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SOME RESULTS ON THE GENERALIZATION OF BERNOULLI, EULER AND GENOCCHI POLYNOMIALS

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Abstract. The present paper deals with generalization of Bernoulli , Euler and Genocchi polynomials. In this paper we study some relations between generalized Bernoulli polynomials with a, b parameters and Euler polynomials with a, b parameters with the methods of generating function and series rearrangement and we derive some basic properties and formulas and consider some interesting applications of generalized Bernoulli, Euler and Genocchi polynomials. Also, we derive multiplication formula related to generalized Genocchi polynomials with a, b parameters of higher order which yields a deeper insight into the effectiveness of this type of generalizations. In final section, we introduce a matrix representation for generalized Bernoulli polynomials with a, b parameters.

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1.Introduction, Definitions and Motivation

The generalized Bernoulli and Euler polynomials play an important role in the calculus of finite differences. In fact, the coefficients in all the usual central-difference formulae for interpolation, numerical differentiation and integration, and differences in terms of derivatives can be expressed in terms of these polynomials. The study of generalized Bernoulli and Euler numbers and their combinatorial relations has received much attention [1-8]. The classical Bernoulli polynomials $B_n(x)$, the classical Euler polynomials $E_n(x)$ and the classical Genocchi polynomials $G_n(x)$, together with their familiar generalizations $B_n^{(\alpha)}(x)$, $E_n^{(\alpha)}(x)$ and $G_n^{(\alpha)}(x)$of (real or complex) order α, are usually defined by means of the following generating functions

$$\left(\frac{z}{e^z-1}\right)^\alpha e^{xz} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x)\frac{z^n}{n!}, (|z| < 2\pi). \tag{1}$$

$$\left(\frac{2}{e^z+1}\right)^\alpha e^{xz} = \sum_{n=0}^{\infty} E_n^{(\alpha)}(x)\frac{z^n}{n!}, (|z| < \pi). \tag{2}$$

and

$$\Big(\frac{2z}{e^z+1}\Big)^{\alpha}e^{xz}=\sum_{n=0}^{\infty}G_n^{(\alpha)}(x)\frac{z^n}{n!},(|z|<\pi). \qquad (3)$$

So that, obviously,

$$B_n(x):=B_n^{(1)}(x),\ E_n(x):=E_n^{(1)}(x)\ and\ G_n(x):=G_n^{(1)}(x) \qquad (4)$$

For the classical Bernoulli numbers B_n, the classical Euler numbers E_n and the classical Genocchi numbers G_n of order n, we have

$$B_n=B_n(0):=B_n^{(1)}(0),\ E_n=E_n(0):=E_n^{(1)}(x)\ and\ G_n=G_n(0):=G_n^{(1)}(0) \qquad (5)$$

respectively.

In 2002, Q. M. Luo and et al. (see [3, 8, 9]) defined the generalization of Bernoulli and Euler polynomials with a,b,c parameters, as follows

$$\frac{tc^{xt}}{b^t-a^t}=\sum_{n=0}^{\infty}\frac{B_n(x;a,b,c)}{n!}t^n,(|t\ln\frac{b}{a}|<2\pi) \qquad (6)$$

$$\frac{2c^{xt}}{b^t+a^t}=\sum_{n=0}^{\infty}E_n(x;a,b,c)\frac{t^n}{n!},(|t\ln\frac{b}{a}|<\pi). \qquad (7)$$

Furthermore in [15] H. Jolany defined generalized Genocchi polynomials with a,b parameters of higher order α as follows

$$\Big(\frac{2t}{b^t+a^t}\Big)^{\alpha}=\sum_{n=0}^{\infty}G_n^{(\alpha)}(x;a,b,c)\frac{t^n}{n!},(|t\ln\frac{b}{a}|<\pi). \qquad (8)$$

For the generalized Bernoulli numbers $B_n(a,b)$ with a,b parameters, the generalized Euler numbers $E_n(a,b)$ with a,b parameters and the generalized Genocchi numbers with a,b parameters $G_n(a,b)$ of order n, we have

$$B_n(a,b):=B_n(0;a,b),\ E_n(a,b):=E_n(0,a,b)\ and\ G_n(a,b)=G_n(0,a,b) \qquad (9)$$

2.Relationships between generalized Bernoulli and Euler numbers

In 2003, Cheon [3] rederived several known properties and relations involving the classical Bernoulli polynomials $B_n(x)$ and the classical Euler polynomials $E_n(x)$ by making use of some standard techniques based upon series rearrangement as well as matrix representation. Srivastava and Pinter [14] followed Cheon's work [3] and established two relations involving the generalized Bernoulli polynomials $B_n^{(\alpha)}(x)$

and the generalized Euler polynomials $E_n^{(\alpha)}(x)$. So, we will study further the relations between generalized Bernoulli polynomials with a, b parameters, generalized Genocchi polynomials with a, b parameters and generalized Euler polynomials with a, b parameters with the methods of generating function and series rearrangement.

