Acta Universitatis Apulensis No. 31/2012
ISSN: 1582-5329 pp. 53-64

APPLICATIONS OF DIFFERENTIAL SUBORDINATION ON CERTAIN CLASS OF MEROMORPHIC P-VALENT FUNCTIONS ASSOCIATED WITH CERTAIN INTEGRAL OPERATOR

R. M. El-Ashwah and M. K. Aouf

Abstract. By making use of the principle of differential subordination, we investigate some interesting properties of certain subclasses of $p-$valent meromorphic functions which are defined by certain integral operator.

2000 *Mathematics Subject Classification* : 30C45.

Keywords and phrases : Meromorphic functions, p-valent, integral operator.

1. Introduction

Let $\Sigma_{p,n}$ denote the class of meromorphically multivalent functions $f(z)$ of the form:

$$f(z) = z^{-p} + \sum_{k=n}^{\infty} a_k z^k \quad (n > -p; p, n \in \mathbb{N} = \{1, 2, ...\}) \tag{1.1}$$

which are analytic in the punctured unit disc $U^* = \{z : z \in \mathbb{C} \text{ and } 0 < |z| < 1\} = U\backslash\{0\}$, we denote for $\Sigma_{p,1-p}$ by Σ_p.

For two functions f and g analytic in U, we say that f is subordinate to g, and write $f \prec g$ in U or $f(z) \prec g(z)$ $(z \in U)$, if there exists a Schwarz function $w(z)$, which is analytic in U with

$$w(0) = 0 \text{ and } |w(z)| < 1 \ \ (z \in U),$$

such that

$$f(z) = g(w(z)) \ \ (z \in U).$$

It is known that

$$f(z) \prec g(z) \Rightarrow f(0) = g(0) \text{ and } f(U) \subset g(U),$$

Furthermore, if the function g is univalent in U, then (see, [5], p.4),

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(U) \subset g(U).$$

Let $\varphi(r,s;z):\mathbb{C}^2\times U\to\mathbb{C}$ and $h(z)$ be univalent in U. If $p(z)$ is analytic in U and satisfies the first order differential subordination:

$$\varphi(p(z),zp^{'}(z);z)\prec h(z) \tag{1.2}$$

then $p(z)$ is a solution of the differential subordination (1.2). The univalent function $q(z)$ is called a dominant of the solutions if $p(z)\prec q(z)$ for all $p(z)$ satisfying (1.2). A univalent dominant $\widetilde{q}$ that satisfies $\widetilde{q}\prec q$ for all dominants of (1.2) is called the best dominant (see [5]).
For functions $f_j(z)\in\Sigma_{p,n}$, given by

$$f_j(z)=z^{-p}+\sum_{k=n}^{\infty}a_{k,j}z^k \quad (j=1,2), \tag{1.3}$$

we define the Hadamard product (or convolution) of $f_1(z)$ and $f_2(z)$ by

$$(f_1*f_2)(z)=z^{-p}+\sum_{k=n}^{\infty}a_{k,1}a_{k,2}z^k=(f_2*f_1)(z).$$

For $p\in\mathbb{N},\alpha>0,\lambda\geq 0$ and $f\in\Sigma_{p,n}$ given by (1.1), we define the following integral operator

$$J_{p,\alpha}^{0}f(z)=f(z)$$

$$J_{p,\alpha}^{\lambda}f(z)=\frac{\alpha^{\lambda}}{z^{\alpha+p}\Gamma(\lambda)}\int_0^z\left(\log\frac{z}{t}\right)^{\lambda-1}t^{\alpha+p-1}f(t)dt \quad (\lambda>0;z\in U),$$

and

$$J_{p,\alpha}f(z)=J_{p,\alpha}^{1}f(z)=\frac{\alpha}{z^{\alpha+p}}\int_0^z t^{\alpha+p-1}f(t)dt \quad (z\in U).$$

Using the elementary integral calculation, it is easy to verify that

$$J_{p,\alpha}^{\lambda}f(z)=z^{-p}+\sum_{k=n}^{\infty}\left(\frac{\alpha}{k+p+\alpha}\right)^{\lambda}a_kz^k \quad (\alpha>0,\lambda\geq 0), \tag{1.4}$$

and

$$J_{p,\alpha}f(z)=z^{-p}+\sum_{k=n}^{\infty}\left(\frac{\alpha}{k+p+\alpha}\right)a_kz^k \qquad (\alpha>0). \tag{1.5}$$

