually denoted by  $\Gamma^2$ .

Consider a double sequence  $x = (x_{ij})$ . The  $(m, n)^{th}$  section  $x^{[m,n]}$  of the sequence is defined by  $x^{[m,n]} = \sum_{i,j=0}^{m,n} x_{ij} \delta_{ij}$  for all  $m, n \in N$ ,

$\delta_{ij} =$  with 1 in the  $(i, j)^{th}$  position and zero other wise. An FK-space (or a metric space)  $X$  is said to have AK property if  $(\delta_{mn})$  is a Schauder basis for  $X$ . Or equivalently  $x^{[m,n]} \rightarrow x$ . We need the following inequality in the sequel of the paper:

**Lemma 1.1** For  $a, b \geq 0$  and  $0 < p < 1$ , we have

$$(a + b)^p \leq a^p + b^p$$

Some initial works on double sequence spaces is found in Bromwich[4]. Later on it was investigated by Hardy[9], Moricz[17], Moricz and Rhoades[18], Basarir and Solankan[2], Tripathy[26], Colak and Turkmenoglu[6], Turkmenoglu[28], and many others.

The notion of difference sequence spaces (for single sequences) was introduced by Kizmaz [14] as follows

$$Z(\Delta) = \{x = (x_k) \in w : (\Delta x_k) \in Z\}$$

for  $Z = c, c_0$  and  $\ell_{\infty}$ , where  $\Delta x_k = x_k - x_{k+1}$  for all  $k \in \mathbb{N}$ . Here  $w, c, c_0$  and  $\ell_{\infty}$  denote the classes of all, convergent, null and bounded scalar valued single sequences respectively. The above spaces are Banach spaces normed by

$$\|x\| = |x_1| + \sup_{k \geq 1} |\Delta x_k|$$

Later on the notion was further investigated by many others. We now introduce the following difference double sequence spaces defined by

$$Z(\Delta) = \{x = (x_{mn}) \in w^2 : (\Delta x_{mn}) \in Z\}$$

where  $Z = \Lambda^2$  and  $\Gamma^2$ . respectively.  $\Delta x_{mn} = (x_{mn} - x_{mn+1}) - (x_{m+1n} - x_{m+1n+1}) = x_{mn} - x_{mn+1} - x_{m+1n} + x_{m+1n+1}$  for all  $m, n \in N$ . Further generalized this notion and introduced the following notion. For  $m, n \geq 1$ ,

$$Z(\Delta_\gamma^\mu) = \{x = (x_{mn} : (\Delta_\gamma^\mu x_{mn}) \in Z\} \text{ for } Z = \Lambda^2 \text{ and } \Gamma^2$$

An Orlicz function is a function  $M : [0, \infty) \rightarrow [0, \infty)$  which is continuous, non-decreasing and convex with  $M(0) = 0$ ,  $M(x) > 0$ , for  $x > 0$  and  $M(x) \rightarrow \infty$  as  $x \rightarrow \infty$ . If convexity of Orlicz function  $M$  is replaced by  $M(x + y) \leq M(x) + M(y)$ , then this function is called modulus function.

**Remark 1.2** *An Modulus function satisfies the inequality  $M(\lambda x) \leq \lambda M(x)$  for all  $\lambda$  with  $0 < \lambda < 1$ .*

In this article are introduce the space  $\Gamma^2(M, A, \Delta)$  of sequences of fuzzy numbers defined by Modulus function.

Throughout the article  $w^2, \Lambda^2$  and  $\Gamma^2$  represent the classes of all, double analytic and double entire sequences of fuzzy numbers respectively.

