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A UNIFIED PRESENTATION OF CERTAIN SUBCLASSES OF P-VALENT PRESTARLIKE FUNCTIONS WITH NEGATIVE COEFFICIENTS

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Abstract. The object of the present paper is to introduce and investigate various properties and characteristics of a unified class $T^p[\alpha,\beta,\sigma]\,(0\le\alpha<p, 0\le\beta<p, p\in N=\{1,2,...\}, 0\le\sigma\le 1)$ of p-valent prestarlike functions with negative coefficients. We obtain a distortion theorem, extreme points and integral operators for functions belonging to the class $T^p[\alpha,\beta,\sigma]$. We also obtain several results for the modified Hadamard products of functions belonging to the class $T^p[\alpha,\beta,\sigma]$.

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1.Introduction

Let $A(p)$ denote the class of functions of the form:

$$f(z)=z^p+\sum_{n=1}^{\infty}a_{p+n}z^{p+n}\quad (p\in N=\{1,2,...\})\ , \tag{1.1}$$

which are analytic and p-valent in the unit disc $U=\{z:|z|<1\}$. A function $f(z)\in A(p)$ is called p-valent starlike of order $\alpha\,(0\le\alpha<p)$ if $f(z)$ satisfies the conditions

$$Re\left\{\frac{zf'(z)}{f(z)}\right\}>\alpha\quad (z\in U) \tag{1.2}$$

and

$$\int_0^{2\pi} Re\left\{\frac{zf'(z)}{f(z)}\right\}d\theta=2\pi p\quad (z\in U)\ . \tag{1.3}$$

We denote by $S^*(p,\alpha)$ the class of p-valent starlike functions of order α. The class $S^*(p,\alpha)$ was introduced by Patil and Thakare [5].

The function

$$s_\alpha^p(z) = \frac{z^p}{(1-z)^{2(p-\alpha)}} \qquad (0 \le \alpha < p; p \in N) \tag{1.4}$$

is the familiar extremal function for the class $S^*(p,\alpha)$. Setting

$$G^p(\alpha,n) = \frac{\prod_{m=2}^{n}[2(p-\alpha)+m-2]}{(n-1)!} \qquad (n \in N\backslash\{1\}; 0 \le \alpha < p)\,, \tag{1.5}$$

$s_\alpha^p(z)$ can be written in the form :

$$s_\alpha^p(z) = z^p + \sum_{n=1}^{\infty} G^p(\alpha,n+1) z^{p+n}\ . \tag{1.6}$$

Clearly, $s_\alpha^p(z) \in S^*(p,\alpha)$ and $G^p(\alpha,n+1)$ is a decreasing function in $\alpha\,(0 \le \alpha \le \frac{2p-1}{2}; p \in N)$ and satisfies

$$\lim_{n\to\infty} G^p(\alpha,n+1) = \begin{cases} \infty & (\alpha < \frac{2p-1}{2}) \\ 1 & (\alpha = \frac{2p-1}{2}) \\ 0 & (\alpha > \frac{2p-1}{2})\ . \end{cases}$$

Let $(f*g)(z)$ denote the Hadamard product (or convolution) of the functions $f(z)$ and $g(z)$, that is, if $f(z)$ is given by (1.1) and $g(z)$ is given by

$$g(z) = z^p + \sum_{n=1}^{\infty} b_{p+n} z^{p+n}\ , \tag{1.7}$$

then

$$(f*g)(z) = z^p + \sum_{n=1}^{\infty}\ a_{p+n}\, b_{p+n}\ z^{p+n}\,. \tag{1.8}$$

A function $f(z) \in A(p)$ is said to be p-valent α-prestarlike function of order β $(0 \le \alpha < p, 0 \le \beta < p, p \in N)$ if

$$(f \otimes s_\alpha^p)(z) \in S^*(p,\beta)\ , \tag{1.9}$$

where $s_\alpha^p(z)$ is defined by (1.4). We denote by $R^p(\alpha,\beta)$ the class of all p-valent α-prestarlike functions of order β. For $\alpha = \frac{2p-1}{2}, 0 \le \beta < p, p \in N, R^p(\frac{2p-1}{2},\beta) = S^*(p,\beta)$. Further let $C^p(\alpha,\beta)$ be the subclass of $A(p)$ consisting of functions $f(z)$ satisfying

$$f(z) \in C^p(\alpha,\beta) \text{ if and only if } \frac{zf'(z)}{p} \in R^p(\alpha,\beta). \tag{1.10}$$

The classes $R^p(\alpha,\beta)$ and $C^p(\alpha,\beta)$ are introduced by Aouf and Silverman [3].

