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THE APPLICATION OF THE ORDINARY DIFFERENTIAL EQUATION IN SOLVING INDEFINITE INTEGRATION

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Abstract. In this paper, we give a new method of solving some integration to avoid complicated calculation, that is, transforming the integration to a corresponding ordinary differential equation, and using the method of setting undetermined coefficient to solve this differential equation.

1. Introduction

In this paper, we consider the solution of following integration

$$\int p_m(t)e^{\alpha t}cos\beta tdt \tag{1}$$

When study the chapter how to solve the integration, we can solve (1) by the method of integrating by parts. However, this needs more than two steps at least,we will feel too tedious. Now we introduce a new method to solve this type integration ,that is, transforming the form of (1) to a corresponding differential equation, then using the method of setting undetermined coefficient to solve this differential equation.

The paper is organized as follows. In Section 2, we discuss the steps of our main new method of solving (1). In Section 3, some examples are given to illustrate our method is applicative.

2.Main method

The steps of solving (1) as follows:
Let

$$x = \int p_m(t)e^{\alpha t}cos\beta tdt \tag{2}$$

then

$$x' = p_m(t)e^{\alpha t}cos\beta t \tag{3}$$

Obviously, solving problem (2) is equivalent to solving problem (3). Now we only need to solve (3) by the method of setting undetermined coefficient.
In order to solve problem (3), we consider the auxiliary equation

$$x' = p_m(t)e^{\alpha+\beta i}t, \tag{4}$$

so we get the characteristic equation of (4) $F(\lambda) = \lambda = 0$,
then set the certain solution of (4)

$$\bar{x} = g(t)e^{\alpha+\beta i}t, \tag{5}$$

where $g(t)$ is undetermined polynomial.
In view of equations (4) and (5), we have $(\alpha + \beta i)g(t) + g'(t) \equiv p_m$, so it is easy to get the value of $g(t)$ by the method of the same with the power coefficient, thus (5) affirmed.
Then we obtain the certain solution of (3) by the knowledge of ordinary differential equation $\bar{x}_1 = Re(\bar{x})$ Consequently, the general solution of (3) is

$$x = \bar{x}_1 + c \tag{6}$$

in which c is a arbitrary constant.
Therefore, (6) is also the solution of (2), i.e. $\int p_m(t)e^{\alpha t}cos\beta tdt = \bar{x}_1 + c$ The result is obtained.

Obviously, we can use this method to solve problems as follows
(a) $\int p_m(t)e^{\alpha t}dt$; (b) $\int p_m(t)cos\beta tdt$;
(c) $\int e^{\alpha t}cos\beta tdt$; (d) $\int p_m(t)e^{\alpha t}sin\beta tdt$.
(a),(b),(c),(d) are concrete types or out of shapes of (1), respectively. In addition, there are some integrations also can be transformed one of the types of above in order to use our new method.

3.Some examples

In this section, we will give four examples, they are the common types of integration.

Example 1 Solve integration $\int(t^2 - 2t + 3)cos2tdt$.

This is a concrete integration, we can get a concrete result of it.
Set $x = \int(t^2 - 2t + 3)cos2tdt$, then

$$x' = (t^2 - 2t + 3)cos2t. \tag{7}$$

Consider the auxiliary equation

$$x' = (t^2 - 2t + 3)e^{2it} \tag{8}$$

we get the characteristic equation of (8) $F(\lambda) = \lambda = 0$.
Then set the certain solution of (8)

$$\bar{x} = (B_0t^2 + B_1t + B_2)e^{2it} \tag{9}$$

where B_0, B_1, B_2 are undetermined coefficients.
By equation (8) and (9), we have

$$2i(B_0t^2 + B_1t + B_2) + 2B_0t + B_1 \equiv t^2 - 2t + 3$$

then the values B_0, B_1, B_2 are obtained,

$$B_0 = -\frac{i}{2}, B_1 = \frac{1+2i}{2}, B_2 = \frac{-2+5i}{4}$$

So $\bar{x} = (-\frac{i}{2}t^2) + \frac{1+2i}{2}t + \frac{-2+5i}{4}e^{2it}$ Thus the certain solution of (7)

$$\bar{x}_1 = Re(\bar{x}) = (\frac{1}{2}t - \frac{1}{2})cos2t + (\frac{1}{2}t^2 + t + \frac{5}{4})sin2t$$

Consequently, the general solution of (7)

$$x = \bar{x}_1 + c = (\frac{1}{2}t - \frac{1}{2})cos2t + (\frac{1}{2}t^2 + t + \frac{5}{4})sin2t + c$$

in which c is a arbitrary constant.
Therefore,

$$\int (t^2 - 2t + 3)cos2tdt = (\frac{1}{2}t - \frac{1}{2})cos2t + (\frac{1}{2}t^2 + t + \frac{5}{4})sin2t + c.$$

Example 2 Solve integration $\int e^{nt}cosmtdt$.