For the general integral operator $J_{p,\alpha}^{\lambda}f(z)$, it is not difficult to deduce from (1.4) that

$$z\left(J_{p,\alpha}^{\lambda}f(z)\right)' = \alpha J_{p,\alpha}^{\lambda-1}f(z) - (\alpha+p)J_{p,\alpha}^{\lambda}f(z), \qquad (\lambda \geq 1). \tag{1.6}$$

We note that

(i) For $n = 0, J_{p,1}^{\lambda}f(z) = P_p^{\lambda}f(z)$ (Aqlan et al. [1]);

(ii) $J_{1,1}^{m}f(z) = J^m f(z)$ (Uralegaddi and Somanatha [8]);

(iii) $J_{1,\alpha}^{\lambda}f(z) = P_{\alpha}^{\lambda}f(z)$ $(\alpha > 0, \lambda > 0)$ (Lashin [3]);

(iv) $J_{1,\alpha}^{1}f(z) = J_{\alpha}f(z) = \frac{1}{z} + \sum\limits_{k=1}^{\infty}\left(\frac{\alpha}{k+\alpha+1}\right)a_k z^k$ $(\alpha > 0)$.

Now we introduce the following subclass of $\Sigma_{p,n}$ associated with the integral operator $J_{p,\alpha}^{\lambda}$.

Definition 1. *For fixed parameter A, B $(-1 \leq B < A \leq 1)$ a function $f(z) \in \Sigma_{p,n}$ is said to be in the class $\sum_{p,n}^{\lambda}(\alpha,\beta,A,B)$ if*

$$-\frac{z^{p+1}}{p}\left\{(1-\beta)(J_{p,\alpha}^{\lambda}f(z))' + \beta(J_{p,\alpha}^{\lambda-1}f(z))'\right\} \prec \frac{1+Az}{1+Bz}, \tag{1.7}$$

where $p \in \mathbb{N}, \beta \geq 0, \alpha > 0$ and $\lambda \geq 1$.

In the present paper, we derive some subordination results for the function class $\sum_{p,n}^{\lambda}(\alpha,\beta,A,B)$ and investigate various other properties of functions belonging to the class $\sum_{p,n}^{\lambda}(\alpha,\beta,A,B)$. Relevant connections of the results presented in this paper with those obtained in earlier works are also pointed out.

2. Preliminaries

Lemma 1 [2]. *Let $h(z)$ be analytic and convex (univalent) in U, $h(0) = 1$, and let*

$$\varphi(z) = 1 + c_{p+n}z^{p+n} + ... \tag{2.1}$$

be analytic in U. If

$$\varphi(z) + \frac{1}{\delta}z\varphi'(z) \prec h(z),$$

then for $\delta \neq 0$ and $Re\delta \geq 0$

$$\varphi(z) \prec \psi(z) = \frac{\delta}{p+n}z^{-\frac{\delta}{p+n}}\int_0^z t^{\frac{\delta}{p+n}-1}h(t)dt \quad (z \in U) \tag{2.2}$$

and ψ is the best dominant of (2.2).

Denote by $P(\gamma)$ the class of functions $\varphi(z)$ given by

$$\varphi(z) = 1 + b_1 z + b_2 z^2 + ...,$$

which are analytic in U and satisfy the following inequality:

$$Re(\varphi(z)) > \gamma \quad (0 \leq \gamma < 1; z \in U).$$

We note that $P(0) = P$.

Lemma 2 [4]. *Let the function* $\varphi(z)$*, given by* (2.1)*, be in the class* P*. Then*

$$Re\{\varphi(z) \geq \frac{1-|z|}{1+|z|}\ (z \in U).$$

Lemma 3 [**7**]. *For* $0 \leq \gamma_1 < \gamma_2 < 1$,

$$P(\gamma_1) * P(\gamma_2) \subset P(\gamma_3) \qquad (\gamma_3 = 1 - 2(1-\gamma_1)(1-\gamma_2)).$$

For real or complex numbers a, b, c ($c \notin \mathbb{Z}_0^- = \{0, -1, -2, ...\}$), the Gaussian hypergeometric function is defined by

$${}_2F_1(a, b, c; z) = 1 + \frac{ab}{c}\frac{z}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2!} + ... \ .$$

We note that the above series converges absolutely for $z \in U$ and hence represents an analytic function in the unit disc U (see, for details, [9, Chapter 14]).