Denoting by $T(p)$ the subclass of $A(p)$ consisting of functions of the form:

$$f(z) = z^p - \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \qquad (a_{p+n} \ge 0; p \in N)\,. \tag{1.11}$$

We denote by $R^p[\alpha,\beta]$ and $C^p[\alpha,\beta]$ the classes obtained by taking intersections, respectively, of the classes $R^p(\alpha,\beta)$ and $C^p(\alpha,\beta)$ with the class $T(p)$. Thus, we have

$$R^p[\alpha,\beta] = R^p(\alpha,\beta) \cap T(p), \tag{1.12}$$

and

$$C^p[\alpha,\beta] = C^p(\alpha,\beta) \cap T(p)\,. \tag{1.13}$$

The classes $R^p[\alpha,\beta]$ and $C^p[\alpha,\beta]$ are studied by Aouf and Silverman [3].

The following results for the classes $R^p[\alpha,\beta]$ and $C^p[\alpha,\beta]$ will be required in our present investigation.

Lemma 1 [3]. *Let the function $f(z)$ be defined by (1.11). Then, $f(z) \in R^p[\alpha,\beta]$ if and only if*

$$\sum_{n=1}^{\infty}(n+p-\beta)G^p(\alpha,n+1)a_{p+n} \le (p-\beta)\,. \tag{1.14}$$

Lemma 2 [3]. *Let the function $f(z)$ be defined by (1.11). Then, $f(z) \in C^p[\alpha,\beta]$ if and only if*

$$\sum_{n=1}^{\infty}(\frac{p+n}{p})\,(n+p-\beta)G^p(\alpha,n+1)a_{p+n} \le (p-\beta)\,. \tag{1.15}$$

In view of Lemma 1 and Lemma 2, it would seem to be natural to introduce and study an interesting unification of the classes $R^p[\alpha,\beta]$ and $C^p[\alpha,\beta]$. Indeed, we say that a function $f(z)$ defined by (1.11) belongs to the class $T^p[\alpha,\beta,\sigma]$, if and only if

$$\sum_{n=1}^{\infty}[1-\sigma+\sigma(\frac{p+n}{n})](n+p-\beta)G^p(\alpha,n+1)a_{p+n} \le (p-\beta)\,, \tag{1.16}$$

where $0 \leq \alpha < p, 0 \leq \beta < p, p \in N$ and $0 \leq \sigma \leq 1$.

Clearly, we have

$$T^p[\alpha, \beta, \sigma] = (1-\sigma)R^p[\alpha, \beta] + \sigma C^p[\alpha, \beta] \quad (0 \leq \sigma \leq 1), \tag{1.17}$$

so that

$$T^p[\alpha, \beta, 0] = R^p[\alpha, \beta] \text{ and } T^p[\alpha, \beta, 1] = C^p[\alpha, \beta] \,. \tag{1.18}$$

We note that :

(i) $T^1[\alpha, \beta, \sigma] = P(\alpha, \beta, \sigma)\,(0 \leq \alpha < 1, 0 \leq \beta \leq 1)$ (Raina and Srivastava [6]);

(ii) $T^1[\alpha, \beta, 0] = R[\alpha, \beta]\,(0 \leq \alpha < 1, 0 \leq \beta \leq 1)$ (Silverman and Silvia [8]), Uralegaddi and Sarangi [10], Aouf and Sâlâgean [2], Aouf et al. [1] and Srivastava and Aouf [9]);

(iii) $T^1[\alpha, \beta, 1] = C[\alpha, \beta]\,(0 \leq \alpha < 1, 0 \leq \beta \leq 1)$ (Owa and Uralegaddi [4]).

The object of this paper is to investigate various properties and characteristics of the general class $T^p[\alpha, \beta, \sigma]$. Also, we obtain several results for the modified Hadamard products of functions belonging to the class $T^p[\alpha, \beta, \sigma]$.

2. Distortion theorem

Theorem 1. *If a function $f(z)$ defined by (1.11) is in the class $T^p[\alpha, \beta, \sigma]$, then*

$$\begin{aligned} &\left\{ \frac{p!}{(p-m)!} - \frac{(p-\beta)(1+p)!}{2[1-\sigma+\sigma(\frac{p+1}{p})](1+p-\beta)(p-\alpha)(1+p-m)!}|z| \right\} |z|^{p-m} \\ \leq\ & \left| f^{(m)}(z) \right| \leq \\ &\left\{ \frac{p!}{(p-m)!} + \frac{(p-\beta)(1+p)!}{2[1-\sigma+\sigma(\frac{p+1}{p})](1+p-\beta)(p-\alpha)(1+p-m)!}|z| \right\} |z|^{p-m} \end{aligned}$$

