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RESTRICTION OF STABLE BUNDLES ON AN ABELIAN SURFACE. THE c_2=1 CASE

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Abstract. In this note we describe the restriction map from the moduli space of stable rank 2 bundle with $c_2 = 1$ on a jacobian X of dimension 2, to the moduli space of stable rank 2 bundles on the corresponding genus 2 curve C embedded in X.

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1. Introduction

Let C a smooth curve of genus 2 and X his jacobian wich is a smooth projective algebraic surface. The base field is the complex number field. We denote by

$$M_{(2,\ C,\ 1)}$$

the moduli space of rank 2 bundle on X with $c_1 = C$ and $c_2 = 1$. Also we denote by

$$M_{(2,\ K)}$$

the moduli space of stable rank 2 bundle on C with determinant K i.e. the canonical class of C. Obviously, for any

$$E \in M_{(2,\ C,\ 1)}$$

the restriction $E_{|C}$ is a rank 2 bundle on C with determinant K. The natural questions wich appear are the followings: is $E_{|C}$ a stable (or at least semi-stable) bundle on C and if yes, what is the induced map

$$M_{(2,\ C,\ 1)} \longrightarrow M_{(2,\ K)}\ ?$$

As we shall see, the restriction is semi-stable, and the restriction map can be described explicitelly, in terms of the generalised theta divisor associated to E, a result announced in [1].

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2. Notations

For X the jacobian of a genus 2 curve C, we denote by $F_0 = \mathcal{O}(C) \otimes \mathcal{J}_0$, where $\mathcal{J}_0$ is the sheaf of ideals of the origin of X. Also, using F_0 we can construct a unique extension

$$0 \longrightarrow \mathcal{O}_X \longrightarrow F_{-1} \longrightarrow F_0 \longrightarrow 0$$

wich has $c_1 = \mathcal{O}_X(C)$ and $c_2 = 1$. The first result we need is the following, proved in [3]:

Theorem 0.1 *For any rank 2 bundle E on X with $c_1 = \mathcal{O}_X(C)$ and $c_2 = 1$ there are uniques $x,\ y \in X$ such that $E \simeq {T_x}^* F_{-1} \otimes P_y$, where ${T_x}^*$ is the pull-back by the x-translation and P_y is the line bundle on X wich correspond to y by the canonical isomorphism $X \longrightarrow \widehat{X}$ defined by the principal polarisation C. As consequence the moduli space is isomorphic with $X \times X$.*

For the moduli space on C we need the following theorem proved in [4]:

Theorem 0.2 *Let F a semi-stable rank 2 bundle on C with determinant equal with the canonical class of C, and x_0 a Weierstrass point of C. Let*

$$D_F = \{\xi \in Pic^1(C) \ \mid H^0(\xi \otimes F \otimes \mathcal{O}(-x_0)) \neq 0\}.$$

With these notations, D_F is a divisor of the linear system $\mid 2C \mid$ on $Pic^1(C)$ and the map $F \longrightarrow D_F$ is an isomorphism between the moduli space of rank two bundles with canonical determinant and $\mathbf{P}^3$.

3. The restriction theorem

Using the previous notations we have the following:

Theorem 0.3 *a) The moduli space of rank 2 bundles on X with $c_1 = \mathcal{O}_X(C)$ and $c_2 = 1$ is isomorphic with X.*
b) For generic $y \in X$ the restriction $E_{|C}$ of $E \simeq {T_{-2y}}^* F_{-1} \otimes P_y$ is semi-stable but not stable.
c) The rational restriction map $X - - \rightarrow \mathbf{P}^3$ is the quotient by the natural involution of X and the image is the Kummer surface.

Proof: a) First of all, for $E \simeq T_x{}^*F_{-1} \otimes P_y$ the determinant is $T^*{}_{x+2y}\mathcal{O}_X(C)$ and it is $\mathcal{O}_X(C)$ iff $x = -2y$. So $M_{(2,\ C,\ 1)}$ is isomorphic with X.

b) For $E_0 = F_{-1}$ the restriction on C is given by the exact sequence

$$0 \longrightarrow \mathcal{O}_C(x_0) \longrightarrow E_0 \mid_C \longrightarrow \mathcal{O}_C(x_0)) \longrightarrow 0.$$

The moduli space on X is fine, so by openess of semi-stability in families, for a generic bundle the restriction is semi-stable.

c) Let $E \simeq E_y \simeq T_{-2y}{}^*F_{-1} \otimes P_y$. The main point here is the calculus of $D_{E_{|C}}$ for a generic E. From the exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow F_{-1} \longrightarrow F_0 \longrightarrow 0$$

we obtain the following for E:

$$0 \longrightarrow P_y \longrightarrow E \longrightarrow T^*_{-2y}F_0 \otimes P_y \longrightarrow 0.$$

For generic y, $P_{y|C} \simeq \mathcal{O}_C(x_0) \otimes y^\vee$ and $T_{-2y}{}^*F_0 \mid_C = y \otimes y$. So

$$E_{y_{|C}} \in \ Ext^1(y \otimes \mathcal{O}_C(x_0), y^\vee \otimes \mathcal{O}_C(x_0)).$$

Now we shall find $D_{E_{|C}}$ for generic $E = E_y$, where

$$D_{E_{|C}} = \{\xi \in Pic^1(C) \mid H^0(\xi \otimes E_{|C} \otimes \mathcal{O}_C(-x_0)) \neq 0\}.$$

The cohomology exact sequence for

$$0 \longrightarrow y^\vee \longrightarrow E' \longrightarrow y \longrightarrow 0,$$

where $E' = E_{|C} \otimes \mathcal{O}(-x_0)$, shows that for generic ξ, the condition $H^0(\xi \otimes E') \neq 0$ is equivalent to the fact that

$$\delta : H^0(y \otimes \xi) \longrightarrow H^1(y^\vee \otimes \xi)$$

is 0. Let's introduce the notations: $z = \mathcal{O}_C(x_0) \otimes \xi \otimes y^\vee$ and $\mathcal{F} = T_{-2y}{}^*F_0$. Now, the fact that $\delta = 0$ is implied by (but not equivalent to) the surjectivity of

$$\varphi : Hom(P_z, \mathcal{F}) \longrightarrow Hom(\xi^\vee, y).$$

Using Propositions 4.4.1 and 4.4.2 from [3] the surjectivity of φ is equivalent to $H^0(\mathcal{F} \otimes P_z{}^*) \neq 0$ wich turn out to be equivalent with $z \in C$. This last fact, using the group structure on X means that

$$C + y \subset D_{E_{|C}}$$

and $D_{E_{|C}}$ beeing in $\mid 2C \mid$, we find that

$$C - y \subset D_{E_{|C}}$$

also, and so

$$D_{E_{|C}} = C - y \cup C + y.$$

This last fact means that on the open set where it is defined, the restriction map is in fact the Kummer involution, proving c).

Also, the image of any bundle has a reducible divisor and so is only semi-stable but not stable, proving also the last part of b).

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