Each of the identities (asserted by Lemma 3 below) is fairly well known (cf. , e.g., [9, Chapter 14]).

Lemma 4 [9]. *For real or complex parameters* a, b, c ($c \notin \mathbb{Z}_0^-$),

$$\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}dt = \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}\,{}_2F_1(a, b; c; z)\ (Re(c) > Re(b) > 0), \tag{2.3}$$

$${}_2F_1(a, b; c; z) = (1-z)^{-a}\ {}_2F_1(a, c-b; c; \frac{z}{z-1}), \tag{2.4}$$

$${}_2F_1(1, 1; 2; \frac{1}{2}) = 2\ell n2, \tag{2.5}$$

and

$${}_2F_1(\alpha_1, \alpha_2, \tfrac{\alpha_1+\alpha_2+1}{2}; \frac{1}{2}) = \frac{\sqrt{\pi}\Gamma(\frac{\alpha_1+\alpha_2+1}{2})}{\Gamma(\frac{\alpha_1+1}{2})\Gamma(\frac{\alpha_2+1}{2})}. \tag{2.6}$$

3. Main results

Unless otherwise mentioned we shall assume throughout the paper that $p \in \mathbb{N}, \alpha, \beta > 0, \lambda \geq 1$ and $-1 \leq B < A \leq 1$.

Theorem 1. *If* $f(z) \in \sum_{p,n}^{\lambda}(\alpha, \beta, A, B)$, *then*

$$-\frac{z^{p+1}}{p}(J_{p,\alpha}^{\lambda}f(z))' \prec Q(z) \prec \frac{1+Az}{1+Bz}\,, \tag{3.1}$$

where the function $Q(z)$ *given by*

$$Q(z) = \begin{cases} \frac{A}{B} + \left(1 - \frac{A}{B}\right)(1+Bz)^{-1}\,{}_2F_1\left(1, 1, \dfrac{\alpha}{\beta(p+n)} + 1; \dfrac{Bz}{1+Bz}\right) & (B \neq 0) \\ 1 + \dfrac{\alpha}{\beta(p+n)+\alpha}Az & (B = 0) \end{cases}$$

is the best dominant of (3.1). *Furthermore*

$$Re\left(-\frac{z^{p+1}}{p}(J_{p,\alpha}^{\lambda}f(z))'\right)^{1/k} > \rho^{1/k} \quad (k \in \mathbb{N};\ z \in U), \tag{3.2}$$

where

$$\rho(\lambda, \alpha, \beta, A.B) = \begin{cases} \frac{A}{B} + \left(1 - \frac{A}{B}\right)(1-B)^{-1}\,{}_2F_1\left(1, 1, \frac{\alpha}{\beta(p+n)} + 1; \frac{B}{B-1}\right) & (B \neq 0) \\ 1 - \dfrac{\alpha}{\beta(p+n)+\alpha}A & (B = 0). \end{cases}$$

The inequality in (3.2) *is the best possible.*

Consider the function $\varphi(z)$ defined by

$$\varphi(z) = -\frac{z^{p+1}}{p}(J_{p,\alpha}^{\lambda}f(z))' \quad (z \in U). \tag{3.3}$$

Then $\varphi(z)$ is analytic in U with $\varphi(0) = 1$. Applying the identity (1.6) in (3.3) and differentiating the resulting equation with respect to z, we get

$$-\frac{z^{p+1}}{p}\left\{(1-\beta)(J_{p,\alpha}^{\lambda}f(z))' + \beta(J_{p,\alpha}^{\lambda-1}f(z))'\right\} = \varphi(z) + \frac{\beta}{\alpha}z\varphi'(z) \prec \frac{1+Az}{1+Bz}.$$