2.1 (1)

$$\left(z \in U; 0 \leq \alpha \leq \frac{2p-1}{2}; 0 \leq \beta < p; 0 \leq \sigma \leq 1; m \in N_0 = N \cup \{0\}; p \in N; p > m\right).$$

The result is sharp for the function $f(z)$ given by

$$f(z) = z^p - \frac{(p-\beta)}{2[1-\sigma+\sigma(\frac{p+1}{p})](1+p-\beta)(p-\alpha)} z^{p+1} \quad (p \in N)\,. \tag{2.2}$$

Since $G^p(\alpha, n+1)$ defined by (1.5) is a decreasing function in $\alpha\,(0 \leq \alpha \leq \frac{2p-1}{2}; p \in N)$, then we find from (1.16) that

$$\frac{2[1-\sigma+\sigma(\frac{p+1}{p})](1+p-\beta)(p-\alpha)}{(p-\beta)(1+p)!}\sum_{n=1}^{\infty}(n+p)!a_{p+n}$$
$$\leq \sum_{n=1}^{\infty}\frac{2[1-\sigma+\sigma(\frac{p+n}{p})](n+p-\beta)G^p(\alpha,n+1)}{(p-\beta)}a_{p+n}\leq 1 \ ,$$

which readily yields

$$\sum_{n=1}^{\infty}(n+p)!a_{p+n}\leq \frac{(p-\beta)(1+p)!}{2[1-\sigma+\sigma(\frac{p+1}{p})](1+p-\beta)(p-\alpha)} \qquad (p\in N)\,. \tag{2.3}$$

Now, by differentiating both sides of (1.11) m times, we have

$$f^{(m)}(z)=\frac{p!}{(p-m)!}z^{p-m}-\sum_{n=1}^{\infty}\frac{(n+p)!}{(n+p-m)!}a_{p+n}z^{n+p-m}$$
$$(n,p\in N; m\in N_0; p>m) \tag{2.4}$$

and Theorem 1 would follows from (2.3) and (2.4).

Finally it is easy to see that, the bounds in (2.1) are attained for the function $f(z)$ fiven by (2.2).

Putting (i) $\sigma=0$ (ii) $\sigma=1$, in Theorem 1 we obtain the following consequences :

Corollary 1. *If a function $f(z)$ defined by (1.11) is in the class $R^p[\alpha,\beta]$, then*

$$\left\{\frac{p!}{(p-m)!}-\frac{(p-\beta)(1+p)!}{2(1+p-\beta)(p-\alpha)(1+p-m)!}|z|\right\}|z|^{p-m}$$
$$\leq \left|f^{(m)}(z)\right| \leq$$
$$\left\{\frac{p!}{(p-m)!}+\frac{(p-\beta)(1+p)!}{2(1+p-\beta)(p-\alpha)(1+p-m)!}|z|\right\}|z|^{p-m}2.5 \qquad (2)$$

$$(z\in U; 0\leq\alpha\leq\frac{2p-1}{2}; 0\leq\beta<p; m\in N_0; p\in N; p>m)\,.$$

The result is sharp for the function $f(z)$ given by

$$f(z)=z^p-\frac{(p-\beta)}{2(1+p-\beta)(p-\alpha)}z^{p+1} \qquad (p\in N)\ . \tag{2.6}$$

Corollary 2. *If a function $f(z)$ defined by (1.11) is in the class $C^p[\alpha,\beta]$, then*

$$\left\{\frac{p!}{(p-m)!}-\frac{(p-\beta)(1+p)!}{2(\frac{p+1}{p})(1+p-\beta)(p-\alpha)(1+p-m)!}|z|\right\}|z|^{p-m}$$
$$\leq \left|f^{(m)}(z)\right| \leq$$
$$\left\{\frac{p!}{(p-m)!}+\frac{(p-\beta)(1+p)!}{2(\frac{p+1}{p})(1+p-\beta)(p-\alpha)(1+p-m)!}|z|\right\}|z|^{p-m} \quad (3)$$

2.7

$$\left(z\in U;0\leq\alpha\leq\frac{2p-1}{2};0\leq\beta<p;m\in N_0;p\in N;p>m\right).$$

The result is sharp for the function $f(z)$ given by

$$f(z)=z^p-\frac{(p-\beta)}{2(\frac{p+1}{p})(1+p-\beta)(p-\alpha)}z^{p+1} \qquad (p\in N)\ . \tag{2.8}$$

3. Extreme points

Theorem 2. *The class $T^p[\alpha,\beta,\sigma]$ is closed under convex linear combinations.*

