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CONCOMITANTS OF ORDER STATISTICS FOR BIVARIATE QUASI-GAUSSIAN DISTRIBUTION

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Abstract. In this paper we have obtained the distribution of the Concomitants of Order Statistics for Bivariate Quasi–Gaussian Distribution alongside the joint distribution of two concomitants. Single and product moments of the resulting distributions have also been derived.

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1. Introduction

Order Statistics are widely used in many areas of statistics. David and Nagaraja (2003) has given a comprehensive review of the subject. Specifically distribution of r–th order statistics is given as:

$$f_{r:n}(x) = \frac{n!}{(r-1)!\,(n-r)!} f(x)\,[F(x)]^{r-1}\,[1-F(x)]^{n-r}, \tag{1}$$

where $F(x)$ is distribution function of the random variable X. Further, joint distribution of r–th and s–th order statistics is:

$$\begin{aligned} f_{r,s:n}(x_1,x_2) &= C_{r,s,n} f(x_1)\,f(x_2)\,[F(x_1)]^{r-1}\,[F(x_2)-F(x_1)]^{s-r-1} \\ &\quad \times [1-F(x_2)]^{n-s}, \end{aligned} \tag{2}$$

where $C_{r,s,n} = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}$.

Concomitants of order statistics has been tremendously used in literature. David and Moeschberger (1978) established some parametric procedures. Harrell and Sen (1979) derived the maximum likelihood estimators of the parameters and the associated large sample covariance matrix for a censored sample from a bivariate Gaussian distribution consisting of $x_{1:n}, x_{2:n}, ..., x_{n:n}$ order random variables and $y_{[1:n]}, y_{[2:n]}, ..., y_{[n:n]}$ their corresponding concomitants variables. The likelihood ratio test for independent and power properties are also studied. It is important to mention here that the bivariate case has been considered by Watterson (1959), Gill and Vanghan (1990) used Tiku's simplified MLE's to deal with two sided censoring data, but allowing for possibly more than one set of concomitants. Specifically the distribution of r–th concomitant is given as, David and Nagaraja (2003):

$$g_{[r:n]}(y) = \sum_{i=n-r+1}^{n} (-1)^{i-n+r-1} \binom{i-1}{n-r} \binom{n}{i} g_{[1:i]}(y), \tag{3}$$

where

$$g_{[1:n]}(y) = \int_x f(y|x) f_{r:n}(x)\, dx, \tag{4}$$

and $f_{r:n}$ is given by (1). Further, the joint distribution of r–th and s–th concomitant is given as:

$$g_{[r,s:n]}(y_1, y_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{x_2} f(y_1|x_1) f(y_2|x_2) f_{r,s:n}(x_1, x_2)\, dx_1 dx_2, \tag{5}$$

where $f_{r,s:n}(x_1, x_2)$ is given in (2).

Concomitants of order statistics has been widely studied in literature. David (1993) has given a comprehensive review of concomitants along with the recent developments made in the field. David et el (1977) has obtained the distribution of rank of concomitant of order statistics. Nagaraja and David (1994) has obtained the distribution of maximum of concomitant of order statistics. Chu et el (1999) has studied the asymptotic behavior of the distribution of concomitant of order statistics. Distribution of concomitant in case of multivariate distributions has been studied by Wang et el (2006).

The Quasi–Gaussian distribution has been used in many statistical and chemical problems. The density function of the distribution is given as:

$$\begin{aligned} f(x, y) \quad = \quad & \frac{1}{2\pi\sqrt{\alpha_1\alpha_2(1-\rho^2)}} \exp\left[-\frac{1}{2(1-\rho^2)}\left\{\left(\frac{x-\alpha_1}{\sqrt{\alpha_1}}\right)^2\right.\right. \\ & \left.\left.\left(\frac{y-\alpha_2}{\sqrt{\alpha_2}}\right)^2 - 2\rho\left(\frac{x-\alpha_1}{\sqrt{\alpha_1}}\right)\left(\frac{y-\alpha_2}{\sqrt{\alpha_2}}\right)\right\}\right]. \end{aligned} \tag{6}$$

Said et el (1984) has studied the distribution in the context of chemical engineering.

In the following section we have obtained the distributions (3) and (5) by using Bivariate Quasi–Gaussian distribution with density (6).

2.Distribution of r-th Concomitant

The distribution of r–th concomitant of order statistics for any random variable is given by (3). In this section we have obtained the distribution of r–th concomitant by using the Bivariate Quasi–Gaussian distribution with density given in (6).