## 2. DEFINITIONS AND PRELIMINARIES

**Definition 2.1** *Let  $A$  denote a four dimensional summability method that maps the complex double sequences  $x$  into the double sequence.  $Ax$  where the  $mn$  – th term to  $Ax$  is as follows*

$$(Ax)_{k,\ell} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{k\ell}^{mn} x_{mn}.$$

In [10] Hardy presented the notion of regularity of two dimensional matrix transformations. The definition is as follows: a two dimensional matrix transformation is said to be regular if it maps every convergent sequence into a convergent sequence with the same limit. In addition, to the numerous theorems characterizing regularity. Hardy also presented the Silvermann-Toeplitz characterization of regularity following this work Robison in 1926 presented a four dimensional analog of regularity for double sequences in which he added an additional assumption of boundedness. This assumption was made because a double sequence which is  $P$ –convergent is not necessarily bounded along these same lines, Robison and Hamilton presented a Silverman-Toeplitz type multidimensional characterization of regularity in [11] and [25]. The definition of regularity for four dimensional matrices will be stated next, followed by the Robison-Hamilton characterization of the regularity of four dimensional matrices.

**Definition 2.2** *The four dimensional matrix  $A$  is said to be RH-regular if it maps every bounded  $p$ –convergent sequence into a  $P$ –convergent sequence with the same  $P$ –limit.*

**Theorem 2.3** *The four dimensional matrix  $A$  is RH-regular if and only if*

$RH_1 : P - \lim_{m,n} a_{kl}^{mn} = 0$  for each  $m$  and  $n$ ;

$RH_2 : P - \lim_{m,n} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{kl}^{mn} = 1$ ;

$RH_3 : P - \lim_{m,n} \sum_{m=1}^{\infty} |a_{kl}^{mn}| = 0$  for each  $n$ ;

$RH_4 : P - \lim_{m,n} \sum_{n=1}^{\infty} |a_{kl}^{mn}| = 0$  for each  $m$ ;

$RH_5 : P - \lim_{m,n} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |a_{kl}^{mn}|$  is  $P$ -convergent; and

$RH_6 : there exist positive numbers  $A$  and  $B$  such that  $\sum_{m,n > B} |a_{kl}^{mn}| < A$ .$

A fuzzy real number  $X$  is a fuzzy set on  $R$ , i.e., a mapping  $X : R \rightarrow I (= [0, 1])$ , associating each real number  $x$  with its grade of membership  $X(x)$ .

A fuzzy real number  $X$  is said to be upper semi-continuous if for each  $\epsilon > 0$ ,  $X^{-1}([0, a + \epsilon])$ , for all  $a \in I$  is open in the usual topology of  $R$ . If there exists  $x \in R$  such that  $X(x) = 1$ , then  $X$  is called normal.

A fuzzy number  $X$  is said to be convex if  $X(x) \geq X(s) \wedge X(r) \min(X(s), X(r))$  where  $s < t < r$ . The class of all upper semi-continuous, normal, convex fuzzy normal is denoted by  $R(I)$ .

The additive identity and multiplicative identity in  $R(I)$  are denoted by  $\bar{0}$  and  $\bar{1}$  respectively.

Let  $C(R^n) = \{A \subset R^n : A \text{ compact and convex}\}$ . The space  $C(R^n)$  has linear structure induced by the operations  $A + B = \{a + b : a \in A, b \in B\}$  and  $\lambda A = \{\lambda a : a \in A\}$  for  $A, B \in C(R^n)$  and  $\lambda \in R$ . The Hausdorff distance between  $A$  and  $B$  of  $C(R^n)$  is defined as

$$\delta_{\infty}(A, B) = \max \{ \sup_{a \in A} \inf_{b \in B} \|a - b\|, \sup_{b \in B} \inf_{a \in A} \|a - b\| \}$$

It is well known that  $(C(R^n), \delta_{\infty})$  is a complete metric space.

