

A GEOMETRIC WAY TO GENERATE BLUNDON TYPE INEQUALITIES

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ABSTRACT. We present a geometric way to generate Blundon type inequalities. Theorem 3.1 gives the formula for $\cos \widehat{POQ}$ in terms of the barycentric coordinates of the points P and Q with respect to a given triangle. This formula implies Blundon type inequalities generated by the points P and Q . Some applications are given in the last section by choosing special points P and Q .

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1. INTRODUCTION

Consider O the circumcenter, I the incenter, G the centroid, N the Nagel point, s the semiperimeter, R the circumradius, and r the inradius of triangle ABC .

Blundon's inequalities express the necessary and sufficient conditions for the existence of a triangle with elements s , R and r :

$$2R^2 + 10Rr - r^2 - 2(R - 2r)\sqrt{R^2 - 2Rr} \leq s^2 \leq 2R^2 + 10Rr - r^2 + 2(R - 2r)\sqrt{R^2 - 2Rr}. \quad (1)$$

Clearly these two inequalities can be written in the following equivalent form

$$|s^2 - 2R^2 - 10Rr + r^2| \leq 2(R - 2r)\sqrt{R^2 - 2Rr}, \quad (2)$$

and in many references this relation is called the fundamental inequality of triangle ABC .

The standard proof is an algebraic one, it was first time given by W.J.Blundon [5] and it is based on the characterization of cubic equations with the roots the length sides of a triangle. For more details we refer to the monograph of D. Mitrinović, J. Pečarić, V. Volenec [16], and to the papers of C.Niculescu [17],[18]. R.A.Satnoianu [20], and S.Wu [22] have obtained some improvements of this important inequality.

The following result was obtained by D.Andrica and C.Barbu in the paper [3] and it contains a simple geometric proof of (1). Assume that the triangle ABC is not equilateral. The following relation holds :

$$\cos \widehat{ION} = \frac{2R^2 + 10Rr - r^2 - s^2}{2(R - 2r)\sqrt{R^2 - 2Rr}}. \quad (3)$$

If we have $R = 2r$, then the triangle must be equilateral and we have equality in (1) and (2). If we assume that $R - 2r \neq 0$, then inequalities (1) are direct consequences of the fact that $-1 \leq \cos \widehat{ION} \leq 1$.

In this geometric argument the main idea is to consider the points O , I and N , and then to get the formula (3). It is a natural question to see what is a similar formula when we kip the circumcenter O and we replace the points I and N by other two points P and Q . In this way we obtain Blundon type inequalities generated by the points P and Q . Section 2 contains the basic facts about the main ingredient helping us to do all the computations, that is the barycentric coordinates. In Section 3 we present the analogous formula to (3), for the triangle POQ , and the we derive the Blundon type inequalities generated in this way. The last section contains some applications of the results in Section 3 as follows: the classical Blundon's inequalities, the dual Blundon's inequalities obtained in the paper [3], the Blundon's inequalities generated by two Cevian points of rank $(k; l; m)$.

2. SOME BASIC RESULTS ABOUT BARYCENTRIC COORDINATES

Let P be a point situated in the plane of the triangle ABC . The Cevian triangle DEF is defined by the intersection of the Cevian lines though the point P and the sides BC, CA, AB of triangle. If the point P has barycentric coordinates $t_1 : t_2 : t_3$, then the vertices of the Cevian triangle DEF have barycentric coordinates given by: $D(0 : t_2 : t_3)$, $E(t_1 : 0 : t_3)$ and $F(t_1 : t_2 : 0)$. The barycentric coordinates were introduced in 1827 by Möbius (see [10]). The using of barycentric coordinates defines a distinct part of Geometry called Barycentric Geometry. More details can be found in the monographs of C. Bradley [10], C. Coandă[11], C. Coşniţă [12], C. Kimberling [14], and in the papers of O. Bottema [9], J. Scott [21], and P. Yiu [23].

It is well-known ([11],[12]) that for every point M in the plane of triangle ABC , then the following relation holds:

$$(t_1 + t_2 + t_3)\overrightarrow{MP} = t_1\overrightarrow{MA} + t_2\overrightarrow{MB} + t_3\overrightarrow{MC}. \quad (4)$$

In the particular case when $M \equiv P$, we obtain

$$t_1\overrightarrow{PA} + t_2\overrightarrow{PB} + t_3\overrightarrow{PC} = \vec{0}.$$

This last relation shows that the point P is the barycenter of the system $\{A, B, C\}$ with the weights $\{t_1, t_2, t_3\}$. The following well-known result is very useful in computing distances from the point M to the barycenter P of the system $\{A, B, C\}$ with the weights $\{t_1, t_2, t_3\}$.