Let the functions

$$f_j(z)=z^p-\sum_{n=1}^{\infty}a_{p+n,j}z^{p+n} \qquad (a_{p+n,j}\geq 0;j=1,2) \tag{3.1}$$

be in the class $T^p[\alpha,\beta,\sigma]$. Then it is sufficient to show that the function $h(z)$ defined by

$$h(z)=tf_1(z)+(1-t)f_2(z) \qquad (0\leq t\leq 1)\ , \tag{3.2}$$

is also in the class $T^p[\alpha,\beta,\sigma]$. Since, for $0\leq t\leq 1$,

$$h(z)=z^p-\sum_{n=1}^{\infty}\{ta_{p+n,1}+(1-t)a_{p+n,2}\}\,z^{p+n}\ , \tag{3.3}$$

with the aid of (1.16), we have

$$\sum_{n=1}^{\infty}\left[1-\sigma+\sigma(\frac{p+n}{p})\right](n+p-\beta)G^p(\alpha,n+1)\{ta_{p+n,1}+(1-t)a_{p+n,2}\}$$
$$\leq(p-\beta) \qquad (0\leq t\leq 1)\ , \tag{3.4}$$

which implies that $h(z) \in T^p[\alpha, \beta, \sigma]$.

As a consequence of Theorem 2, there exist the extreme points of the class $T^p[\alpha, \beta, \sigma]$.

Theorem 3. *Let*

$$f_p(z) = z^p \tag{3.5}$$

and

$$f_{p+n}(z) = z^p - \frac{(p-\beta)}{[1-\sigma+\sigma(\frac{p+n}{p})](n+p-\beta)G^p(\alpha, n+1)} z^{p+n} \quad (p, n \in N). \tag{3.6}$$

Then, $f(z) \in T^p[\alpha, \beta, \sigma]$ if and only if it can be expressed in the form

$$f(z) = \sum_{n=0}^{\infty} \mu_{p+n} f_{p+n}(z)\,, \tag{3.7}$$

where $\mu_{p+n} \geq 0$ and $\sum\limits_{n=0}^{\infty} \mu_{p+n} = 1$.

Suppose that

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} \mu_{p+n} f_{p+n}(z) \\ &= z^p - \sum_{n=1}^{\infty} \frac{(p-\beta)}{[1-\sigma+\sigma(\frac{p+n}{p})](n+p-\beta)G^p(\alpha, n+1)} \mu_{p+n} z^{p+n}\,. \\ & 3.8 \end{aligned} \tag{4}$$

Then it follows that

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{[1-\sigma+\sigma(\frac{p+n}{p})](n+p-\beta)G^p(\alpha, n+1)}{(p-\beta)}\,. \\ & . \frac{(p-\beta)}{[1-\sigma+\sigma(\frac{p+n}{p})](n+p-\beta)G^p(\alpha, n+1)} \mu_{p+n} \\ & = \sum_{n=1}^{\infty} \mu_{p+n} = 1 - \mu_p \leq 1\,. \end{aligned} \tag{3.9}$$

Therefore, by (1.16), $f(z) \in T^p[\alpha, \beta, \sigma]$.

Conversely, assume that the function $f(z)$ defined by (1.11) belongs to the class $T^p[\alpha, \beta, \sigma]$. Then

$$a_{p+n} \leq \frac{(p-\beta)}{[1-\sigma+\sigma(\frac{p+n}{p})](n+p-\beta)G^p(\alpha, n+1)} \qquad (p, n \in N)\,. \tag{3.10}$$

Setting

$$\mu_{p+n} = \frac{[1-\sigma+\sigma(\frac{p+n}{p})](n+p-\beta)G^p(\alpha,n+1)}{(p-\beta)} a_{p+n} \qquad (p,n \in N) \tag{3.11}$$

and

$$\mu_p = 1 - \sum_{n=1}^{\infty} \mu_{p+n} \,. \tag{3.12}$$

Hence, we can see that $f(z)$ can be expressed in the form (3.7). This completes the proof of Theorem 3.

Corollary 3. *The extreme points of the class $T^p[\alpha,\beta,\sigma]$ are the functions $f_p(z)$ and $f_{p+n}(z)$ given by (3.5) and (3.6), respectively.*

4.Integral operators

Theorem 4. *Let the function $f(z)$ defined by* (1.11) *be in the class $T^p[\alpha,\beta,\sigma]$, and let c be a real number such that $c > -p$. Then the function $F(z)$ defined by*

$$F(z) = \frac{c+p}{z^c} \int_0^z t^{c-1} f(t)\, dt \tag{4.1}$$

also belongs to the class $T^p[\alpha,\beta,\sigma]$.