The marginal distribution of random variable X for (6) is:

$$\begin{aligned} f(x) &= \frac{1}{2\pi\sqrt{\alpha_1\alpha_2(1-\rho^2)}}\exp\left[-\frac{1}{2(1-\rho^2)}\left(\frac{x-\alpha_1}{\sqrt{\alpha_1}}\right)^2\right] \\ &\int_{-\infty}^{\infty}\exp\left[-\frac{1}{2(1-\rho^2)}\left\{\left(\frac{y-\alpha_2}{\sqrt{\alpha_2}}\right)^2-2\rho\left(\frac{x-\alpha_1}{\sqrt{\alpha_1}}\right)\left(\frac{y-\alpha_2}{\sqrt{\alpha_2}}\right)\right\}\right]dy. \end{aligned}$$

After some calculus, we have:

$$f(x)=\frac{1}{\sqrt{2\pi\alpha_1}}\exp\left[-\frac{1}{2}\left(\frac{x-\alpha_1}{\sqrt{\alpha_1}}\right)^2\right]=\phi\left(\frac{x-\alpha_1}{\sqrt{\alpha_1}}\right), \tag{7}$$

where $\phi(z)$ is density of standard normal distribution.

The conditional distribution of random variable Y given $X=x$ is:

$$f(y|x)=\frac{1}{\sqrt{2\pi\alpha_2(1-\rho^2)}}\exp\left[-\frac{1}{2(1-\rho^2)}\left\{\left(\frac{y-\alpha_2}{\sqrt{\alpha_2}}\right)-\rho\left(\frac{x-\alpha_1}{\sqrt{\alpha_1}}\right)\right\}^2\right]. \tag{8}$$

Using $r=1$ in (1) and normal distribution, the distribution of 1st order statistics for (7) is:

$$g_{1:n}(x)=(n-1)\,\phi\left(\frac{x-\alpha_1}{\sqrt{\alpha_1}}\right)\left[1-\Phi\left(\frac{x-\alpha_1}{\sqrt{\alpha_1}}\right)\right]^{n-1}, \tag{9}$$

where $\Phi(z)$ is cumulative standard normal distribution function. Using (8) and (9) in (4) the distribution of 1st concomitant is given by:

$$\begin{aligned} g_{[1:n]}(y) &= \frac{n}{\sqrt{2\pi\alpha_2(1-\rho^2)}}E_{\phi\left(\frac{x-\alpha_1}{\sqrt{\alpha_1}}\right)}\left[\left\{1-\Phi\left(\frac{x-\alpha_1}{\sqrt{\alpha_1}}\right)\right\}^{n-1}\right. \\ &\left.\times\exp\left\{-\frac{1}{2(1-\rho^2)}\left\{\left(\frac{Y-\alpha_2}{\sqrt{\alpha_2}}\right)-\rho\left(\frac{X-\alpha_1}{\sqrt{\alpha_1}}\right)\right\}^2\right\}\right], \end{aligned} \tag{10}$$

where $E_{g(x)}(X) = \int_{-\infty}^{\infty} xg(x)\,dx$. Using (10) in (3) the distribution of r–th concomitant is:

$$\begin{aligned} g_{[r:n]}(y) &= \sum_{i=n-r+1}^{n} (-1)^{i-n+r-1} \binom{i-1}{n-r} \binom{n}{i} \frac{i}{\sqrt{2\pi\alpha_2(1-\rho^2)}} \\ &\quad \times E_{\phi\left(\frac{x-\alpha_1}{\sqrt{\alpha_1}}\right)} \left[\left\{ 1 - \Phi\left(\frac{x-\alpha_1}{\sqrt{\alpha_1}}\right) \right\}^{n-1} \right. \\ &\quad \left. \times \exp\left\{ -\frac{1}{2(1-\rho^2)} \left\{ \left(\frac{Y-\alpha_2}{\sqrt{\alpha_2}}\right) - \rho\left(\frac{X-\alpha_1}{\sqrt{\alpha_1}}\right) \right\}^2 \right\} \right]. \end{aligned} \tag{11}$$

The distribution given in (11) can be used for certain problems where one is interested in concomitants from bivariate Quasi–Gaussian distribution.