The fuzzy number is a function  $X$  from  $R^n$  to  $[0, 1]$  which is normal, fuzzy convex, upper semi-continuous and the closure of  $\{x \in R^n : X(x) > 0\}$  is compact. These properties imply that for each  $0 < \alpha \leq 1$ , the  $\alpha$ -level set  $[X]^{\alpha} = \{x \in R^n : X(x) \geq \alpha\}$  is a nonempty compact convex subset of  $R^n$ , with support  $X^c = \{x \in R^n : X(x) > 0\}$ . Let  $L(R^n)$  denote the set of all fuzzy numbers. The linear structure of  $L(R^n)$  induces the addition  $X + Y$  and scalar multiplication  $\lambda X$ ,  $\lambda \in R$ , in terms of  $\alpha$ -level sets, by  $|X + Y|^{\alpha} = |X|^{\alpha} + |Y|^{\alpha}$ ,  $|\lambda X|^{\alpha} = \lambda |X|^{\alpha}$  for each  $0 \leq \alpha \leq 1$ . Define, for each  $1 \leq q < \infty$ ,

$$d_q(X, Y) = \left( \int_0^1 \delta_{\infty}(X^{\alpha}, Y^{\alpha})^q d\alpha \right)^{1/q}, \text{ and } d_{\infty} = \sup_{0 \leq \alpha \leq 1} \delta_{\infty}(X^{\alpha}, Y^{\alpha}),$$

where  $\delta_{\infty}$  is the Hausdorff metric. Clearly  $d_{\infty}(X, Y) = \lim_{q \rightarrow \infty} d_q(X, Y)$  with  $d_q \leq d_r$ , if  $q \leq r$ . Throughout the paper,  $d$  will denote  $d_q$  with  $1 \leq q \leq \infty$ .

A fuzzy double sequence is a double infinite array of fuzzy real numbers. We denote a fuzzy real-valued double sequence by  $(X_{mn})$ , where  $X_{mn}$  are fuzzy real numbers for each  $m, n \in N$ . Let  $s''$  denote the set of all double sequences of fuzzy numbers.

We give the following definitions for fuzzy double sequences.

**Definition 2.4** A double sequence  $X = (X_{mn})$  of fuzzy numbers is said to be entire in the Pringsheim's sense or  $P$ -convergent to a fuzzy number  $\bar{0}$ , such that

$$d\left(|X_{mn}|^{1/m+n}, \bar{0}\right) = 0 \text{ as } m, n \rightarrow \infty,$$

and we denote by  $P - \lim X = \bar{0}$ . The number  $\bar{0}$  is called the Pringsheim limit to  $X_{mn}$ .

Let  $\Gamma^2(F)$  denote the set of all double entire sequences of fuzzy numbers.

**Definition 2.5** A double sequence  $X = (X_{mn})$  of fuzzy numbers is analytic if there exists a positive number  $M$  such that  $d\left(|X_{mn}|^{1/m+n}, X_0\right) < M$  for all  $m$  and  $n$ ,

$$d(X, Y) = \sup_{m,n} \left\{ |X_{mn} - Y_{mn}|^{1/m+n} : m, n = 1, 2, 3, \dots \right\} = \sup_{m,n} d\left(|X_{mn}|^{1/m+n}, X_0\right) < \infty.$$

We will denote the set of all analytic double sequences by  $\Lambda^2(F)$ .

In this paper we introduce and study the concept of strong double  $\Gamma^2(M, A, \Delta)$  - summable and also some properties of this sequence space is examined.

Before we can state our main results, first we shall present the following definition by combining a four dimensional matrix transformation  $A$  and modulus function.