From (1.11) and the representation (4.1) of $F(z)$, it follows that

$$F(z) = z^p - \sum_{n=1}^{\infty} b_{p+n}\, z^{p+n}, \tag{4.2}$$

where

$$b_{p+n} = (\frac{c+p}{c+p+n}) a_{p+n} \,.$$

Therefore, we have

$$\begin{aligned} &\sum_{n=1}^{\infty}[1-\sigma+\sigma(\frac{p+n}{p})](n+p-\beta)G^p(\alpha,n+1)b_{p+n} \\ = &\sum_{n=1}^{\infty}[1-\sigma+\sigma(\frac{p+n}{p})](n+p-\beta)G^p(\alpha,n+1)(\frac{c+p}{c+p+n})a_{p+n} \\ \le &\sum_{n=1}^{\infty}[1-\sigma+\sigma(\frac{p+n}{p})](n+p-\beta)G^p(\alpha,n+1)a_{p+n} \le (p-\beta)\,, \end{aligned}$$

since $f(z) \in T^p[\alpha, \beta, \sigma]$. Hence, by (1.16), $F(z) \in T^p[\alpha, \beta, \sigma]$.

Theorem 5. *Let the function* $F(z) = z^p - \sum_{n=1}^{\infty} a_{p+n} z^{p+n} (a_{p+n} \geq 0)$ *be in the class* $T^p[\alpha, \beta, \sigma]$ *and let* c *be a real number such that* $c > -p$. *Then the function* $f(z)$ *involved in (4.1) is p-valent in* $|z| < R_p^*$, *where*

$$R_p^* = \inf_n \left\{ \frac{p(c+p)[1-\sigma+\sigma(\frac{p+n}{p})](n+p-\beta)G^p(\alpha, n+1)}{(p+n)(c+p+n)(p-\beta)} \right\}^{\frac{1}{n}} \quad (n \in N) \ . \tag{4.3}$$

The result is sharp.

From (4.1), we have

$$\begin{aligned} f(z) &= \frac{z^{1-c}[z^c F(z)]'}{(c+p)} \quad (c > -p) \\ &= z^p - \sum_{n=1}^{\infty} (\frac{c+p+n}{c+p}) a_{p+n} z^{p+n} \ . \end{aligned}$$

To prove the result, it sufficies to show that

$$\left| \frac{f'(z)}{z^{p-1}} - p \right| \leq p \quad \text{for} \quad |z| < R_p^* \ ,$$

where R_p^* is defined by (4.3). Now

$$\begin{aligned} \left| \frac{f'(z)}{z^{p-1}} - p \right| &= \left| - \sum_{n=1}^{\infty} (p+n)(\frac{c+p+n}{c+p}) a_{p+n} z^n \right| \\ &\leq \sum_{n=1}^{\infty} (p+n)(\frac{c+p+n}{c+p}) a_{p+n} |z|^n \ . \end{aligned}$$

Thus $\left| \frac{f'(z)}{z^{p-1}} - p \right| \leq p$ if

$$\sum_{n=1}^{\infty} (\frac{p+n}{p})(\frac{c+p+n}{c+p}) a_{p+n} |z|^n \leq 1 \ . \tag{4.4}$$

But (1.16) confirms that

$$\sum_{n=1}^{\infty} \frac{[1-\sigma+\sigma(\frac{p+n}{p})](n+p-\beta)G^p(\alpha, n+1)}{(p-\beta)} a_{p+n} \leq 1 \ . \tag{4.5}$$

Thus (4.4) will be satisfied if

$$\frac{(p+n)(c+p+n)}{p(c+p)}|z|^n \leq \frac{[1-\sigma+\sigma(\frac{p+n}{p})](n+p-\beta)G^p(\alpha,n+1)}{(p-\beta)}$$

or if

$$|z| \leq \left\{\frac{p(c+p)[1-\sigma+\sigma(\frac{p+n}{p})](n+p-\beta)G^p(\alpha,n+1)}{(p+n)(c+p+n)(p-\beta)}\right\}^{\frac{1}{n}} \qquad (n \in N)\ . \qquad (4.6)$$

The required result follows now from (4.6). The result is sharp for the function $f(z)$ in the form:

$$f(z) = z^p - \frac{(c+p+n)(p-\beta)}{(c+p)[1-\sigma+\sigma(\frac{p+n}{p})](n+p-\beta)G^p(\alpha,n+1)} z^{p+n} \qquad (p,n \in N)\ . \qquad (4.7)$$

5. Modified Hadamard products

Let the functions $f_j(z)(j=1,2)$ be defined by (3.1). Then the modified Hadamard product of $f_1(z)$ and $f_2(z)$ is defined by

$$(f_1 * f_2)(z) = z^p - \sum_{n=1}^{\infty} a_{p+n,1}a_{p+n,2}\ z^{p+n}. \qquad (5.1)$$

Throughout this section, we assume that $0 \leq \alpha \leq \frac{2p-1}{2}$, $0 \leq \beta < p$, $0 \leq \sigma \leq 1$ and $p,n \in N$.