Now by using Lemma 1 for $\gamma = \dfrac{\alpha}{\beta}$, we deduce that

$$-\frac{z^{p+1}}{p}(J_{p,\alpha}^{\lambda}f(z))' \prec Q(z)$$

$$= \frac{\alpha}{\beta(p+n)} z^{-\left\{\frac{\alpha}{\beta(p+n)}\right\}} \int_0^z t^{\left\{\frac{\alpha}{\beta(p+n)}\right\}-1}\left(\frac{1+At}{1+Bt}\right)dt$$

$$= \begin{cases} \frac{A}{B} + \left(1 - \frac{A}{B}\right)(1+Bz)^{-1} {}_2F_1\left(1, 1, \frac{\alpha}{\beta(p+n)} + 1; \frac{Bz}{1+Bz}\right) & (B \neq 0), \\ 1 + \frac{\alpha}{\beta(p+n)+\alpha} Az & (B = 0), \end{cases}$$

by change of variables followed by the use of identities (2.3) and (2.4) (with $b = \frac{\alpha}{\beta(p+n)}$ and $c = b + 1$). This proves assertion (3.1) of Theorem 1.

Next, in order to prove the assertion (3.2) of Theorem 1, it suffices to show that,

$$\inf_{|z|<1} \{Re(Q(z))\} = Q(-1). \tag{3.4}$$

Indeed, for $|z| \leq r < 1$,

$$Re\left(\frac{1+Az}{1+Bz}\right) \geq \frac{1-Ar}{1-Br} \quad (|z| \leq r < 1).$$

Upon setting

$$G(s,z) = \frac{1+Asz}{1+Bsz}$$

and

$$d\nu(s) = \frac{\alpha}{\beta(p+n)} s^{\left\{\frac{\alpha}{\beta(p+n)} - 1\right\}} ds \ (0 \leq s \leq 1),$$

which is a positive measure on the closed interval [0,1], we get

$$Q(z) = \int_0^1 G(s,z) d\nu(s),$$

so that

$$Re\{Q(z)\} \geq \int_0^1 \frac{1-Asr}{1-Bsr} d\nu(s) = Q(-r) \quad (|z| \leq r < 1).$$

Letting $r \to 1^-$ in the above inequality, and using the aid of the elementary inequality

$$Re(w^{\frac{1}{k}}) \geq (Re(w))^{\frac{1}{k}} \quad (Re(w) > 0; k \in \mathbb{N}),$$

the estimate (3.2) follows.

Finally, the estimate in (3.2) is the best possible as the function $Q(z)$ is the best dominant of (3.1)

Putting $\beta = \frac{\sigma\alpha}{1-2\sigma} (0 < \sigma < \frac{1}{2})$ in Theorem 1, we obtain the following corollary.

Corollary 1. *If* $f \in \Sigma_{p,n}$ *satisfies*

$$-\frac{z^{p+1}}{p}\frac{\left[(1+\sigma(p-1))(J_{p,\alpha}^{\lambda}f(z))'+\sigma z(J_{p,\alpha}^{\lambda}f(z))''\right]}{1-2\sigma}\prec\frac{1+Az}{1+Bz},$$

then

$$-\frac{z^{p+1}}{p}\left[(J_{p,\alpha}^{\lambda}f(z))'\right]\prec Q^*(z)\prec\frac{1+Az}{1+Bz},$$

where the function $Q^*(z)$ *given by*

$$Q^*(z)=\begin{cases}\frac{A}{B}+(1-\frac{A}{B})(1+Bz)^{-1}\,{}_2F_1(1,1,\frac{1-2\sigma}{\sigma(p+n)}+1;\frac{Bz}{Bz+1}) & (B\neq 0)\\ 1-\frac{1-2\sigma}{1-2\sigma+\sigma(p+n)}Az & (B=0),\end{cases}$$

is the best dominant . Furthermore,

$$Re\left(-\frac{z^{p+1}}{p}(J_{p,\alpha}^{\lambda}f(z))'\right)>\rho^*(z\in U)$$

$$\rho^*(\sigma,A,B)=\begin{cases}\frac{A}{B}+\left(1-\frac{A}{B}\right)(1-B)^{-1}\,{}_2F_1\left(1,1;\frac{1-2\sigma}{\sigma(p+n)}+1;\frac{B}{B-1}\right) & (B\neq 0)\\ 1-\frac{1-2\sigma}{1-2\sigma+\sigma(p+n)}A & (B=0).\end{cases}$$

The result is the best possible.

Remark 1. Putting $\lambda=0$ *and* $p=1$ *in Corollary 1, we obtain the result obtained by Patel and Sahoo [6, Theorem 1].*

Putting $A=1-2\delta$ $(0\leq\delta<1)$, $B=-1$ and $\beta=\alpha$ in Theorem 1, we get the following corollary.

