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TWO NEW TYPE OF IRRESOLUTE FUNCTIONS VIA b-OPEN SETS

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Abstract. The purpose of this paper is to give two new types of irresolute functions called, completely b-irresolute functions and completely weakly b-irresolute functions in topological spaces. Some characterization and interesting properties of these functions are discussed.

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1. Introduction and Preliminaries

Functions and of course irresolute functions stand among the most important and most researched points in the whole of mathematical science. Various interesting problems arise when one considers irresoluteness. Its importance is significant in various areas of mathematics and related sciences. In 1996, Andrijevic [3] introduced a new class of generalized open sets called b-open sets into the field of topology. Andrijevic studied several fundamental and interesting properties of b-open sets. Quite recently, Caldas and Jafari [5] obtained some applications of b-open sets in topological spaces. In this paper, we introduce and characterize the concepts of completely b-irresolute and completely weakly b-irresolute functions. Throughout this paper, spaces always means topological spaces on which no separation axioms are assumed unless otherwise mentioned and $f : (X, \tau) \to (Y, \sigma)$ (or simply $f : X \to Y$) denotes a function f of a space (X, τ) into a space (Y, σ). Let A be a subset of a space X. The closure and the interior of A are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. A subset A of a topological space (X, τ) is said to be regular open [19] (resp. preopen [13], b-open [3]) if $A = \mathrm{Int}(\mathrm{Cl}(A))$ (resp. $A \subset \mathrm{Int}(\mathrm{Cl}(A))$, $A \subset \mathrm{Int}(\mathrm{Cl}(A)) \cup \mathrm{Cl}(\mathrm{Int}(A))$). The complement of regular open (resp. b-open) set is called regular closed (resp. b-closed). A set $A \subset X$ is said to be δ-open [21] if it is the union of regular open sets of X. The complement of a δ-open set is called δ-closed. The intersection of all δ-closed sets of (X, τ) containing A is called the δ-closure [21] of A and is denoted by $\mathrm{Cl}_\delta(A)$. The union of all b-open sets contained in $A \subset X$ is called the b-interior of A, and is denoted by $b\,\mathrm{Int}(A)$. The intersection of all b-closed sets of X containing A is called the b-closure of A, and is denoted by $b\,\mathrm{Cl}(A)$. The family of all regular open (resp. b-open, b-closed, regular closed, α-open) set of (X, τ) is denoted by $RO(X)$ (resp. $BO(X)$, $BC(X)$, $RC(X)$, $\alpha O(X)$). We set $BO(X, x) = \{V \in BO(X) | x \in V\}$ for $x \in X$.

2. Completely b-irresolute functions

Definition 1 *A function $f : (X, \tau) \to (Y, \sigma)$ is said to be:*

(i) strongly continuous [10] if $f^{-1}(V)$ is both open and closed in X for each subset V of Y;

(ii) completely continuous [4] if $f^{-1}(V)$ is regular open in X for every open set V of Y;

(iii) b-irresolute [5] if $f^{-1}(V)$ is b-closed (resp. b-open) in X for every b-closed (resp. b-open) subset V of Y;

(iv) completely b-irresolute if the inverse image of each b-open subset of Y is regular open in X.

Remark 2 *Clearly, every strongly continuous function is completely b-irresolute and every completely b-irresolute function is b-irresolute. But the converse of the implications are not true in general as seen from the following examples.*

Example 3 *Let $X = \{a, b, c\} = Y$, $\tau = \{\varnothing, \{a\}, \{b, c\}, X\}$ and $\sigma = \{\varnothing, \{a\}, Y\}$. Clearly, the identity function $f : (X, \tau) \to (Y, \sigma)$ is completely b-irresolute but not strongly continuous.*

Example 4 *Let $X = \{a, b, c\} = Y$, $\tau = \{\varnothing, \{a, b\}, X\}$ and $\sigma = \{\varnothing, \{a\}, \{a, b\}, Y\}$. Then the identity function $f : (X, \tau) \to (Y, \sigma)$ is b-irresolute but not completely b-irresolute.*

Theorem 5 *The following statements are equivalent for a function $f : X \to Y$:*

(i) f is completely b-irresolute;

(ii) $f^{-1}(F)$ is regular closed in X for every b-closed set F of Y.

Proof: (i)⇔(ii): Let F be any b-closed set of Y. Then $Y \setminus F \in BO(Y)$. By (i), $f^{-1}(Y \setminus F) = X \setminus f^{-1}(F) \in RO(X)$. We have $f^{-1}(F) \in RC(X)$.
Converse is similar.

Lemma 6 *[11] Let S be an open subset of a topological space (X, τ). Then the following hold:*

(i) If U is regular open in X, then so is $U \cap S$ in the subspace (S, τ_s).