Theorem 1. *The following relationship holds true:*

$$B_n(x+y;a,b) = \frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}[B_k(y;a,b)+B_k(y+1;a,b)]E_{n-k}(x) \tag{10}$$

between the generalized Bernoulli polynomials with a, b parameters and Euler polynomials

Proof. By applying following assertions

$$B_n(x+y;a,b) = \sum_{k=0}^{n}\binom{n}{k}B_k(y;a,b)x^{n-k} \tag{11}$$

$$x^n = \frac{1}{2}\Big[E_n(x) + \sum_{k=0}^{n}\binom{n}{k}E_k(x)\Big] \tag{12}$$

we get

$$B_n(x+y;a,b) = \frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}B_k(y;a,b)\Big[E_{n-k}(x) + \sum_{j=0}^{n-k}\binom{n-k}{j}E_j(x)\Big] \tag{13}$$

$$\begin{aligned}
B_n(x+y;a,b) &= \frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}B_k(y;a,b)E_{n-k}(x) + \frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}B_k(y;a,b)\sum_{j=0}^{n-k}\binom{n-k}{j}E_j(x) \\
&= \frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}B_k(y;a,b)E_{n-k}(x) + \frac{1}{2}\sum_{j=0}^{n}\binom{n}{j}E_j(x)\sum_{k=0}^{n-j}\binom{n-j}{k}B_k(y;a,b) \\
&= \frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}B_k(y;a,b)E_{n-k}(x) + \frac{1}{2}\sum_{j=0}^{n}\binom{n}{j}E_j(x)B_{n-j}(y+1,a,b)
\end{aligned}$$

So we get

$$B_n(x+y;a,b) = \frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}[B_k(y;a,b)+B_k(y+1;a,b)]E_{n-k}(x)$$

GI-Sang Cheon and H. M. Srivastava in [3, 14] investigated the classical relationship between Bernoulli and Euler polynomials as follows

Corollary 1. *For all $n \geq 0$ we have*

$$B_n(x) = \sum_{\substack{k=0 \\ k\neq 1}}^{n} \binom{n}{k} B_k E_{n-k}(x) \tag{14}$$

Proof. By applying $b = e, a = 1, y = 0$ in (10) proof is complete.

Theorem 2. *The following relationship holds true:*

$$E_n(x+y;a,b) = \sum_{j=0}^{n} \frac{1}{n-j+1} \binom{n}{j} [E_{n-j+1}(y+1;a,b) - E_{n-j+1}(y;a,b)] B_j(x) \tag{15}$$

between the generalized Euler polynomials with a, b parameters and Bernoulli polynomials

Proof. By comparing the coefficients of the Taylor expansion of the two sides of the following identity we obtain desired result.

$$2e^{(x+y)t}/(b^t + a^t) = (te^{xt}/(e^t - 1))(((2e^{(y+1)t}/(b^t + a^t)) - (2e^{yt}/(b^t + a^t)))/t)$$

So proof is complete.

By applying a similar method we obtain the following assertions for generalized Bernoulli polynomials with a, b parameters and Genocchi polynomials

Corollary 2. *We have*

$$B_n(x+y,a,b) = \frac{1}{2} \sum_{k=0}^{n} \frac{1}{n-k+1} \binom{n}{k} [B_k(y,a,b) + B_k(y+1,a,b)] G_{n-k}(x) \tag{16}$$

Now in next theorem we introduce and derive multiplication formula related to the generalized Genocchi polynomials with a, b, c parameters. Here our method is similar to that of [7].

Theorem 3. *For $m \in \mathbf{N}$ (m is odd) the generalized Genocchi polynomials $G_n(x;a,b,c)$ satisfy the following multiplication formula*

$G_n^{(\alpha)}(mx; a, b, c) =$

$m^{n-\alpha} \sum_{\nu_1,\nu_2,\nu_3,\ldots,\nu_{m-1}\geq 0} \binom{\alpha}{\nu_1,\nu_2,\nu_3,\ldots,\nu_{m-1}} (-1)^r G_n^{(\alpha)}\Big(x + \frac{r(\ln b-\ln a)+\alpha(m-1)\ln a}{m\ln c}; a, b, c\Big)$
where $r = \nu_1 + 2\nu_2 + 3\nu_3 + \ldots + (m-1)\nu_{m-1}$ and $\binom{\alpha}{\nu_1,\nu_2,\nu_3,\ldots,\nu_{m-1}} = \frac{\alpha!}{\nu_1!\nu_2!\nu_3!\ldots\nu_{m-1}!}$

Proof. It is easy to observe that for m odd we have

$$\frac{2t}{b^t + a^t} = 2te^{-t\ln a}\frac{\sum_{k=0}^{m-1}(-e^{t(\ln b-\ln a)})^k}{1 + e^{mt(\ln b-\ln a)}} \tag{17}$$

By using (8) and (17), we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty} G_n^{(\alpha)}(mx; a, b, c)\frac{t^n}{n!} &= \Big(\frac{2te^{-t\ln a}}{1 + e^{mt(\ln b-\ln a)}}\Big)^{\alpha}\left(\sum_{k=0}^{m-1}\Big(-e^{t(\ln b-\ln a)}\Big)^k\right)^{\alpha} e^{mxt\ln c} \\
&= \sum_{\nu_1,\nu_2,\nu_3,\ldots,\nu_{m-1}\geq 0}\binom{\alpha}{\nu_1,\nu_2,\nu_3,\ldots,\nu_{m-1}}(-1)^r\left(\frac{2te^{-t\ln a}}{1 + e^{mt(\ln b-\ln a)}}\right)^{\alpha} \\
&\times \quad e^{mt\ln c\left(x+\frac{r(\ln b-\ln a)}{m\ln c}\right)}
\end{aligned}$$

By comparing the coefficients of t^n on both sides in the above equation, we arrive at the desired result.