Corollary 2. *If* $f(z)\in\Sigma_{p,n}$ *satisfies the following inequality:*

$$Re\left\{-\frac{z^{p+1}}{p}[(2+p)\left(J_{p,\alpha}^{\lambda}f(z)\right)'+z(J_{p,\alpha}^{\lambda}f(z)'']\right\}>\delta\quad(0\leq\delta<1;z\in U)$$

then

$$Re\left\{-\frac{z^{p+1}}{p}\left(J_{p,\alpha}^{\lambda}f(z)\right)'\right\}>1+2(1-\delta)\left[\frac{1}{2}\,{}_2F_1\left(1,1;\frac{1}{(p+n)}+1;\frac{1}{2}\right)-1\right]\quad(z\in U).$$

The result is the best possible.

Remark 2. *Putting* $p=1$ *and* $n=0$ *in Corollary 2, we obtain the result obtained by Lashin [3, Corollary 2].*

Taking $A = 1 - 2\delta$ $(0 \leq \delta < 1), B = -1$ and $\beta = 2\alpha$ in Theorem 1, we obtain the following corollary.
Corollary 3. If $f(z) \in \Sigma_{p,n}$ *satisfies the following inequality*

$$Re\left\{-\frac{z^{p+1}}{p}[(5+2(p-1))\left(J^{\lambda}_{p,\alpha}f(z)\right)' + 2z(J^{\lambda}_{p,\alpha}f(z)'']\right\} > -\frac{\pi-2}{4-\pi} \qquad (z \in U),$$

then

$$Re\left\{-\frac{z^{p+1}}{p}\left(J^{\lambda}_{p,\alpha}f(z)\right)'\right\} > \left(-\tfrac{\pi}{4-\pi}\right) + \left(\tfrac{2}{4-\pi}\right)\ {}_2F_1\left(1,1;\frac{1}{(p+n)}+1;\frac{1}{2}\right) \ (z \in U).$$

The result is the best possible.
Remark 3 *Putting $p = 1$ and $n = 0$ in Corollary 3, we obtain the result obtained by Lashin [3, Corollary 3].*
Theorem 2. *If $f(z) \in \Sigma_{p,n}$ satisfies*

$$z^p\left[(1-\beta)J^{\lambda}_{p,\alpha}f(z) + \beta J^{\lambda-1}_{p,\alpha}f(z)\right] \prec \frac{1+Az}{1+Bz},$$

then

$$z^p J^{\lambda}_{p,\alpha}f(z) \prec Q(z) \prec \frac{1+Az}{1+Bz},$$

and

$$Re\left\{z^p J^{\lambda}_{p,\alpha}f(z)\right\} > \rho(\lambda,\alpha,\beta,A,B) \qquad (z \in U),$$

where Q and $\rho(\lambda,\alpha,\beta,A,B)$ are given as in Theorem 1. The result is the best possible.

The proof following by replacing $\varphi(z)$ by $z^p J^{\lambda}_{p,\alpha}f(z)$ in (3.3) and using the same lines as in the proof of Theorem 1.
Theorem 3. *Let $-1 \leq B_i < A_i \leq 1 (i = 1, 2)$. If each of the functions $f_i(z) \in \Sigma_p$ satisfies the following subordination condition*

$$(1-\beta)z^p J^{\lambda}_{p,\alpha}f_i(z) + \beta z^p J^{\lambda-1}_{p,\alpha}f_i(z) \prec \frac{1+A_i z}{1+B_i z} \qquad (i = 1, 2), \tag{3.5}$$

then

$$(1-\beta)z^p J^{\lambda}_{p,\alpha}G(z) + \beta z^p J^{\lambda-1}_{p,\alpha}G(z) \prec \frac{1+(1-2\eta)z}{1-z},$$

where

$$G(z) = J^{\lambda}_{p,\alpha}(f_1 * f_2)(z)$$

and

$$\eta = 1 - \frac{4(A_1 - B_1)(A_2 - B_2)}{(1 - B_1)(1 - B_2)} \left[1 - {}_2F_1(1, 1, \frac{\alpha}{\beta} + 1; \frac{1}{2}) \right].$$

The result is the best possible when $B_1 = B_2 = -1$.

