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SOME SIMPLE SUFFICIENT CONDITIONS FOR A CLASS OF ANALYTIC FUNCTIONS CONCERNING SUBORDINATION

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Abstract. Making use of subordination authors obtain some interesting conditions for the expression $\frac{D^{n+1}f(z)-(1-\gamma)D^n f(z)}{z}$ belongs to the class $S(n, 1-\gamma)$. Relevant connections of the results presented here with various known results are briefly indicated.

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1. Introduction

Let A denote the class of functions f of the form

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j \tag{1}$$

which are analytic in the open unit disc $U = \{z : |z| < 1\}$ and normalized by the condition $f(0) = f'(0) - 1 = 0$.

Now, for $0 \le \alpha < 1$, a function $f \in A$ is said to be in the class $S(n, \alpha)$ if

$$Re\left\{\frac{D^{n+1}f(z)}{D^n f(z)}\right\} > \alpha, \qquad z \in U, \tag{2}$$

and in the class $\widetilde{S}(n, \alpha)$, if and only if

$$\left|arg\left[\frac{D^{n+1}f(z)}{D^n f(z)}\right]\right| < \frac{\alpha\pi}{2}, \qquad z \in U, \tag{3}$$

where D^n stands for the Salagean derivative introduced by Salagean in [8].

The class $S(n, \alpha)$ was introduced and studied by Kadioğlu [2].

Here $S(0,\alpha) = S^*(\alpha)$, $S(1,\alpha) = K(\alpha)$, $\widetilde{S}(0,\alpha) = \widetilde{S}(\alpha)$ and $\widetilde{S}(1,\alpha) = \widetilde{K}(\alpha)$ are the classes of starlike, convex, strongly starlike and strongly convex functions of order α in U, respectively and $S(0,0) = S^*(0) = \widetilde{S}(0,1) = S^*$, $K(1,0) = K(0) = \widetilde{K}(1) = K$ are the classes of starlike and convex functions in the unit disc U, respectively. For detailed study see [1].

The function $f(z)$ is subordinate to the function $g(z)$, written as $f(z) \prec g(z)$, if there exist an analytic function $\omega(z)$ defined on U with $\omega(0) = 0$ and $|\omega(z)| < 1$ such that $f(z) = g(\omega(z))$. In particular, if $g(z)$ is univalent in U, $f(z) \prec g(z)$ if and only if $f(0) = g(0)$ and $f(U) \subseteq g(U)$.

In the present paper the expression

$$\frac{D^{n+1}f(z) - (1-\gamma)D^n f(z)}{z} \tag{4}$$

is studied and sufficient conditions that will place $f(z)$ in the classes defined above are given. The special cases for $n = 0, n = 1$, $n = 0$ with $\gamma = 0, 1$ and $n = 1$ with $\gamma = 0, 1$ were earlier studied by Tuneski [12], Mocanu [4], [5], Singh and Tuneski [10], Tuneski [11].

The following lemma is due to a special case of Theorem 2 of [3].

Lemma 1. *Let the functions $F(z)$ and $G(z)$ be analytic functions in the unit disc, $\gamma \geq 0$ and $G'(0) = 0$. For $\gamma = 0$, furthermore $F(0) = G(0) = 0$. If*

$$Re\left\{1 + \frac{zG''(z)}{G'(z)}\right\} > K(\gamma) = \begin{cases} -\frac{\gamma}{2}, & \gamma \leq 1 \\ -\frac{1}{2\gamma}, & \gamma \geq 1 \end{cases} \tag{5}$$

for all $z \in U$ and $F(z) \prec G(z)$ then

$$\frac{1}{z^\gamma}\int_0^z t^{\gamma-1}F(t)dt \prec \frac{1}{z^\gamma}\int_0^z t^{\gamma-1}G(t)dt. \tag{6}$$

For $F(z) = 1 - \gamma p(z) - zp'(z)$ we obtain the following lemma. The detailed proof can be found in [10].