3. Moments of The Concomitants

The distribution of 1st concomitant for bivariate Quasi–Gaussian distribution is given in (10). Now to find the moments of $y_{[r:n]}$ we first find the k–th moment of $y_{[1:n]}$ by using:

$$\mu^k_{[1:n]} = \int_{-\infty}^{\infty} y^k g_{[1:n]}(y)\,dy. \tag{12}$$

Using (10) in (12) we have:

$$\begin{aligned} \mu^k_{[1:n]} &= \frac{n}{\sqrt{2\pi\alpha_2(1-\rho^2)}} \int_{-\infty}^{\infty} \phi\left(\frac{x-\alpha_1}{\sqrt{\alpha_1}}\right) \left\{ 1 - \Phi\left(\frac{x-\alpha_1}{\sqrt{\alpha_1}}\right) \right\}^{n-1} \\ &\quad \times \int_{-\infty}^{\infty} y^k \exp\left[-\frac{1}{2(1-\rho^2)} \left\{ \left(\frac{Y-\alpha_2}{\sqrt{\alpha_2}}\right) - \rho\left(\frac{X-\alpha_1}{\sqrt{\alpha_1}}\right) \right\}^2 \right] dydx. \end{aligned}$$

After some calculus we have:

$$\begin{aligned} \mu^k_{[1:n]} &= n\alpha_2^k \sum_{h=0}^{k} \binom{k}{h} \left(\frac{1}{\sqrt{\alpha_2}}\right)^h \sum_{j=0}^{h} \binom{h}{2j} \left(1-\rho^2\right)^j \frac{(2j)!}{2^j j!} \rho^{h-2j} \\ &\quad \times E_{\phi\left(\frac{x-\alpha_1}{\sqrt{\alpha_1}}\right)} \left[\left(\frac{X-\alpha_1}{\sqrt{\alpha_1}}\right)^{h-2j} \left\{ 1 - \Phi\left(\frac{x-\alpha_1}{\sqrt{\alpha_1}}\right) \right\}^{n-1} \right]. \end{aligned} \tag{13}$$

Using (13) in

$$\mu^k_{[r:n]} = \sum_{i=n-r+1}^{n} (-1)^{i-n+r-1} \binom{i-1}{n-r} \binom{n}{i} \mu^k_{[1:i]}$$

we can obtain the expression for k–th moment of r–th concomitant. The mean and variance of r–the concomitant can be obtained by using the k–th moment of r–th order statistics. Special case of the moments of concomitants for bivariate Quasi–Gaussian distribution can be obtained by using $\rho = 0$. The k–th moment of r–th concomitant in this is given as:

$$\mu_{[r:n]}^{k} = \alpha_2^k \sum_{j=0}^{k} \binom{k}{2j} \left(\frac{1}{\sqrt{\alpha_2}}\right)^{2j} \frac{(2j)!}{2^j j!}. \tag{14}$$

The mean and variance can be easily obtained from (14).

4. Joint Distribution of the Concomitants

In this section the joint distribution of r–th and s–th concomitant of order statistics for bivariate Quasi–Gaussian distribution has been obtained. The joint distribution of two concomitants is given by (5). Now to derive the joint distribution of two concomitants we consider (6). The marginal distribution of X for (6) is given in (7). The distribution of r–th and s–th order statistics for (7) is:

$$\begin{aligned} f_{r,s:n}(x_1, x_2) &= C_{r,s:n} \phi\left(\frac{x_1-\alpha_1}{\sqrt{\alpha_1}}\right) \phi\left(\frac{x_2-\alpha_1}{\sqrt{\alpha_1}}\right) \left[\Phi\left(\frac{x_1-\alpha_1}{\sqrt{\alpha_1}}\right)\right]^{r-1} \\ &\times \left[\Phi\left(\frac{x_2-\alpha_1}{\sqrt{\alpha_1}}\right) - \Phi\left(\frac{x_1-\alpha_1}{\sqrt{\alpha_1}}\right)\right]^{s-r-1} \\ &\times \left[1 - \Phi\left(\frac{x_2-\alpha_1}{\sqrt{\alpha_1}}\right)\right]^{n-s}. \end{aligned} \tag{15}$$