Theorem 2.1. *If M is a point situated in the plane of triangle ABC , then*

$$(t_1+t_2+t_3)^2 MP^2 = (t_1 MA^2 + t_2 MB^2 + t_3 MC^2)(t_1+t_2+t_3) - (t_2 t_3 a^2 + t_3 t_1 b^2 + t_1 t_2 c^2). \quad (5)$$

Proof. Using the scalar product of two vectors, from (4) we obtain:

$$(t_1 + t_2 + t_3)^2 MP^2 = t_1^2 MA^2 + t_2^2 MB^2 + t_3^2 MC^2 + 2t_1 t_2 \overrightarrow{MA} \cdot \overrightarrow{MB} + 2t_1 t_3 \overrightarrow{MA} \cdot \overrightarrow{MC} + 2t_2 t_3 \overrightarrow{MB} \cdot \overrightarrow{MC},$$

that is

$$(t_1 + t_2 + t_3)^2 MP^2 = t_1^2 MA^2 + t_2^2 MB^2 + t_3^2 MC^2 + t_1 t_2 (MA^2 + MB^2 - AB^2) + t_1 t_3 (MA^2 + MC^2 - AC^2) + t_2 t_3 (MB^2 + MC^2 - BC^2),$$

hence,

$$(t_1+t_2+t_3)^2 MP^2 = (t_1 MA^2 + t_2 MB^2 + t_3 MC^2)(t_1+t_2+t_3) - (t_2 t_3 a^2 + t_3 t_1 b^2 + t_1 t_2 c^2).$$

To get the last relation we have used the definition of the scalar product and the Cosine Law as follows

$$\begin{aligned} 2\overrightarrow{MA} \cdot \overrightarrow{MB} &= 2MA \cdot MB \cos \widehat{AMB} = \\ 2MA \cdot MB \cdot \frac{MA^2 + MB^2 - AB^2}{2MA \cdot MB} &= MA^2 + MB^2 - AB^2. \end{aligned}$$

If we consider that t_1, t_2, t_3 , and $t_1 + t_2 + t_3$ are nonzero real numbers, then the relation (5) becomes the Lagrange's relation

$$MP^2 = \frac{t_1 MA^2 + t_2 MB^2 + t_3 MC^2}{t_1 + t_2 + t_3} - \frac{t_1 t_2 t_3}{(t_1 + t_2 + t_3)^2} \left(\frac{a^2}{t_1} + \frac{b^2}{t_2} + \frac{c^2}{t_3} \right). \quad (6)$$

If we consider in (6) $M \equiv O$, the circumcenter of the triangle, then it follows

$$R^2 - OP^2 = \frac{t_1 t_2 t_3}{(t_1 + t_2 + t_3)^2} \left(\frac{a^2}{t_1} + \frac{b^2}{t_2} + \frac{c^2}{t_3} \right). \quad (7)$$

The following version of Cauchy-Schwarz inequality is also known in the literature as Bergström's inequality (see [6], [7], [8], [19]): If $x_k, a_k \in \mathbb{R}$ and $a_k > 0$, $k = 1, 2, \dots, n$, then

$$\frac{x_1^2}{a_1} + \frac{x_2^2}{a_2} + \dots + \frac{x_n^2}{a_n} \geq \frac{(x_1 + x_2 + \dots + x_n)^2}{a_1 + a_2 + \dots + a_n},$$

with equality if and only if

$$\frac{x_1}{a_1} = \frac{x_2}{a_2} = \dots = \frac{x_n}{a_n}.$$

Using Bergström's inequality and relation (4), we obtain

$$R^2 - OP^2 \geq \frac{t_1 t_2 t_3}{(t_1 + t_2 + t_3)^2} \cdot \frac{(a + b + c)^2}{t_1 + t_2 + t_3},$$

that is in any triangle with semiperimeter s the following inequality holds:

$$R^2 - OP^2 \geq \frac{4s^2 t_1 t_2 t_3}{(t_1 + t_2 + t_3)^3},$$

where $t_1 : t_2 : t_3$ are the barycentric coordinates of P and $t_1, t_2, t_3 > 0$. Equality holds if and only if $t_1 = a, t_2 = b, t_3 = c$, that is $P \equiv I$, the incenter of the triangle ABC .