Theorem 6. *Let the functions* $f_j(z)(j=1,2)$ *defined by* (3.1) *be in the class* $T^p[\alpha,\beta,\sigma]$. *Then* $(f_1 * f_2)(z) \in T^p[\alpha,\gamma,(\alpha,\beta,\sigma),\sigma]$, *where*

$$\gamma(\alpha,\beta,p,\sigma) = p - \frac{(p-\beta)^2}{2[1-\sigma+\sigma(\frac{p+1}{p})](1+p-\beta)^2(p-\alpha)-(p-\beta)^2}\ . \qquad (5.2)$$

The result is sharp.

. Employing the technique used earlier by Schild and Silverman [7], we need to find the largest $\gamma = \gamma(\alpha,\beta,p,\sigma)$ such that

$$\sum_{n=1}^{\infty} \frac{[1-\sigma+\sigma(\frac{p+n}{p})](n+p-\gamma)G^p(\alpha,n+1)}{(p-\gamma)} a_{p+n,1}\ a_{p+n,2} \leq 1\ . \qquad (5.3)$$

Since $f_j(z) \in T^p[\alpha, \beta, \sigma]$ $(j = 1, 2)$, we readily see that

$$\sum_{n=1}^{\infty} \frac{[1-\sigma+\sigma(\frac{p+n}{p})](n+p-\gamma)G^p(\alpha, n+1)}{(p-\beta)} a_{p+n,1} \leq 1 \tag{5.4}$$

and

$$\sum_{n=1}^{\infty} \frac{[1-\sigma+\sigma(\frac{p+n}{p})](n+p-\gamma)G^p(\alpha, n+1)}{(p-\beta)} a_{p+n,2} \leq 1\,. \tag{5.5}$$

Therefore, by the Cauchy - Schwarz inequality, we obtain

$$\sum_{n=1}^{\infty} \frac{[1-\sigma+\sigma(\frac{p+n}{p})](n+p-\gamma)G^p(\alpha, n+1)}{(p-\beta)} \sqrt{a_{p+n,1}\ a_{p+n,2}} \leq 1. \tag{5.6}$$

Thus it is sufficient to show that

$$\frac{[1-\sigma+\sigma(\frac{p+n}{p})](n+p-\gamma)G^p(\alpha, n+1)}{(p-\gamma)} a_{p+n,1}\ a_{p+n,2}$$

$$\leq \frac{[1-\sigma+\sigma(\frac{p+n}{p})](n+p-\gamma)G^p(\alpha, n+1)}{(p-\beta)} \sqrt{a_{p+n,1}\ a_{p+n,2}}\ , \tag{5.7}$$

or, equivalently, that

$$\sqrt{a_{p+n,1}\ a_{p+n,2}} \leq \frac{(p-\gamma)(n+p-\beta)}{(p-\beta)(n+p-\gamma)} \qquad (n \in N)\,. \tag{5.8}$$

Note that

$$\sqrt{a_{p+n,1}\ a_{p+n,2}} \leq \frac{(p-\beta)}{[1-\sigma+\sigma(\frac{p+n}{p})](n+p-\beta)G^p(\alpha, n+1)}\ . \tag{5.9}$$

Consequently, we need only to prove that

$$\frac{(p-\beta)}{[1-\sigma+\sigma(\frac{p+n}{p})](n+p-\beta)G^p(\alpha, n+1)} \leq \frac{(p-\gamma)(n+p-\beta)}{(p-\beta)(n+p-\gamma)}\,, \tag{5.10}$$

or, equivalently , that

$$\gamma \leq p - \frac{n(p-\beta)^2}{[1-\sigma+\sigma(\frac{p+n}{p})](n+p-\beta)^2 G^p(\alpha, n+1) - (p-\beta)^2}\,. \tag{5.11}$$

Since

$$A(n) = p - \frac{n(p-\beta)^2}{[1-\sigma+\sigma(\frac{p+n}{p})](n+p-\beta)^2 G^p(\alpha, n+1) - (p-\beta)^2} \tag{5.12}$$

is an increasing function of n for $0 \leq \alpha \leq \dfrac{2p-1}{2}$, $0 \leq \beta < p, 0 \leq \sigma \leq 1$ and $p \in N$, letting $n = 1$ in (5.12), we obtain

$$\gamma \leq A(1) = p - \frac{(p-\beta)^2}{2[1-\sigma+\sigma(\frac{p+n}{p})](1+p-\beta)^2(p-\alpha)-(p-\beta)^2} \tag{5.13}$$

which completes the proof of Theorem 6.