Suppose that each of the functions $f_i(z) \in \Sigma_p (i = 1, 2)$ satisfies the condition (3.5). Then by letting

$$\varphi_i(z) = (1 - \beta) z^p J^{\lambda}_{p,\alpha} f_i(z) + \beta z^p J^{\lambda-1}_{p,\alpha} f_i(z) \ (i = 1, 2), \tag{3.6}$$

we have

$$\varphi_i(z) \in P(\gamma_i) \ \ (\gamma_i = \frac{1 - A_i}{1 - B_i}; i = 1, 2).$$

By making use of the identity (1.6) in (3.6), we observe that

$$J^{\lambda}_{p,\alpha} f_i(z) = \frac{\alpha}{\beta} z^{-p-\frac{\alpha}{\beta}} \int_0^z t^{\frac{\alpha}{\beta}-1} \varphi_i(t) dt \ \ (i = 1, 2)$$

which, in view of the definition of $G(z)$, we have,

$$J^{\lambda}_{p,\alpha} G(z) = \frac{\alpha}{\beta} z^{-p-\frac{\alpha}{\beta}} \int_0^z t^{\frac{\alpha}{\beta}-1} \varphi_0(t) dt, \tag{3.7}$$

where, for convenience,

$$\varphi_0(z) = z^p \left\{ (1 - \beta) J^{\lambda}_{p,\alpha} G(z) + \beta J^{\lambda-1}_{p,\alpha} G(z) \right\}$$

$$= \frac{\alpha}{\beta} z^{-\frac{\alpha}{\beta}} \int_0^z t^{\frac{\alpha}{\beta}-1} (\varphi_1 * \varphi_2)(t) dt. \tag{3.8}$$

Since $\varphi_i(z) \in P(\gamma_i)$ $(i = 1, 2)$ it follows from Lemma 3, that

$$(\varphi_1 * \varphi_2)(z) \in P(\gamma_3) \ \ (\gamma_3 = 1 - 2(1 - \gamma_1)(1 - \gamma_2)). \tag{3.9}$$

Now, by using (3.9) in (3.8) and then appealing to Lemma 2 and Lemma 4, we get

$$Re\{\varphi_0(z)\} = \frac{\alpha}{\beta} \int_0^1 u^{\frac{\alpha}{\beta}-1} Re(\varphi_1 * \varphi_2)(uz) du$$

$$\geq \frac{\alpha}{\beta} \int_0^1 u^{\frac{\alpha}{\beta}-1} \left(2\gamma_3 - 1 + \frac{2(1 - \gamma_3)}{1 + u\,|z|} \right) du$$

$$> \frac{\alpha}{\beta}\int_0^1 u^{\frac{\alpha}{\beta}-1}\left(2\gamma_3 - 1 + \frac{2(1-\gamma_3)}{1+u}\right) du$$

$$= 1 - \frac{4(A_1-B_1)(A_2-B_2)}{(1-B_1)(1-B_2)}\left(1 - \tfrac{\alpha}{\beta}\int_0^1 u^{\frac{\alpha}{\beta}-1}(1+u)^{-1} du\right)$$

$$= 1 - \frac{4(A_1-B_1)(A_2-B_2)}{(1-B_1)(1-B_2)}\left[1 - \frac{1}{2}\ {}_2F_1(1,1,\frac{\alpha}{\beta}+1;\frac{1}{2})\right]$$

$$= \eta \quad (z \in U).$$

When $B_1 = B_2 = -1$, we consider the functions $f_i(z) \in \Sigma_p (i = 1, 2)$, which satisfy the hypothesis (3.5) of Theorem 3 and are defined by

$$J^{\lambda}_{p,\alpha} f_i(z) = \frac{\alpha}{\beta} z^{-p-\frac{\alpha}{\beta}} \int_0^z t^{\frac{\alpha}{\beta}-1}\left(\frac{1+A_i t}{1-t}\right) dt \qquad (i = 1, 2).$$

Thus it follows from (3.8) and Lemma 4 that

$$\varphi_0(z) = \frac{\alpha}{\beta}\int_0^1 u^{\frac{\alpha}{\beta}-1}\left(1 - (1+A_1)(1+A_2) + \frac{(1+A_1)(1+A_2)}{1-uz}\right) du$$

$$= 1 - (1+A_1)(1+A_2) + (1+A_1)(1+A_2)(1-z)^{-1}_2 F_1(1,1,\frac{\alpha}{\beta}+1;\frac{z}{z-1})$$

$$\to 1 - (1+A_1)(1+A_2) + \frac{1}{2}(1+A_1)(1+A_2)_2F_1(1,1,\frac{\alpha}{\beta}+1;\frac{1}{2}) \text{ as } z \to -1,$$

which evidently completes the proof of Theorem 3.