Theorem 2.2. ([11], [12]). *If the points P and Q have barycentric coordinates $t_1 : t_2 : t_3$, and $u_1 : u_2 : u_3$, respectively, with respect to the triangle ABC , and $u = u_1 + u_2 + u_3$, $t = t_1 + t_2 + t_3$, then*

$$PQ^2 = -\alpha\beta\gamma \left(\frac{a^2}{\alpha} + \frac{b^2}{\beta} + \frac{c^2}{\gamma} \right) \quad (8)$$

where the numbers α, β, γ are defined by

$$\alpha = \frac{u_1}{u} - \frac{t_1}{t}; \beta = \frac{u_2}{u} - \frac{t_2}{t}; \gamma = \frac{u_3}{u} - \frac{t_3}{t}.$$

3. BLUNDON TYPE INEQUALITIES GENERATED BY TWO POINTS

Theorem 3.1. *Let P and Q be two points diferent from the circumcircle O , having the barycentric coordinates $t_1 : t_2 : t_3$, and $u_1 : u_2 : u_3$ with respect to the triangle ABC and let $u = u_1 + u_2 + u_3$, $t = t_1 + t_2 + t_3$. If $u_1, u_2, u_3, t_1, t_2, t_3 \neq 0$, then the following relation holds*

$$\cos \widehat{POQ} = \frac{2R^2 - \frac{t_1 t_2 t_3}{t^2} \left(\frac{a^2}{t_1} + \frac{b^2}{t_2} + \frac{c^2}{t_3} \right) - \frac{u_1 u_2 u_3}{t^2} \left(\frac{a^2}{u_1} + \frac{b^2}{u_2} + \frac{c^2}{u_3} \right) + \alpha \beta \gamma \left(\frac{a^2}{\alpha} + \frac{b^2}{\beta} + \frac{c^2}{\gamma} \right)}{2\sqrt{\left[R^2 - \frac{t_1 t_2 t_3}{t^2} \left(\frac{a^2}{t_1} + \frac{b^2}{t_2} + \frac{c^2}{t_3} \right) \right] \cdot \left[R^2 - \frac{u_1 u_2 u_3}{u^2} \left(\frac{a^2}{u_1} + \frac{b^2}{u_2} + \frac{c^2}{u_3} \right) \right]}} \quad (9)$$

where a, b, c are the length sides of the triangle and

$$\alpha = \frac{u_1}{u} - \frac{t_1}{t}; \beta = \frac{u_2}{u} - \frac{t_2}{t}; \gamma = \frac{u_3}{u} - \frac{t_3}{t}. \quad (10)$$

Proof. Applying the relation (7) for the points P and Q , we have

$$OP^2 = R^2 - \frac{t_1 t_2 t_3}{t^2} \left(\frac{a^2}{t_1} + \frac{b^2}{t_2} + \frac{c^2}{t_3} \right) \quad (11)$$

and

$$OQ^2 = R^2 - \frac{u_1 u_2 u_3}{u^2} \left(\frac{a^2}{u_1} + \frac{b^2}{u_2} + \frac{c^2}{u_3} \right) \quad (12)$$

We use the Law of Cosines in the triangle POQ to obtain

$$\cos \widehat{POQ} = \frac{OP^2 + OQ^2 - PQ^2}{2OP \cdot OQ}, \quad (13)$$

and from relations (8), (11), (12) and (13) we obtain the relation (9).

Theorem 3.2. *Let P and Q be two points diferent from the circumcircle O , having the barycentric coordinates $t_1 : t_2 : t_3$, and $u_1 : u_2 : u_3$ with respect to the triangle ABC and let $u = u_1 + u_2 + u_3$, $t = t_1 + t_2 + t_3$. If $u_1, u_2, u_3, t_1, t_2, t_3 \neq 0$, then the following relation holds*

$$\begin{aligned} & -2\sqrt{\left[R^2 - \frac{t_1 t_2 t_3}{t^2} \left(\frac{a^2}{t_1} + \frac{b^2}{t_2} + \frac{c^2}{t_3}\right)\right]} \cdot \sqrt{\left[R^2 - \frac{u_1 u_2 u_3}{u^2} \left(\frac{a^2}{u_1} + \frac{b^2}{u_2} + \frac{c^2}{u_3}\right)\right]} \leq \\ & \alpha\beta\gamma \left(\frac{a^2}{\alpha} + \frac{b^2}{\beta} + \frac{c^2}{\gamma}\right) + 2R^2 - \left[\frac{t_1 t_2 t_3}{t^2} \left(\frac{a^2}{t_1} + \frac{b^2}{t_2} + \frac{c^2}{t_3}\right) + \frac{u_1 u_2 u_3}{u^2} \left(\frac{a^2}{u_1} + \frac{b^2}{u_2} + \frac{c^2}{u_3}\right)\right] \leq \\ & 2\sqrt{\left[R^2 - \frac{t_1 t_2 t_3}{t^2} \left(\frac{a^2}{t_1} + \frac{b^2}{t_2} + \frac{c^2}{t_3}\right)\right]} \cdot \sqrt{\left[R^2 - \frac{u_1 u_2 u_3}{u^2} \left(\frac{a^2}{u_1} + \frac{b^2}{u_2} + \frac{c^2}{u_3}\right)\right]} \quad (14) \end{aligned}$$

where a, b, c are the length sides of the triangle and the numbers α, β, γ are defined by (10).