As a direct result of Theorem3, we get following well known assertion about Genocchi polynomials

Corollary 3.*For m odd, we have*

$$G_n(mx) = m^{n-1}\sum_{k=0}^{m-1}(-1)^k G_n\Big(x + \frac{k}{m}\Big) \tag{18}$$

*Proof.*By substituting $\alpha = a = 1, b = c = e$ in Theorem 3, proof is complete.

3.Matrix representation for $B_n(x; a, b)$ and $B_n(a, b)$

In this section by using basic linear algebra and properties of determinant we introduce a new definition of generalized Bernoulli polynomials with a, b parameters.

Theorem 4.*For $a \neq b$, we have*

$$B_n(x;a,b) = A\begin{pmatrix} (\ln b-\ln a) & 0 & 0 & \dots & 0 & 1 \\ \frac{(\ln b-\ln a)^2}{2!} & (\ln b-\ln a) & 0 & \dots & 0 & \frac{(x-\ln a)}{1!} \\ \frac{(\ln b-\ln a)^3}{3!} & \frac{(\ln b-\ln a)^2}{2!} & (\ln b-\ln a) & \dots & 0 & \frac{(x-\ln a)^2}{2!} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{(\ln b-\ln a)^n}{n!} & \frac{(\ln b-\ln a)^{n-1}}{(n-1)!} & \frac{(\ln b-\ln a)^{n-2}}{(n-2)!} & \dots & (\ln b-\ln a) & \frac{(x-\ln a)^{n-1}}{(n-1)!} \\ \frac{(\ln b-\ln a)^{n+1}}{(n+1)!} & \frac{(\ln b-\ln a)^n}{n!} & \frac{(\ln b-\ln a)^{n-1}}{(n-1)!} & \dots & \frac{(\ln b-\ln a)^2}{2!} & \frac{(x-\ln a)^n}{n!} \end{pmatrix}$$

where $A = \frac{n!}{(\ln b-\ln a)^{n+1}}$

Proof. By applying Taylor expansion, we get

$$\begin{aligned} \sum_{n=0}^{\infty} B_n(x;a,b)\frac{t^n}{n!} &= \frac{te^{t(x-\ln a)}}{e^{t(\ln b-\ln a)}-1} \\ &= \frac{1+t(x-\ln a)+\frac{t^2(x-\ln a)^2}{2!}+...+\frac{t^n(x-\ln a)^n}{n!}+...}{(\ln b-\ln a)+\frac{t(\ln b-\ln a)^2}{2!}+\frac{t^2(\ln b-\ln a)^3}{3!}+...+\frac{t^{n-1}(\ln b-\ln a)^n}{n!}+...} \end{aligned} \quad (19)$$

So, by multiplying both sides of (19) by the dominator of the right side formula, we get
$1+t(x-\ln a)+\frac{t^2(x-\ln a)^2}{2!}+...+\frac{t^n(x-\ln a)^n}{n!}+... =$
$\left(\frac{B_0(x;a,b)}{0!}+\frac{B_1(x;a,b)}{1!}t+...\frac{B_n(x;a,b)}{n!}t^n+...\right)\left((\ln b-\ln a)+\frac{t(\ln b-\ln a)^2}{2!}+\frac{t^2(\ln b-\ln a)^3}{3!}+ ...+\frac{t^{n-1}(\ln b-\ln a)^n}{n!}+...\right)$

This equation leads to the following system of infinite equations.

$$(\ln b-\ln a)c_0(x;a,b)=1$$

$$\frac{(\ln b-\ln a)^2}{2!}c_0(x;a,b)+c_1(x;a,b)(\ln b-\ln a)=\frac{x-\ln a}{1!}$$

$$\frac{(\ln b-\ln a)^3}{3!}c_0(x;a,b)+c_1(x;a,b)\frac{(\ln b-\ln a)^2}{2!}+c_2(x;a,b)(\ln b-\ln a)=\frac{(x-\ln a)^2}{2!}$$

$$\vdots$$

$$\frac{(\ln b-\ln a)^{n+1}}{(n+1)!}c_0(x;a,b)+c_1(x;a,b)\frac{(\ln b-\ln a)^n}{n!}+...+c_n(x;a,b)(\ln b-\ln a)=\frac{(x-\ln a)^n}{n!}$$

where $c_i(x;a,b) = \frac{B_i(x;a,b)}{i!}$

The matrix of the coefficients of the above system is lower triangular and with $\ln b - \ln a$ along the main diagonal. Since $\ln b - \ln a \neq 0$, this matrix is invertible and by the Crammer method and some elementary calculation proof is complete.

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