Finally , by taking the functions

$$f_j(z) = z^p - \frac{(p-\beta)}{2[1-\sigma+\sigma(\frac{p+n}{p})](1+p-\beta)(p-\alpha)} z^{p+1} \qquad (j = 1, 2) \tag{5.14}$$

we can see that the result is sharp.

Corollary 4. *For* $f_j(z)$ $(j = 1, 2)$ *as in Theorem 6, we have*

$$h(z) = z^p -_{n=1}^{\infty} \sqrt{a_{p+n,1}\ a_{p+n,2}}\, z^{p+n} \tag{5.15}$$

belongs to the class $T^p[\alpha, \beta, \sigma]$.

The result follows from the inequality (5.6). It is sharp for the same functions as in Theorem 6.

Theorem 7. *Let the function* $f_1(z)$ *defined by (3.1) be in the class* $T^p[\alpha, \beta, \sigma]$ *and the function* $f_2(z)$ *defined by (3.1) be in the class* $T^p[\alpha, \tau, \sigma]$*. Then* $(f_1 * f_2)(z) \in T^p[\alpha, \xi(\alpha, \beta, \tau, p, \sigma), \sigma]$*, where*

$\xi(\alpha, \beta, \tau, p, \sigma) = p-$

$$\frac{(p-\beta)(p-\tau)}{2[1-\sigma+\sigma(\frac{p+1}{p})](1+p-\beta)(1+p-\tau)(p-\alpha)-(p-\beta)(p-\tau)}. \tag{5.16}$$

The result is sharp.

Proceeding as in the proof of Theorem 6, we get

$\xi \leq B(n) = p-$

$$\frac{n(p-\beta)(p-\tau)}{[1-\sigma+\sigma(\frac{p+n}{p})](n+p-\beta)(n+p-\tau)G^p(\alpha, n+1)-(p-\beta)(p-\tau)}. \tag{5.17}$$

Since the function $B(n)$ is an increasing function of $n(n \in N)$ for $0 \leq \alpha \leq \dfrac{2p-1}{2}$, $0 \leq \beta < p, 0 \leq \tau < p, 1 \leq \sigma \leq 1$ and $p \in N$, letting $n = 1$ in (5.17), we obtain

$\xi \leq B(1) = p-$

$$\frac{(p-\beta)(p-\tau)}{2[1-\sigma+\sigma(\frac{p+1}{p})](1+p-\beta)(1+p-\tau)(p-\alpha)-(p-\beta)(p-\tau)} \tag{5.18}$$

which evidently proves Theorem 7.

Finally, the result is the best possible for the functions

$$f_1(z) = z^p - \frac{(p-\beta)}{2[1-\sigma+\sigma(\frac{p+1}{p})](1+p-\beta)(p-\alpha)} z^{p+1} \tag{5.19}$$

and

$$f_2(z) = z^p - \frac{(p-\tau)}{2[1-\sigma+\sigma(\frac{p+1}{p})](1+p-\tau)(p-\alpha)} z^{p+1}\,. \tag{5.20}$$

Corollary 5. *Let the functions $f_j(z)\,(j=1,2,3)$ defined by (3.1) be in the class $T^p[\alpha,\beta,\sigma]$. Then $(f_1 * f_2 * f_3)(z) \in T^p[\alpha,\eta(\alpha,\beta,p,\sigma),\sigma]$, where*

$\eta(\alpha,\beta,p,\sigma) = p-$

$$\frac{(p-\beta)^3}{4[1-\sigma+\sigma(\frac{p+1}{p})](1+p-\beta)^3(p-\alpha)^2-(p-\beta)^3}\,. \tag{5.21}$$

The result is the best possible for the functions

$$f_j(z) = z^p - \frac{(p-\beta)}{2[1-\sigma+\sigma(\frac{p+1}{p})](1+p-\beta)(p-\alpha)} z^{p+1} \qquad (j=1,2,3; p\in N)\,. \tag{5.22}$$

From Theorem 6, we have $(f_1 * f_2)(z) \in T^p[\alpha,\gamma(\alpha,\beta,p,\sigma),\sigma]$, where γ is given by (5.2). Using now Theorem 7, we get $(f_1 * f_2 * f_3)(z) \in R^p[\alpha,\eta(\alpha,\beta,p,\sigma),\sigma]$, where

$\eta((\alpha,\beta,p,\sigma) = p-$

$$\begin{aligned} & \frac{(p-\beta)(p-\gamma)}{2[1-\sigma+\sigma(\frac{p+1}{p})](1+p-\beta)(1+p-\gamma)(p-\alpha)-(p-\beta)(p-\gamma)} \\ = & \frac{(p-\beta)^3}{4[1-\sigma+\sigma(\frac{p+1}{p})](1+p-\beta)^3(p-\alpha)^2-(p-\beta)^3}\,. \end{aligned}$$

This completes the proof of Corollary 5.