Taking $A_i = 1 - 2\alpha_i (0 \le \alpha_i < 1)$ $B_i = -1 (i = 1, 2)$ and $\frac{\beta}{\alpha} = \tau$, in Theorem 3, we obtain the following corollary.

Corollary 4. *If the functions $f_i(z) \in \Sigma_p (i = 1, 2)$ satisfy the following inequality:*

$$Re\left\{(1+\tau p)z^p J^{\lambda}_{p,\alpha} f_i(z) + \tau z^{p+1}\left(J^{\lambda}_{p,\alpha} f_i(z)\right)'\right\} > \alpha_i \ (\alpha_i < 1,\ i = 1, 2; z \in U), \tag{3.10}$$

then

$$Re\left\{(1+\tau p)z^p (J^{\lambda}_{p,\alpha} f_1 * J^{\lambda}_{p,\alpha} f_2)(z) + \tau z^{p+1}(J^{\lambda}_{p,\alpha} f_1 * J^{\lambda}_{p,\alpha} f_2)(z))'\right\} > \gamma \ (z \in U),$$

where

$$\gamma = 1 - 4(1-\alpha_1)(1-\alpha_2)\left[1 - \frac{1}{2}\ {}_2F_1(1,1,\frac{1}{\tau}+1;\frac{1}{2})\right].$$

The result is the best possible.
Remark 4. (i) *Putting* $p = 1$ *in Corollary 4, we obtain the result obtained by Lashin [3, Corollary 4];*
(ii) Putting $\lambda = 0$ *in Corollary 4, we obtain the result obtained by Yang [10, Theorem 4].*

References

[1] E. Aqlan, J. M. Jahangiri and S. R. Kulkarni, Certain integral operators applied to meromorphic p-valent functions, J. Nat. Geom., 24(2003), 111-120.

[2] D. J. Hallenbeck and St. Ruscheweyh, Subordinations by convex functions, Proc. Amer. Math. Soc., 52(1975), 191-195.

[3] A. Y. Lashin, On certain subclass of meromorphic functions associated with certain integral operators, Comput. Math Appl., 59(2010), no.1, 524-531.

[4] T. H. MacGregor, Functions whose derivative has a positive real part, Trans. Amer. Math. Soc., 104(1962), 532-537.

[5] S. S. Miller and P. T. Mocanu, Differential subordinations : Theory and Applications, Series on Monographs and Textbooks in Pure and Appl. Math. No.225 Marcel Dekker, Inc., New York, 2000.

[6] J. Patel and P. Sahoo, On certain subclass of meromorphically univalent functions, Kyungpook Math. J., 42(2002), 121-132.

[7] J. Stankiewicz and Z. Stankiewicz, Some applications of the Hadamard convolution in the theory of functions, Ann. Univ. Mariae Curie-Sklodwska Sect. A, 40 (1986), 251-265

[8] B. A. Uralegaddi and C. Somanatha, New criteria for memorphic starlike univalent functions, Bull. Austral. Math. Soc., 43(1991), 137-140.

[9]E. T. Whittaker and G. N. Watson, A Course on Modern Analysis : An Introduction to the General Theory of Infinite Processes and of Analytic Functions; With an Account of the Principal Transcendental Functions, Fourth Edition, Cambridige Univ. Press, Cambridge ,1927.

[10] D.-G. Yang, Certain convolution operators for meromorphic functions, Southeast Asian Bull. Math., 25(2001), 175-186.

R. M. El-Ashwah
Department of Mathematics
Faculty of Science
Damietta University
New Damietta 34517, Egypt

E-mail: r_elashwah@yahoo.com
M. K. Aouf
Department of Mathematics
Faculty of Science
Mansoura University
Mansoura 35516, Egypt
E-mail: mkaouf127@yahoo.com