Proof. The inequalities (14) are simple direct consequences of the fact that $-1 \leq \cos \widehat{POQ} \leq 1$. The equality in the right inequality holds if and only if $\widehat{POQ} = 0$, that is the points O, P, Q are collinear in the order O, P, Q or O, Q, P . The equality in the left inequality holds if and only if $\widehat{POQ} = \pi$, that is the points O, P, Q are collinear in the order P, O, Q or Q, O, P .

From Theorem 3.1. it follows that it is a natural and important problem to construct the triangle ABC from the points O, P, Q , when we know their barycentric coordinates. In the special case when $P \equiv I$ and $Q \equiv N$ we know that that points I, G, N are collinear, determining the Nagel line of triangle, and the centroid G lies on the segment IN such that $IG = \frac{1}{3}IN$. Then, using the Euler's line of the triangle, we get the orthocenter H on the ray $(OG$ such that $OH = 3OG$. In this case the problem is reduced to the famous Euler's determination problem i.e. to construct a triangle from its incenter I , circumcenter O , and orthocenter H (see the paper of P.Yiu [24] for details and results). This is a reason to call the problem as the general determination problem.

4. APPLICATIONS

The formula (3) and the classical Blundon's inequalities (1) can be obtained from (9) and (14) by considering $P = I$, the incenter, and $Q = N$, the Nagel point of the triangle. Indeed, the barycentric coordinates of incenter I and of Nagel's point N are $(t_1, t_2, t_3) = (a, b, c)$, and $(u_1, u_2, u_3) = (s - a, s - b, s - c)$, respectively. We have

$$u = u_1 + u_2 + u_3 = s, \quad u_1 u_2 u_3 = r^2 s, \quad (15)$$

and

$$t = t_1 + t_2 + t_3 = 2s, \quad t_1 t_2 t_3 = abc = 4Rrs. \quad (16)$$

We obtain

$$\alpha = \frac{s-a}{s} - \frac{a}{2s} = \frac{2s-3a}{2s}, \quad \beta = \frac{2s-3b}{2s}, \quad \gamma = \frac{2s-3c}{2s}. \quad (17)$$

Therefore

$$\begin{aligned} \alpha\beta\gamma \left(\frac{a^2}{\alpha} + \frac{b^2}{\beta} + \frac{c^2}{\gamma} \right) &= \sum_{cyc} \beta\gamma a^2 = \sum_{cyc} \left(1 - \frac{3b}{2s} \right) \left(1 - \frac{3c}{2s} \right) a^2 = \\ &= \sum_{cyc} a^2 - \frac{3}{2s} \sum_{cyc} [a^2(a+b+c) - a^3] + \frac{9abc}{4s^2} \sum_{cyc} a = \\ &= \sum_{cyc} a^2 - 3 \sum_{cyc} a^2 + \frac{3}{2s} \sum_{cyc} a^3 + \frac{9abc}{2s} = \\ &= -2(2s^2 - 2r^2 - 8Rr) + 3(s^2 - 3r^2 - 6Rr) + 18Rr \end{aligned}$$

that is

$$\alpha\beta\gamma \left(\frac{a^2}{\alpha} + \frac{b^2}{\beta} + \frac{c^2}{\gamma} \right) = -s^2 - 5r^2 + 16Rr. \quad (18)$$

Now, using (16) and (17) we get

$$\frac{t_1 t_2 t_3}{t} \left(\frac{a^2}{t_1} + \frac{b^2}{t_2} + \frac{c^2}{t_3} \right) = \frac{4Rrs}{4s^2} \cdot 2s = 2Rr, \quad (19)$$

and

$$\begin{aligned} \frac{u_1 u_2 u_3}{u} \left(\frac{a^2}{u_1} + \frac{b^2}{u_2} + \frac{c^2}{u_3} \right) &= \frac{r^2 s}{s^2} \left(\frac{a^2}{s-a} + \frac{b^2}{s-b} + \frac{c^2}{s-c} \right) = \\ &= \frac{r^2}{s} \cdot \frac{\sum_{cyc} a^2 (s-b)(s-c)}{r^2 s} = \\ &= \frac{1}{s^2} \left\{ s^2 \sum_{cyc} a^2 - s \sum_{cyc} [a^2(a+b+c) - a^3] + abc \sum_{cyc} a \right\} = \\ &= \frac{1}{s^2} \left(s \sum_{cyc} a^3 - s^2 \sum_{cyc} a^2 + 8Rrs^2 \right) = \\ &= \frac{1}{s^2} [2s^2(s^2 - 3r^2 - 6Rr) - s^2(2s^2 - 2r^2 - 8Rr) + 8Rrs^2] = 4Rr - 4r^2 \quad (20) \end{aligned}$$

Using the relations (18)-(20) in (9) we obtain the relation (3). These computations are similar to those given by complex numbers in [1].