**Definition 2.6** Let  $M$  be an modulus function and  $A = (a_{k\ell}^{mn})$  be a non-negative RH-regular summability matrix method. We now present the following double sequence spaces:

$$\Gamma^2(M, A, \Delta, p)(F) = \left\{ X \in s'' : P - \lim_{k,\ell} \sum_{m,n=0,\infty}^{\infty,\infty} a_{k\ell}^{mn} \left[ M \left( \frac{d(|\Delta_{11} X_{mn}|^{1/m+n}, \bar{0})}{\rho} \right) \right]^{p_{m,n}} = 0, \text{ for some } \rho > 0 \right\},$$

$$\Lambda^2(M, A, \Delta, p)(F) = \left\{ X \in s'' : \sup_{k,\ell,m,n} \sum_{m,n=0,\infty}^{\infty,\infty} a_{k\ell}^{mn} \left[ M \left( \frac{d(|\Delta_{11} X_{mn}|^{1/m+n}, X_0)}{\rho} \right) \right]^{p_{m,n}} < \infty, \text{ for some } \rho > 0 \right\}$$

where  $\Delta_{11} = \sup_{r,s=1 \text{ and } \setminus \text{ or } 0} \{d(X_{m,n}, X_{m-r,n-s})\}$   $\bar{0}(x) = 1$ , if  $x = (0, 0, \dots = 0)$ , = 0, otherwise.

Let us consider the few special cases of the above definition:

(1)  $M(X) = X$  then we have

$$\Gamma^2(A, \Delta, p)(F) = \left\{ X \in s'' : P - \lim_{k,\ell} \sum_{m,n=0,0}^{\infty,\infty} a_{k\ell}^{mn} \left( d(|\Delta_{11} X_{mn}|)^{1/m+n}, \bar{0} \right)^{p_{m,n}} = 0 \right\},$$

$$\Lambda^2(M, A, \Delta, p)(F) = \left\{ X \in s'' : \sup_{k,\ell,m,n} \sum_{m,n=0,0}^{\infty,\infty} a_{k\ell}^{mn} \left( d(|\Delta_{11} X_{mn}|)^{1/m+n}, X_0 \right)^{p_{m,n}} < \infty \right\}.$$

(2) If  $p_{mn} = 1$  for all  $(m, n)$  we have

$$\Gamma^2(M, A, \Delta)(F) = \left\{ X \in s'' : P - \lim_{k,\ell} \sum_{m,n=0,0}^{\infty,\infty} a_{k\ell}^{mn} \left[ M \left( \frac{d(|\Delta_{11} X_{mn}|)^{1/m+n}, \bar{0}}{\rho} \right) \right] = 0, \text{ for some } \rho > 0 \right\},$$

$$\Lambda^2(M, A, \Delta)(F) = \left\{ X \in s'' : \sup_{k,\ell,m,n} \sum_{m,n=0,0}^{\infty,\infty} a_{k\ell}^{mn} \left[ M \left( \frac{d(|\Delta_{11} X_{mn}|)^{1/m+n}, X_0}{\rho} \right) \right] < \infty, \text{ for some } \rho > 0 \right\}.$$

(3) If we take  $M(X) = X$  and  $p_{mn} = 1$  for all  $(m, n)$  then we have

$$\Gamma^2(A, \Delta)(F) = \left\{ X \in s'' : P - \lim_{k,\ell} \sum_{m,n=0,0}^{\infty,\infty} a_{k\ell}^{mn} \left( d(|\Delta_{11} X_{mn}|)^{1/m+n}, \bar{0} \right) = 0 \right\},$$

$$\Lambda^2(A, \Delta)(F) = \left\{ X \in s'' : \sup_{k,\ell,m,n} \sum_{m,n=0,0}^{\infty,\infty} a_{k\ell}^{mn} \left( d(|\Delta_{11} X_{mn}|)^{1/m+n}, X_0 \right) < \infty \right\}.$$

(4) If we take  $A = (C, 1, 1)$  we have

$$\Gamma^2(M, \Delta, p)(F) = \left\{ X \in s'' : P - \lim_{k,\ell} \frac{1}{k\ell} \sum_{m,n=0,0}^{k-1,\ell-1} \left[ M \left( \frac{d(|\Delta_{11} X_{mn}|)^{1/m+n}, \bar{0}}{\rho} \right) \right]^{p_{m,n}} = 0, \text{ for some } \rho > 0 \right\},$$

$$\Lambda^2(M, \Delta, p)(F) = \left\{ X \in s'' : \sup_{k,\ell,m,n} \frac{1}{k\ell} \sum_{m,n=0,0}^{k-1,\ell-1} \left[ M \left( \frac{d(|\Delta_{11} X_{mn}|)^{1/m+n}, X_0}{\rho} \right) \right]^{p_{m,n}} < \infty, \text{ for some } \rho > 0 \right\}.$$