Now using conditional distribution from (8) and distribution of r–th and s–th order statistics from (15), the joint distribution of r–th and s–th concomitant of order statistics is given as:

$$\begin{aligned} g_{[r,s:n]}(y_1, y_2) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\alpha_2(1-\rho^2)}} \exp\left[-\frac{1}{2(1-\rho^2)}\left\{\left(\frac{y_1-\alpha_2}{\sqrt{\alpha_2}}\right)\right.\right. \\ &\left.\left. -\rho\left(\frac{x_1-\alpha_1}{\sqrt{\alpha_1}}\right)\right\}^2\right] \times \frac{1}{\sqrt{2\pi\alpha_2(1-\rho^2)}} \\ &\times \exp\left[-\frac{1}{2(1-\rho^2)}\left\{\left(\frac{y_2-\alpha_2}{\sqrt{\alpha_2}}\right) - \rho\left(\frac{x_2-\alpha_1}{\sqrt{\alpha_1}}\right)\right\}^2\right] \end{aligned}$$

$$
\begin{aligned}
&\times C_{r,s:n}\phi\left(\frac{x_1-\alpha_1}{\sqrt{\alpha_1}}\right)\phi\left(\frac{x_2-\alpha_1}{\sqrt{\alpha_1}}\right)\left[\Phi\left(\frac{x_1-\alpha_1}{\sqrt{\alpha_1}}\right)\right]^{r-1}\\
&\times\left[\Phi\left(\frac{x_2-\alpha_1}{\sqrt{\alpha_1}}\right)-\Phi\left(\frac{x_1-\alpha_1}{\sqrt{\alpha_1}}\right)\right]^{s-r-1}\\
&\times\left[1-\Phi\left(\frac{x_2-\alpha_1}{\sqrt{\alpha_1}}\right)\right]^{n-s}dx_1dx_2.
\end{aligned}
$$

If $\rho = 0$ then the joint distribution is:

$$
\begin{aligned}
g_{[r,s:n]}\left(y_1,y_2\right) &= C_{r,s:n}\sum_{i=0}^{s-r-1}\binom{s-r-1}{i}\frac{(-1)^i}{r+i}\frac{\Gamma\left(n-s+1\right)\Gamma\left(s-r-1\right)}{\Gamma\left(n-r-i+1\right)}\\
&\quad\times\phi\left(\frac{y_1-\alpha_2}{\sqrt{\alpha_2}}\right)\phi\left(\frac{y_2-\alpha_2}{\sqrt{\alpha_2}}\right) \qquad (16)
\end{aligned}
$$

From (16) it can be readily seen that the joint distribution of r–th and s–th concomitant of order statistics for bivariate Quasi–Gaussian distribution when $\rho = 0$ is a weighted sum of marginal distributions of the Y variables.

References

[1] S. J. Chu, W. J. Huang, H. Chen, *A study of Asymptotic distribution of Concomitants of certain Order Statistics*, Statistics Sinica, 9, (1999), 811–830.

[2] H. A. David, *Multiple Comparisons, Selection and Application in Biometry*, chapter Concomitants of Order Statistics: Review and Recent Developments, Dekker, 1993, 507–518.

[3] H. A. David, M.L. Moeschberger, *The Theory of Competing Risks*, Griffin, London, 1978.

[4] H. A. David, H. Nagaraja, *Order Statistics*, 3rd Edn., John Wiley & Sons, New York, 2003.

[5] H. A. David, M. J. O'Connell, S. S. Yang, *Distribution and Expected value of the Rank of a Concomitant of an Order Statistics*, Annals Of Statistics, 5(1), (1977), 216–233.

[6] P.S. Gill, M.L. Tiku, D.C. Vanghan, *Inference Problems in Life Testing under Multivariate Normality*, J. Appl. Statist., 17,(1990), 133-147.

[7] F.E. Harrell, P. Sen, *Statistical Inference for Censored Bivariate Normal Distributions Based on Induced order Statistics*, Biometrika, 66, (1979), 293-298.

[8] H. Nagaraja, H. A. David, *Distribution of the Maximum of Concomitants of selected Order Statistics*, Annals of Statistics, 22(1), (1994), 478–494.

[9] A. S. Said, H. A. Ali, E. Hamad, *Mathematics for Chromatography. Part I: Concentration Distributions and their moments*, Journal of High Resolution Chromatography, 5(6), (1984), 306–310.

[10] X. Wang, L. Stocks, J. Lim, M. Chen, *Concomitant of Multivariate Order Statistics with application to Judgment Post–Stratification*, Technical report of University of Texas, 2006.

[11] G.A. Watterson, *Linear Estimation in Censored Samples from Multivariate Normal Populations*, Ann. Math. Statist., 30, (1959), 814-824.

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