Lemma 2. *Let $p(z)$ and $G(z)$ be analytic functions in the unit disc, $\gamma \geq 0$ and $G'(0) \neq 0$. If*

$$Re\left\{1 + \frac{zG''(z)}{G'(z)}\right\} > K(\gamma) \tag{7}$$

for all $z \in U$ and

$$1 - \gamma p(z) - zp'(z) \prec G(z) \tag{8}$$

then

$$p(z) - \frac{C}{z^\gamma} \prec \frac{1}{z^\gamma}\int_0^z t^{\gamma-1}G(t)dt, \tag{9}$$

where $C = p(0)$ *for* $\gamma = 0$ *and* $C = 0$ *for* $\gamma > 0$.

2. Main Results

In this section we give sufficient condition for expression (4).

Theorem 2.1. *If* $f \in A$, $\gamma \geq 0$ *and* $\lambda > 0$. *If*

$$\left| \frac{D^{n+1}f(z) - (1-\gamma)D^n f(z)}{z} - \gamma \right| < \lambda, \tag{10}$$

for all $z \in U$, *then*

$$\left| \frac{D^n f(z)}{z} - 1 \right| < \frac{\lambda}{1+\gamma}, \tag{11}$$

and

$$|D^n f(z)| < 1 + \frac{\lambda}{1+\gamma}, \tag{12}$$

for all $z \in U$. *The result is sharp.*

Proof. Let us define functions $p(z) = \frac{D^n f(z)}{z}$ and $G(z) = 1-\gamma+\lambda z$. The $p(z)$ and $G(z)$ are analytic functions in the unit disc, $p(0) = 1$ and $G'(0) = \lambda > 0$. Further,

$$Re\left\{1 + \frac{zG''(z)}{G'(z)}\right\} = 1 > K(\gamma) \tag{13}$$

for all $z \in U$ and

$$1 - \gamma p(z) - zp'(z) = 1 - \frac{D^{n+1}f(z)}{z} + (1-\gamma)\frac{D^n f(z)}{z}.$$

Thus, inequality (10) is equivalent to the subordination (8) and Lemma 2 implies

$$\frac{D^n f(z)}{z} - \frac{C}{z^\gamma} \prec \frac{1}{z^\gamma}\int_0^z t^{\gamma-1}G(t)dt = 1 + \frac{\lambda}{1+\gamma}z, \tag{14}$$

where $C = p(0) = 1$ for $\gamma = 0$ and $C = 0$ for $\gamma > 0$, i.e., $\frac{C}{z^\gamma} = 1$ for $\gamma = 0$ and $Cz^{-\gamma} = 0$ for $\gamma > 0$. So, we have obtained that if the conditions of the theorem hold then

$$\frac{D^n f(z)}{z} \prec 1 + \frac{\lambda}{1+\gamma}z.$$

which is equivalent to (11).

Finally, for all $z \in U$,

$$|D^n f(z)| < \left|\frac{D^n f(z)}{z}\right| < 1 + \frac{\lambda}{1+\gamma}. \tag{15}$$

The sharpness of the result is due to the function

$$f(z) = z + \frac{\lambda}{1+\gamma}\frac{z^2}{2^n}. \tag{16}$$

Remark 1. In Theorem 2.1, we consider only $\lambda > 0$ because functions $1-\gamma-|\lambda|z$ and $1-\gamma+|\lambda|z$ map the unit disc U into the same region.

Next, using Theorem 2.1 we will prove a condition for a function belonging to the class $S(n, 1-\gamma)$.

Theorem 2.2. *If $f \in A$, $\gamma \in (0,1]$ and $\lambda \in (0, \lambda_1]$, $\lambda_1 = \frac{\gamma(1+\gamma)}{\sqrt{(1+\gamma)^2+\gamma^2}}$. If*

$$\left|\frac{D^{n+1}f(z) - (1-\gamma)D^n f(z)}{z} - \gamma\right| < \lambda, \tag{17}$$

for all $z \in U$, then $f \in S(n, 1-\gamma)$.