Now, consider the excenters I_a, I_b, I_c , and N_a, N_b, N_c the adjoint points to the Nagel point N . For the definition and some properties of the adjoint points N_a, N_b, N_c we refer to the paper of D.Andrica and K.L.Nguyen [2]. Let s, R, r, r_a, r_b, r_c be the semiperimeter, circumradius, inradius, and exradii of triangle ABC , respectively. Considering the triangle I_aON_a , D.Andrica and C.Barbu [3] have proved the following formula

$$\cos \widehat{I_aON_a} = \frac{R^2 - 3Rr_a - r_a^2 - \alpha}{(R + 2r_a)\sqrt{R^2 + 2Rr_a}}, \quad (21)$$

where $\alpha = \frac{a^2+b^2+c^2}{4}$.

Using formula (21), we get the dual form of Blundon's inequalities given in the paper [3]

$$0 \leq \frac{a^2 + b^2 + c^2}{4} \leq R^2 - 3Rr_a - r_a^2 + (R + 2r_a)\sqrt{R^2 + 2Rr_a}. \quad (22)$$

There are similar inequalities involving the exradii r_b and r_c .

We know that the barycentric coordinates of the excenter I_a are $(t_1, t_2, t_3) = (-a, b, c)$, and of the adjoint Nagel point N_a are $(u_1, u_2, u_3) = (s, c - s, b - s)$. Using formula (9) we can obtain the relation (21) and then the dual form of the classical Blundon's inequalities (22).

We have

$$u = u_1 + u_2 + u_3 = s - a, \quad u_1u_2u_3 = s(s - b)(s - c)$$

and

$$t = t_1 + t_2 + t_3 = 2(s - a), \quad t_1t_2t_3 = -abc = -4Rrs.$$

We obtain

$$\begin{aligned} \alpha &= \frac{2s + a}{2(s - a)} = 1 + \frac{3a}{2(s - a)}, \\ \beta &= \frac{2c - 2s - b}{2(s - a)} = 1 - \frac{3b}{2(s - a)}, \\ \gamma &= \frac{2b - 2s - c}{2(s - a)} = 1 - \frac{3c}{2(s - a)}. \end{aligned}$$

Therefore,

$$\frac{t_1t_2t_3}{t^2} \left(\frac{a^2}{t_1} + \frac{b^2}{t_2} + \frac{c^2}{t_3} \right) = \frac{-4Rrs}{4(s - a)^2} \cdot 2(s - a) = -2Rr_a, \quad (23)$$

and

$$\begin{aligned}
 \frac{u_1 u_2 u_3}{u^2} \left(\frac{a^2}{u_1} + \frac{b^2}{u_2} + \frac{c^2}{u_3} \right) &= \frac{s(s-c)(s-b)}{(s-a)^2} \left(\frac{a^2}{s} - \frac{b^2}{s-c} - \frac{c^2}{s-b} \right) = \\
 &= - \left(-a^2 \cdot \frac{s-b}{s-a} \cdot \frac{s-c}{s-a} + b^2 \cdot \frac{s}{s-a} \cdot \frac{s-b}{s-a} + c^2 \cdot \frac{s}{s-a} \cdot \frac{s-c}{s-a} \right) = \\
 &= - \left(-a^2 \cdot \frac{r_a}{r_b} \cdot \frac{r_a}{r_c} + b^2 \cdot \frac{r_a}{r} \cdot \frac{r_a}{r_b} + c^2 \cdot \frac{r_a}{r} \cdot \frac{r_a}{r_c} \right) = \\
 &= -r_a^2 \left(\frac{-a^2}{r_b r_c} + \frac{b^2}{r r_b} + \frac{c^2}{r r_c} \right) = -r_a^2 \left(\frac{4R}{r_a} + 4 \right) = -4Rr_a - 4r_a^2, \tag{24}
 \end{aligned}$$

where we have used the relation $\frac{-a^2}{r_b r_c} + \frac{b^2}{r r_b} + \frac{c^2}{r r_c} = \frac{4R}{r_a} + 4$ (see [2], p. 134).