Theorem 8. *Let the functions $f_j(z)\,(j=1,2)$ defined by (3.1) be in the class $T^p[\alpha,\beta,\sigma]$. Then the function $h(z)$ defined by*

$$h(z)=z^p-\sum_{n=1}^{\infty}(a_{p+n,1}^2+a_{p+n,2}^2)z^{p+n} \tag{5.23}$$

belongs to the class $R^p[\alpha,\varphi(\alpha,\beta,p,\sigma),\sigma]$, where
$\varphi(\alpha,\beta,p,\sigma)=p-$

$$\frac{(p-\beta)^2}{[1-\sigma+\sigma(\frac{p+1}{p})](1+p-\beta)^2(p-\alpha)-(p-\beta)^2}\,. \tag{5.24}$$

The result is sharp for the functions $f_j(z)\,(j=1,2)$ defined by (5.14).

By virture of (1.16), we obtain

$$\sum_{n=1}^{\infty}\left\{\frac{[1-\sigma+\sigma(\frac{p+n}{p})](n+p-\beta)G^p(\alpha,n+1)}{(p-\beta)}\right\}^2 a_{p+n,1}^2$$

$$\leq\left\{\sum_{n=1}^{\infty}\frac{[1-\sigma+\sigma(\frac{p+n}{p})](n+p-\beta)G^p(\alpha,n+1)}{(p-\beta)}a_{p+n,1}\right\}^2\leq 1 \tag{5.25}$$

and

$$\sum_{n=1}^{\infty}\left\{\frac{[1-\sigma+\sigma(\frac{p+n}{p})](n+p-\beta)G^p(\alpha,n+1)}{(p-\beta)}\right\}^2 a_{p+n,2}^2$$

$$\leq\left\{\sum_{n=1}^{\infty}\frac{[1-\sigma+\sigma(\frac{p+n}{p})](n+p-\beta)G^p(\alpha,n+1)}{(p-\beta)}a_{p+n,2}\right\}^2\leq 1\,. \tag{5.26}$$

It follows from (5.25) and (5.26) that

$$\sum_{n=1}^{\infty}\frac{1}{2}\left\{\frac{[1-\sigma+\sigma(\frac{p+n}{p})](n+p-\beta)G^p(\alpha,n+1)}{(p-\beta)}\right\}^2(a_{p+n,1}^2+a_{p+n,2}^2)\leq 1\,. \tag{5.27}$$

Therefore, we need to find the largest φ such that

$$\frac{[1-\sigma+\sigma(\frac{p+n}{p})](n+p-\varphi)G^p(\alpha,n+1)}{(p-\varphi)}$$

$$\leq \frac{1}{2}\left\{\frac{[1-\sigma+\sigma(\frac{p+n}{p})](n+p-\beta)G^p(\alpha,n+1)}{(p-\beta)}\right\}^2, \quad (5.28)$$

that is, that

$$\varphi \leq p - \frac{2n(p-\beta)^2}{[1-\sigma+\sigma(\frac{p+n}{p})](n+p-\beta)^2G^p(\alpha,n+1)-2(p-\beta)^2}. \quad (5.29)$$

Since

$$D(n) = p - \frac{2n(p-\beta)^2}{[1-\sigma+\sigma(\frac{p+n}{p})](n+p-\beta)^2G^p(\alpha,n+1)-2(p-\beta)^2}.$$

is an increasing function of $n (n \in N)$ for $0 \leq \alpha \leq \dfrac{2p-1}{2}, 0 \leq \beta < p, 0 \leq \sigma \leq 1$ and $p \in N$, we readily have

$$\varphi \leq D(1) = p - \frac{(p-\beta)^2}{[1-\sigma+\sigma(\frac{p+1}{p})](1+p-\beta)^2(p-\alpha)^2-(p-\beta)^2}, \quad (5.30)$$

which completes the proof of Theorem 8.

Remarks. (i) Putting $\sigma = 0$ in Theorem 6, we obtain the result obtained by Aouf and Silverman [3, Theorem 12];

(ii) Putting $\sigma = 1$ in Theorem 6, we obtain the result obtained by Aouf and Silverman [3, Corollary 9];

(iii) Putting $\sigma = 0$ and $\sigma = 1$ in Theorem 7, Corollary 5 and Theorem 8, respectively, we obtain the corresponding results for the classes $R^p[\alpha,\beta]$ and $C^p[\alpha,\beta]$, respectively.

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