(5) If we take  $A = (C, 1, 1)$  and  $p_{mn} = 1$  for all  $(m, n)$  then we have

$$\Gamma^2(M, \Delta)(F) = \left\{ X \in s'' : P - \lim_{k,\ell} \frac{1}{k\ell} \sum_{m,n=0,0}^{k-1,\ell-1} \left[ M \left( \frac{d(|\Delta_{11} X_{mn}|)^{1/m+n}, \bar{0}}{\rho} \right) \right] = 0, \text{ for some } \rho > 0 \right\},$$

$$\Lambda^2(M, \Delta)(F) = \left\{ X \in s'' : \sup_{k,\ell,m,n} \frac{1}{k\ell} \sum_{m,n=0,0}^{k-1,\ell-1} \left[ M \left( \frac{d(|\Delta_{11}X_{mn}|)^{1/m+n}, X_0}{\rho} \right) \right] < \infty, \text{ for some } \rho > 0 \right\}.$$

(6) If we take  $A = (C, 1, 1)$   $M(X) = X$  and  $p_{mn} = 1$  for all  $(m, n)$  then we have

$$\Gamma^2(\Delta)(F) = \left\{ X \in s'' : P - \lim_{k,\ell} \frac{1}{k\ell} \sum_{m,n=0,0}^{k-1,\ell-1} \left( d(|\Delta_{11}X_{mn}|)^{1/m+n}, \bar{0} \right) = 0 \right\},$$

$$\Lambda^2(\Delta)(F) = \left\{ X \in s'' : \sup_{k,\ell,m,n} \frac{1}{k\ell} \sum_{m,n=0,0}^{k-1,\ell-1} \left( d(|\Delta_{11}X_{mn}|)^{1/m+n}, X_0 \right) < \infty \right\}.$$

(7) Let us consider the following notations and definition. The double sequence  $\theta_{r,s} = \{(m_r, n_s)\}$  is called double lacunary if there exists two increasing of integers sequences such that

$$m_0 = 0, h_r = m_r - m_{r-1} \rightarrow \infty \text{ as } r \rightarrow \infty,$$

$$n_0 = 0, h_s = n_s - n_{s-1} \rightarrow \infty \text{ as } s \rightarrow \infty,$$

and let  $h_{r,s} = h_r h_s$ ,  $\theta_{r,s}$  is determine by  $I_{r,s} = \{(i, j) : k_{r-1} < i \leq k_r \ \& \ n_{s-1} < j \leq n_s\}$ . If we take

$$a_{r,s}^{m,n} = \begin{cases} \frac{1}{h_{r,s}}, & \text{if } (m, n) \in I_{r,s}; \\ 0, & \text{otherwise} \end{cases}$$

We are granted

$$\Gamma^2(\theta, M, \Delta, p)(F) = \left\{ X \in s'' : P - \lim_{r,s} \frac{1}{h_{r,s}} \sum_{(m,n) \in I_{r,s}} \left[ M \left( \frac{d(|\Delta_{11}X_{mn}|)^{1/m+n}, \bar{0}}{\rho} \right) \right]^{p_{m,n}} = 0, \text{ for some } \rho > 0 \right\},$$

$$\Lambda^2(\theta, M, \Delta, p)(F) = \left\{ X \in s'' : \sup_{r,s,k,\ell} \frac{1}{h_{r,s}} \sum_{(m,n) \in I_{r,s}} \left[ M \left( \frac{d(|\Delta_{11}X_{mn}|)^{1/m+n}, X_0}{\rho} \right) \right]^{p_{m,n}} < \infty, \text{ for some } \rho > 0 \right\}.$$