Now, we will calculate the expression:

$$\begin{aligned}
 E &= \alpha\beta\gamma \left(\frac{a^2}{\alpha} + \frac{b^2}{\beta} + \frac{c^2}{\gamma} \right) + \frac{a^2 + b^2 + c^2}{2} = \\
 &= a^2\beta\gamma + \frac{a^2}{2} + b^2\alpha\gamma + \frac{b^2}{2} + c^2\alpha\beta + \frac{c^2}{2} = \\
 &= a^2 \left[1 - \frac{3(b+c)}{2(s-a)} + \frac{9bc}{4(s-a)^2} \right] + \frac{a^2}{2} + \\
 &= b^2 \left[1 + \frac{3(a-c)}{2(s-a)} - \frac{9ca}{4(s-a)^2} \right] + \frac{b^2}{2} + \\
 &= c^2 \left[1 + \frac{3(a-b)}{2(s-a)} - \frac{9ab}{4(s-a)^2} \right] + \frac{c^2}{2},
 \end{aligned}$$

that is

$$\begin{aligned}
 E &= a^2 \left[\frac{-3s}{2(s-a)} + \frac{9bc}{4(s-a)^2} \right] + b^2 \left[\frac{3(s-c)}{2(s-a)} - \frac{9ca}{4(s-a)^2} \right] + \\
 &= c^2 \left[\frac{3(s-b)}{2(s-a)} - \frac{9ab}{4(s-a)^2} \right] = \\
 &= \frac{3}{2(s-a)} [-a^2s + b^2(s-c) + c^2(s-b)] + \frac{9abc}{4(s-a)^2} (a-b-c) = \\
 &= \frac{3}{2(s-a)} [s(-a^2 + b^2 + c^2) - bc(b+c)] - 18Rr_a. \tag{25}
 \end{aligned}$$

We have

$$s(-a^2 + b^2 + c^2) - bc(b+c) = 2sbc \cos A - 2bcs + abc =$$

$$\begin{aligned} 2sbc(\cos A - 1) + abc &= abc - 4sbc \sin^2 \frac{A}{2} = \\ abc - 4s(p-b)(p-c) &= abc - 4Sr_a = 4S(R - r_a), \end{aligned} \quad (26)$$

where S denotes the area of triangle ABC . From relations (25) and (26) we get

$$E = \frac{3}{2(s-a)} \cdot 4S(R - r_a) - 18Rr_a = 6r_a(R - r_a) - 18Rr_a = -12Rr_a - 6r_a^2,$$

therefore

$$\alpha\beta\gamma \left(\frac{a^2}{\alpha} + \frac{b^2}{\beta} + \frac{c^2}{\gamma} \right) = -12Rr_a - 6r_a^2 - \frac{a^2 + b^2 + c^2}{2} \quad (27)$$

Using formulas (23), (24) and (27) in the general formula (9) we obtain the relation (21).

In the paper [13], N. Minculete and C. Barbu have introduced the Cevians of rank $(k; l; m)$. The line AD is called *ex-Cevian of rank $(k; l; m)$* or *exterior Cevian of rank $(k; l; m)$* , if the point D is situated on side (BC) of the non-isosceles triangle ABC and the following relation holds:

$$\frac{BD}{DC} = \left(\frac{c}{b} \right)^k \cdot \left(\frac{s-c}{s-b} \right)^l \cdot \left(\frac{a+b}{a+c} \right)^m.$$

In the paper [13] it is proved that the Cevians of rank $(k; l; m)$ are concurrent in the point $I(k, l, m)$ called *the Cevian point of rank $(k; l; m)$* and the barycentric coordinates of $I(k, l, m)$ are:

$$a^k(s-a)^l(b+c)^m : b^k(s-b)^l(a+c)^m : c^k(s-c)^l(a+b)^m.$$

In the case $l = m = 0$, we obtain the Cevian point of rank k .

Let I_1, I_2 be two Cevian points with barycentric coordinates:

$$I_i[a^{k_i}(s-a)^{l_i}(b+c)^{m_i} : b^{k_i}(s-b)^{l_i}(a+c)^{m_i} : c^{k_i}(s-c)^{l_i}(a+b)^{m_i}], \quad i = 1, 2.$$

Denote $t_i^1 = a^{k_i}(s-a)^{l_i}(b+c)^{m_i}, t_i^2 = b^{k_i}(s-b)^{l_i}(a+c)^{m_i}, t_i^3 = c^{k_i}(s-c)^{l_i}(a+b)^{m_i}, i = 1, 2$. From formula (9) we obtain