Set

$$x = \int e^{nt}cosmtdt,$$

then

$$x' = e^{nt}cosmt \tag{10}$$

Consider the auxiliary equation

$$x' = e^{(n+mi)t} \tag{11}$$

we get the characteristic equation of (11) $F(\lambda) = \lambda = 0$,
Then set the certain solution of (11)

$$\bar{x} = B_0 e^{(n+mi)t} \tag{12}$$

where B_0 is undetermined coefficient.
By equation (11) and (12), we have

$$(n + mi)B_0 \equiv 1$$

then the values $B_0 = \frac{n-mi}{n^2+m^2}$ So

$$\bar{x} = \frac{n - mi}{n^2 + m^2} e^{(n+mi)t}$$

Thus the certain solution of (10)

$$\bar{x}_1 = Re(\bar{x}) = e^{nt}(\frac{n}{n^2 + m^2} cosmt + \frac{m}{n^2 + m^2 sinmt})$$

Consequently, the general solution of (10)

$$x = \bar{x}_1 + c = e^{nt}(\frac{n}{n^2 + m^2} cosmt + \frac{m}{n^2 + m^2 sinmt}) + c$$

in which c is a arbitrary constant.
Therefore,

$$\int e^{nt} cosmtdt = e^{nt}(\frac{n}{n^2 + m^2} cosmt + \frac{m}{n^2 + m^2 sinmt}) + c.$$

Example 3 Solve integration $\int t^2 e^t cos3tdt$.

Let

$$x = \int t^2 e^t cos3tdt,$$

then

$$x' = \int t^2 e^t cos3t \tag{13}$$

Consider the auxiliary equation

$$x' = t^2 e^{(1+3i)t} \tag{14}$$

we get the characteristic equation of (14) $F(\lambda) = \lambda = 0$.
Then set the certain solution of (14)

$$\bar{x} = (B_0 t^2 + B_1 t + B_2) e^{(1+3i)t} \tag{15}$$

where B_0, B_1, B_2 are undetermined coefficients.
By equation (14) and (15), we have

$$(1+3i)(B_0t^2+B_1t+B_2)+2B_0t+B_1 \equiv t^2$$

then the values

$$B_0=\frac{1-3i}{10}, B_1=\frac{4+3i}{25}, B_2=\frac{-13+9i}{250}$$

So

$$\bar{x}=(\frac{1-3i}{10}t^2+\frac{4+3i}{25}t+\frac{-13+9i}{250})e^{(1+3i)t}$$

Thus the certain solution of (10)

$$\bar{x}_1=Re(\bar{x})=e^t[(\frac{1}{10}t^2+\frac{4}{25}t-\frac{13}{250})cos3t+(\frac{3}{10}t^2+\frac{3}{25}t-\frac{9}{250})sin3t]$$

Consequently, the general solution of (10)

$$x=\bar{x}_1+c=e^t[(\frac{1}{10}t^2+\frac{4}{25}t-\frac{13}{250})cos3t+(\frac{3}{10}t^2+\frac{3}{25}t-\frac{9}{250})sin3t]+c$$

in which c is a arbitrary constant.
Therefore,

$$\int t^2e^tcos3tdt=e^t[(\frac{1}{10}t^2+\frac{4}{25}t-\frac{13}{250})cos3t+(\frac{3}{10}t^2+\frac{3}{25}t-\frac{9}{250})sin3t]+c.$$

Example 4 Solve integration $\int t^2e^tsin3tdt$.

Let

$$x=\int t^2e^tsin3tdt,$$

then

$$x'=t^2e^tsin3t \tag{16}$$

Consider the auxiliary equation

$$x'=t^2e^{(1+3i)t} \tag{17}$$

we get the characteristic equation of (17)

$$F(\lambda)=\lambda=0,$$

Then set the certain solution of (17)

$$\bar{x}=(B_0t^2+B_1t+B_2)e^{(1+3i)t} \tag{18}$$

where B_0, B_1, B_2 are undetermined coefficients.
By equation (17) and (18), we have

$$(1+3i)(B_0t^2+B_1t+B_2)+2B_0t+B_1 \equiv t^2$$

then the values

$$B_0=\frac{1-3i}{10}, B_1=\frac{4+3i}{25}, B_2=\frac{-13+9i}{250}$$

So

$$\bar{x}=(\frac{1-3i}{10}t^2+\frac{4+3i}{25}t+\frac{-13+9i}{250})e^{(1+3i)t}$$

Thus the certain solution of (10)

$$\bar{x}_1=Im(\bar{x})=e^t[(\frac{1}{10}t^2+\frac{4}{25}t-\frac{13}{250})sin3t+(\frac{3}{10}t^2+\frac{3}{25}t-\frac{9}{250})cos3t]$$

Consequently, the general solution of (10)

$$x=\bar{x}_1+c=e^t[(\frac{1}{10}t^2+\frac{4}{25}t-\frac{13}{250})cos3t+(\frac{3}{10}t^2+\frac{3}{25}t-\frac{9}{250})sin3t]+c$$

in which c is a arbitrary constant.
Therefore,

$$\int t^2e^tsin3tdt=e^t[(\frac{1}{10}t^2+\frac{4}{25}t-\frac{13}{250})sin3t+(\frac{3}{10}t^2+\frac{3}{25}t-\frac{9}{250})cos3t]+c.$$

From all of above examples we can know that the new method introduced in this paper is simple to solve some types of integration, it simplify the calculation.

References

[1] Vladimir Igorevich Arnol'd, Mark Levi. Geometrical methods in the theory of ordinary differential equations , 1998.

[2] Vladimir Igorevich Arnol'd. Ordinary differential equations , 1992.

[3] Philip E. Protter. Stochastic integration and differential equations . 2004.

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