(8) As a final illustration let

$$a_{i,j}^{m,n} = \begin{cases} \frac{1}{\lambda_{i,j}}, & \text{if } m \in I_i = [i - \lambda_i + 1, i] \text{ and } n \in I_j = [j - \lambda_j + 1, j]; \\ 0, & \text{otherwise} \end{cases}$$

where  $\bar{\lambda}_{i,j}$  by  $\lambda_i \mu_j$ . Let  $\lambda = (\lambda_i)$  and  $\mu = (\mu_j)$  be two non-decreasing sequences of positive real numbers such that each tending to  $\infty$  and  $\lambda_{i+1} \leq \lambda_i + 1, \lambda_1 = 0$  and  $\mu_{j+1} \leq \mu_j + 1, \mu_1 = 0$ . The our definitions reduce to the following

$$\Gamma^2(\lambda, M, \Delta, p)(F) = \left\{ X \in s'' : P - \lim_{i,j} \frac{1}{\lambda_{i,j}} \sum_{(m,n) \in I_{i,j}} \left[ M \left( \frac{d(|\Delta_{11} X_{mn}|)^{1/m+n}, \bar{0}}{\rho} \right) \right]^{p_{m,n}} = 0, \text{ for some } \rho > 0 \right\},$$

$$\Lambda^2(\lambda, M, \Delta, p)(F) = \left\{ X \in s'' : \sup_{i,j,k,\ell} \frac{1}{\lambda_{i,j}} \sum_{(m,n) \in I_{i,j}} \left[ M \left( \frac{d(|\Delta_{11} X_{mn}|)^{1/m+n}, X_0}{\rho} \right) \right]^{p_{m,n}} < \infty, \text{ for some } \rho > 0 \right\}.$$

The following inequalities will be used throughout the paper. Let  $p = (p_{m,n})$  be a double sequence of positive real numbers with  $0 < p_{m,n} \leq \sup_{m,n} p_{m,n} = H$  and let  $c = \max \{1, 2^{H-1}\}$ .

### 3. MAIN RESULTS

**Theorem 3.1** *If  $M$  be an modulus function then  $\Gamma^2(M, A, \Delta, p)(F) \subset \Lambda^2(M, A, \Delta, p)(F)$ .*

**Proof:** Let us choose  $X$  in  $\Gamma^2(M, A, \Delta, p)(F)$  then there exists some positive number  $\rho_1$  such that

$$P - \lim_{k,\ell} \sum_{m,n=0,0}^{\infty,\infty} a_{k\ell}^{mn} \left[ M \left( \frac{d(|\Delta_{11} X_{mn}|)^{1/m+n}, \bar{0}}{\rho} \right) \right]^{p_{m,n}} = 0.$$

Let define  $\rho = 2\rho_1$ . Since  $M$  is non-decreasing and convex, we obtain the following :

$$\begin{aligned} & \sum_{m,n=0,0}^{\infty,\infty} a_{k\ell}^{mn} \left[ M \left( \frac{d(|\Delta_{11} X_{mn}|)^{1/m+n}, \bar{0}}{\rho} \right) \right]^{p_{m,n}} \leq \\ & \sum_{m,n=0,0}^{\infty,\infty} \frac{1}{2^{p_{m,n}}} a_{k\ell}^{mn} \left[ M \left( \frac{d(|\Delta_{11} X_{mn}|)^{1/m+n}, X_0}{\rho} \right) + M \left( \frac{d(X_0, \bar{0})}{\rho} \right) \right]^{p_{m,n}} \\ & \leq C \sum_{m,n=0,0}^{\infty,\infty} a_{k\ell}^{mn} \left[ M \left( \frac{d(|\Delta_{11} X_{mn}|)^{1/m+n}, X_0}{\rho} \right) \right]^{p_{m,n}} + C \sum_{m,n=0,0}^{\infty,\infty} a_{k\ell}^{mn} M \left( \frac{d(X_0, \bar{0})}{\rho} \right)^{p_{m,n}} \\ & \leq C \sum_{m,n=0,0}^{\infty,\infty} a_{k\ell}^{mn} \left[ M \left( \frac{d(|\Delta_{11} X_{mn}|)^{1/m+n}, X_0}{\rho} \right) \right]^{p_{m,n}} + \\ & \quad C \max \left\{ 1, \left[ M \left( \frac{d(X_0, \bar{0})}{\rho} \right) \right]^H \sum_{m,n=0,0}^{\infty,\infty} a_{k\ell}^{mn} \right\}. \end{aligned}$$