$$\cos \widehat{I_1 O I_2} = \frac{2R^2 - \frac{t_1^1 t_1^2 t_1^3}{(T_1)^2} \left(\frac{a^2}{t_1^1} + \frac{b^2}{t_1^2} + \frac{c^2}{t_1^3} \right) - \frac{t_2^1 t_2^2 t_2^3}{(T_2)^2} \left(\frac{a^2}{t_2^1} + \frac{b^2}{t_2^2} + \frac{c^2}{t_2^3} \right) + \alpha\beta\gamma \left(\frac{a^2}{\alpha} + \frac{b^2}{\beta} + \frac{c^2}{\gamma} \right)}{2\sqrt{\left[R^2 - \frac{t_1^1 t_1^2 t_1^3}{(T_1)^2} \left(\frac{a^2}{t_1^1} + \frac{b^2}{t_1^2} + \frac{c^2}{t_1^3} \right) \right] \cdot \left[R^2 - \frac{t_2^1 t_2^2 t_2^3}{(T_2)^2} \left(\frac{a^2}{t_2^1} + \frac{b^2}{t_2^2} + \frac{c^2}{t_2^3} \right) \right]}}, \quad (28)$$

where $T_1 = t_1^1 + t_1^2 + t_1^3, T_2 = t_2^1 + t_2^2 + t_2^3$, and for $i = 1, 2$, we have

$$\frac{t_i^1 t_i^2 t_i^3}{(T_i)^2} \left(\frac{a^2}{t_i^1} + \frac{b^2}{t_i^2} + \frac{c^2}{t_i^3} \right) = \frac{\prod_{cyc} a^{k_i} (s-a)^{l_i} (b+c)^{m_i}}{\sum_{cyc} a^{k_i} (s-a)^{l_i} (b+c)^{m_i}} \cdot \sum_{cyc} \frac{a^2}{a^{k_i} (s-a)^{l_i} (b+c)^{m_i}},$$

and

$$\begin{aligned} \alpha &= \frac{a^{k_1} (s-a)^{l_1} (b+c)^{m_1}}{\sum_{cyc} a^{k_1} (s-a)^{l_1} (b+c)^{m_1}} - \frac{a^{k_2} (s-a)^{l_2} (b+c)^{m_2}}{\sum_{cyc} a^{k_2} (s-a)^{l_2} (b+c)^{m_2}}, \\ \beta &= \frac{b^{k_1} (s-b)^{l_1} (a+c)^{m_1}}{\sum_{cyc} a^{k_1} (s-a)^{l_1} (b+c)^{m_1}} - \frac{b^{k_2} (s-b)^{l_2} (a+c)^{m_2}}{\sum_{cyc} a^{k_2} (s-a)^{l_2} (b+c)^{m_2}}, \\ \gamma &= \frac{c^{k_1} (s-c)^{l_1} (a+b)^{m_1}}{\sum_{cyc} a^{k_1} (s-a)^{l_1} (b+c)^{m_1}} - \frac{c^{k_2} (s-c)^{l_2} (a+b)^{m_2}}{\sum_{cyc} a^{k_2} (s-a)^{l_2} (b+c)^{m_2}}. \end{aligned}$$

If I_1, I_2 are Cevian points of rank k_1, k_2 , then formula (28) becomes

$$\cos \widehat{I_1 O I_2} = \frac{2R^2 - (abc)^{k_1} \frac{S_{2-k_1}}{(S_{k_1})^2} - (abc)^{k_2} \frac{S_{2-k_2}}{(S_{k_2})^2} + \sum_{cyc} \left(\frac{b^{k_1}}{S_{k_1}} - \frac{b^{k_2}}{S_{k_2}} \right) \left(\frac{c^{k_1}}{S_{k_1}} - \frac{c^{k_2}}{S_{k_2}} \right) a^2}{2\sqrt{[R^2 - (abc)^{k_1} \frac{S_{2-k_1}}{(S_{k_1})^2}][R^2 - (abc)^{k_2} \frac{S_{2-k_2}}{(S_{k_2})^2}]}}}, \quad (29)$$

where $S_l = a^l + b^l + c^l$.

Here are few special cases of formula (29). For $k_1 = 0$ and $k_2 = 1$ we get the centroid G and the incenter I of barycentric coordinates $(1; 1; 1)$ and $(a; b; c)$, respectively. Formula (29) becomes

$$\cos \widehat{GOI} = \frac{6R^2 - s^2 - r^2 + 2Rr}{2\sqrt{9R^2 - 2s^2 + 2r^2 + 8Rr} \cdot \sqrt{R^2 - 2Rr}}, \quad (30)$$

where $abc = 4sRr, S_0 = 3, S_1 = 2s, S_2 = 2(s^2 - r^2 - 4Rr)$.

For $k_2 = 2$ we obtain the Lemoine point L of triangle ABC , of barycentric coordinates $(a^2; b^2; c^2)$, and other two formulas are generated

$$\cos \widehat{GOL} = \frac{6R^2 S_2 - S_2^2 + S_4}{2\sqrt{9R^2 - S_2} \cdot \sqrt{R^2 S_2^2 - 48(Rrs)^2}}, \quad (31)$$

where $S_4 = S_2^2 - 2[(s^2 + r^2 + 4Rr)^2 - 16Rrs^2]$, and

$$\cos \widehat{IOL} = \frac{RS_2 + rS_2 - 4rs^2}{2\sqrt{R^2 - 2Rr} \cdot \sqrt{S_2^2 - 48r^2 s^2}}. \quad (32)$$

Each of the formulas (30), (31), (32) generates a Blundon type inequality, but these inequalities have not nice geometric interpretations.