Proof. Let the function $f(z)$ satisfy the condition of the theorem. Then, there exists a function $\omega(z)$ that is analytic in the unit disc with the following properties:

$$\omega(0) = 0, \quad |\omega(z)| < 1, \ forall\ z \in U$$

and

$$\frac{D^{n+1}f(z)}{D^n f(z)} - (1-\gamma) = \frac{z}{D^n f(z)}[\gamma + \lambda\omega(z)].$$

Also, by Theorem 2.1

$$\left|\frac{D^n f(z)}{z} - 1\right| < \frac{\lambda}{1+\gamma} \left|\arg\left[\frac{D^{n+1}f(z)}{D^n f(z)} - (1-\gamma)\right]\right| \le \left|\arg\frac{z}{D^n f(z)}\right| + |\arg[\gamma + \lambda\omega(z)]|$$

$$\le \arcsin\frac{\lambda}{1+\gamma} + \arcsin\frac{\lambda}{\gamma} \le \arcsin\frac{\lambda_1}{1+\gamma} + \arcsin\frac{\lambda_1}{\gamma} = \arcsin\left\{\frac{\lambda_1}{\gamma}\sqrt{1-\frac{\lambda_1^2}{(1+\gamma)^2}} + \frac{\lambda_1}{1+\gamma}\sqrt{1-\frac{\lambda_1^2}{\gamma^2}}\right\}$$

$$= \frac{\pi}{2}$$

i.e., $Re\left\{\frac{D^{n+1}f(z)}{D^n f(z)}\right\} > 1-\gamma$ for all $z \in U$ and $f \in S(n, 1-\gamma)$.

If we put $\gamma = 1$ in Theorem 2.1 and 2.2, we obtain following corollary.

Corollary 2.1. If $f \in A$, $\lambda > 0.$ and

$$\left|\frac{D^{n+1}f(z)}{z}\right| < \lambda, \tag{18}$$

for all $z \in U$, then

$$(i) \quad \left|\frac{D^n f(z)}{z} - 1\right| < \frac{\lambda}{2},\ forall z \in U, \tag{19}$$

$$(ii)\ If f \in \widetilde{S}(n,\gamma_1), for \lambda \leq \frac{2}{\sqrt{5}}, where \tag{20}$$

$\gamma_1 = \frac{2}{\pi}\arcsin\left(\lambda\sqrt{1-\frac{\lambda^2}{4}} + \frac{\lambda}{2}\sqrt{1-\lambda^2}\right)$

Remark 2. If we put $n = 0$ in Corollary 2.1, we obtain corresponding results of Mocanu [5] and Tuneski [11].

Now, again using Theorem 2.1 we obtain another interesting condition on $S(n, 1-\gamma)$.

Theorem 2.3. Let $f \in A$, $\gamma \in (0, \frac{1}{2})$ and $\lambda \in (0, \lambda_2]$, $\lambda_2 = \frac{(1+\gamma)(1-2\gamma)}{2}$, if

$$\left|\frac{D^{n+1}f(z) - (1-\gamma)D^n f(z)}{z} - \gamma\right| < \lambda$$

for all $z \in U$, then

$$\left|\frac{D^{n+1}f(z)}{D^n f(z)} - (1-\gamma)\right| < \frac{(\gamma+\lambda)(1+\gamma)}{1+\gamma-\lambda} = r, \tag{21}$$

for all $z \in U$ and further $f \in S(n, 1-\gamma)$ and $f \in \widetilde{S}(n,\alpha_2)$, $\alpha_2 = \frac{2}{\pi}\arcsin\frac{\gamma}{1-\gamma}$.

Proof. Simple calculus shows that for γ, λ and λ_2 satisfying the conditions of the theorem we have $\lambda_2 > 0$ and $0 < r \leq 1-\gamma$, i.e., the condition of the theorem is well formulated. Further, from Theorem we obtain $\left|\frac{D^n f(z)}{z} - 1\right| < \frac{\lambda}{1+\gamma}$, i.e.,

$$1 - \frac{\lambda}{1+\gamma} < \left|\frac{D^n f(z)}{z}\right| < 1 + \frac{\lambda}{1+\gamma},$$

for all $z \in U$. Therefore

$$\begin{aligned}
&\left(1-\tfrac{\lambda}{1+\gamma}\right)\left|\tfrac{D^{n+1}f(z)}{D^n f(z)} - (1+\gamma)\right| \\
&< \left|\tfrac{D^n f(z)}{z}\right|\left|\tfrac{D^{n+1}f(z)}{D^n f(z)} - (1+\gamma)\right| \\
&< \gamma + \lambda
\end{aligned}$$

and (17) holds for any $z \in U$ which complete the proof of above theorem.

Remark 3. If we put $n = 0, n = 1$ in Theorem 2.1-2.3, we obtain corresponding results of Tuneski [12].

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