Since  $A$  is RH-regular we are granted that  $X \in \Lambda^2(M, A, \Delta, p)(F)$ . and thus completes the proof.

**Theorem 3.2** *If  $0 < \inf p_{m,n} \leq p_{m,n} < 1$  then  $\Gamma^2(M, A, \Delta, p)(F) \subset \Gamma^2(M, A, \Delta)(F)$   
If  $1 \leq p_{m,n} \leq \sup p_{m,n} < \infty$  then  $\Gamma^2(M, A, \Delta)(F) \subset \Gamma^2(M, A, \Delta, p)(F)$*

$$\sum_{m,n=0,0}^{\infty,\infty} a_{k\ell}^{mn} \left[ M \left( \frac{d(|\Delta_{11} X_{mn}|)^{1/m+n}, \bar{0}}{\rho} \right) \right] \leq \sum_{m,n=0,0}^{\infty,\infty} a_{k\ell}^{mn} \left[ M \left( \frac{d(|\Delta_{11} X_{mn}|)^{1/m+n}, \bar{0}}{\rho} \right) \right]^{p_{m,n}}$$

and hence  $X \in \Gamma^2(M, A, \Delta, p)(F)$ . Let  $p_{m,n} \geq 1$  for each  $(m, n)$  and  $\sup_{m,n} p_{m,n} < \infty$ . Let  $X \in \Gamma^2(M, A, \Delta)(F)$ . Then for each  $0 < \epsilon < 1$  there exists a positive integer  $N$  such that

$$\sum_{m,n=0,0}^{\infty,\infty} a_{k\ell}^{mn} \left[ M \left( \frac{d(|\Delta_{11} X_{mn}|)^{1/m+n}, \bar{0}}{\rho} \right) \right] \leq \epsilon < 1$$

for all  $m, n \geq N$ . This implies that

$$\sum_{m,n=0,\infty}^{\infty,\infty} a_{kl}^{mn} \left[ M \left( \frac{d(|\Delta_{11} X_{mn}|)^{1/m+n}, \bar{0}}{\rho} \right) \right]^{p_{m,n}} \leq \sum_{m,n=0,\infty}^{\infty,\infty} a_{kl}^{mn} \left[ M \left( \frac{d(|\Delta_{11} X_{mn}|)^{1/m+n}, \bar{0}}{\rho} \right) \right].$$

Thus  $X \in \Gamma^2(M, A, \Delta, p)(F)$ . This completes the proof.

The following corollary follows immediately from the above theorem

**Corollary 3.4** *Let  $A = (C, 1, 1)$  double Cesàro matrix and let  $M$  be an modulus function.*

(1) *If  $0 < \inf p_{m,n} \leq p_{m,n} < 1$  then  $\Gamma^2(M, \Delta, p)(F) \subset \Gamma^2(M, \Delta)(F)$ .*

(2) *If  $1 \leq p_{m,n} \leq \sup p_{m,n} < \infty$  then  $\Gamma^2(M, \Delta)(F) \subset \Gamma^2(M, \Delta, p)(F)$ .*

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