Let I_1, I_2, I_3 be three Cevian points of rank $(k; l; m)$ with barycentric coordinates as follows:

$$a^{k_i}(s-a)^{l_i}(b+c)^{m_i} : b^{k_i}(s-b)^{l_i}(a+c)^{m_i} : c^{k_i}(s-c)^{l_i}(a+b)^{m_i}, \quad i = 1, 2, 3,$$

and let $t_i^1 = a^{k_i}(s-a)^{l_i}(b+c)^{m_i}$, $t_i^2 = b^{k_i}(s-b)^{l_i}(a+c)^{m_i}$, $t_i^3 = c^{k_i}(s-c)^{l_i}(a+b)^{m_i}$. Now, consider the numbers

$$\alpha_{ij} = \frac{t_j^1}{t_j^1 + t_j^2 + t_j^3} - \frac{t_i^1}{t_i^1 + t_i^2 + t_i^3},$$

and

$$\beta_{ij} = \frac{t_j^2}{t_j^1 + t_j^2 + t_j^3} - \frac{t_i^2}{t_i^1 + t_i^2 + t_i^3},$$

and

$$\gamma_{ij} = \frac{t_j^3}{t_j^1 + t_j^2 + t_j^3} - \frac{t_i^3}{t_i^1 + t_i^2 + t_i^3},$$

for all $i, j \in \{1, 2, 3\}$. Applying the relation (5) we obtain

$$I_i I_j^2 = -\alpha_{ij} \cdot \beta_{ij} \cdot \gamma_{ij} \cdot \left(\frac{a^2}{\alpha_{ij}} + \frac{b^2}{\beta_{ij}} + \frac{c^2}{\gamma_{ij}} \right),$$

for all $i, j \in \{1, 2, 3\}$. Using the Cosine Law in triangle $I_1 I_2 I_3$ it follows

$$\begin{aligned} \cos \widehat{I_1 I_2 I_3} &= \frac{I_1 I_2^2 + I_2 I_3^2 - I_3 I_1^2}{2 I_1 I_2 \cdot I_2 I_3} = \\ &= \frac{-a^2(\beta_{12}\gamma_{12} + \beta_{23}\gamma_{23} - \beta_{31}\gamma_{31}) - b^2(\gamma_{12}\alpha_{12} + \gamma_{23}\alpha_{23} - \gamma_{31}\alpha_{31}) + c^2(\alpha_{12}\beta_{12} + \alpha_{23}\beta_{23} - \alpha_{31}\beta_{31})}{2\sqrt{-\beta_{12}\gamma_{12}a^2 - \gamma_{12}\alpha_{12}b^2 - \alpha_{12}\beta_{12}c^2} \cdot \sqrt{-\beta_{23}\gamma_{23}a^2 - \gamma_{23}\alpha_{23}b^2 - \alpha_{23}\beta_{23}c^2}} \end{aligned} \quad (33)$$

Theorem 4.1. *The following inequalities hold*

$$\begin{aligned} &-2\sqrt{-\beta_{12}\gamma_{12}a^2 - \gamma_{12}\alpha_{12}b^2 - \alpha_{12}\beta_{12}c^2} \cdot \sqrt{-\beta_{23}\gamma_{23}a^2 - \gamma_{23}\alpha_{23}b^2 - \alpha_{23}\beta_{23}c^2} \leq \\ &-a^2(\beta_{12}\gamma_{12} + \beta_{23}\gamma_{23} - \beta_{31}\gamma_{31}) - b^2(\gamma_{12}\alpha_{12} + \gamma_{23}\alpha_{23} - \gamma_{31}\alpha_{31}) + c^2(\alpha_{12}\beta_{12} + \alpha_{23}\beta_{23} - \alpha_{31}\beta_{31}) \leq \\ &2\sqrt{-\beta_{12}\gamma_{12}a^2 - \gamma_{12}\alpha_{12}b^2 - \alpha_{12}\beta_{12}c^2} \cdot \sqrt{-\beta_{23}\gamma_{23}a^2 - \gamma_{23}\alpha_{23}b^2 - \alpha_{23}\beta_{23}c^2} \end{aligned} \quad (34)$$

Proof. The inequalities (34) are simple direct consequences of the inequalities $-1 \leq \cos \widehat{I_1 I_2 I_3} \leq 1$.

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