(ii) If $B \subset S$ is regular open in (S, τ_s), then there exists a regular open set U in (X, τ) such that $B = U \cap S$.

Theorem 7 *If $f : (X, \tau) \to (Y, \sigma)$ is a completely b-irresolute function and A is any open subset of X, then the restriction $f|_A$: $A \to Y$ is completely b-irresolute.*

Proof: Let F be a b-open subset of Y. By hypothesis $f^{-1}(F)$ is regular open in X. Since A is open in X, it follows from the previous Lemma that $(f|_A)^{-1}(F) = A \cap f^{-1}(F)$, which is regular open in A. Therefore, $f|_A$ is completely b-irresolute.

Lemma 8 *[2] Let Y be a preopen subset of a topological space (X, τ). Then $Y \cap U$ is regular open in Y for each regular open subset U of X.*

Theorem 9 *If $f : (X, \tau) \to (Y, \sigma)$ is completely λ-irresolute function and A is preopen subset of X, then $f|_A$: $A \to Y$ is completely λ-irresolute.*

Proof: Similar to the Proof of Theorem 7.

Theorem 10 *The following hold for functions $f : (X, \tau) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \eta)$:*

(i) If f is completely b-irresolute and g is b-irresolute, then $g \circ f$ is completely b-irresolute;

(ii) If f is completely continuous and g is completely b-irresolute, then $g \circ f$ is completely b-irresolute.

(iii) If f is completely b-irresolute and g is b-continuous, then $g \circ f$ is completely continuous function.

Proof: The proof of the theorem is easy and hence omitted.

Definition 11 *A space X is said to be almost connected [7] (resp. b-connected [5]) if there does not exist disjoint regular open (resp. b-open) sets A and B such that $A \cup B = X$.*

Theorem 12 *If $f : (X, \tau) \to (Y, \sigma)$ is completely b-irresolute surjective function and X is almost connected, then Y is b-connected.*

Proof: Suppose that Y is not b-connected. Then there exist disjoint b-open sets A and B of Y such that $A \cup B = Y$. Since f is completely b-irresolute surjective, $f^{-1}(A)$ and $f^{-1}(B)$ are regular open sets in X. Moreover, $f^{-1}(A) \cup f^{-1}(B) = X$, $f^{-1}(A) \neq \varnothing$ and $f^{-1}(B) \neq \varnothing$. This shows that X is not almost connected, which is a contradiction to the assumption that X is almost connected. By contradiction, Y is b-connected.

Definition 13 *A topological space X is said to be*

(i) nearly compact [16] if every regular open cover of X has a finite subcover;

(ii) nearly countably compact [8] if every cover by regular open sets has a countable subcover;

(iii) nearly Lindelof [7] if every cover of X by regular open sets has a countable subcover;

(iv) b-compact if every b-open cover of X has a finite subcover;

(v) countably b-compact if every b-open countable cover of X has a finite subcover;

(vi) b-Lindelof if every cover of X by b-open sets has a countable subcover.

Theorem 14 *Let $f : (X, \tau) \to (Y, \sigma)$ be a completely b-irresolute surjective function. Then the following statements hold:*

(i) If X is nearly compact, then Y is b-compact.

(ii) If X is nearly Lindelof, then Y is b-Lindelof.

(i) If X is nearly countably compact, then Y is countably b-compact.

Proof: (i) Let $f : X \to Y$ be a completely b-irresolute function of nearly compact space X onto a space Y. Let $\{U_\alpha : \alpha \in \triangle\}$ be any b-open cover of Y. Then, $\{f^{-1}(U_\alpha) : \alpha \in \triangle\}$ is a regular open cover of X. Since X is nearly compact, there exists a finite subfamily, $\{f^{-1}(U_{\alpha_i}) | i = 1, 2, ...n\}$ of $\{f^{-1}(U_\alpha) : \alpha \in \triangle\}$ which cover X. It follows then that $\{U_{\alpha_i} : i = 1, 2, ...n\}$ is a finite subfamily of $\{U_\alpha : \alpha \in \triangle\}$ which cover Y. Hence, space Y is a b-compact space.
The proof of other cases are similar.

Definition 15 *A topological space (X, τ) is said to be:*

(i) S-closed [20] (resp. b-closed compact) if every regular closed (resp. b-closed) cover of X has a finite subcover;

(ii) countably S-closed-compact [1] (resp. countably b-closed compact) if every countable cover of X by regular closed (resp. b-closed) sets has a finite subcover;

(iii) S-Lindelof [12] (resp. b-closed Lindelof) if every cover of X by regular closed (resp. b-closed) sets has a countable subcover.

Theorem 16 *Let $f : (X, \tau) \to (Y, \sigma)$ be a completely b-irresolute surjective function. Then the following statements hold:*

(i) If X is S-closed, then Y is b-closed compact.

(ii) If X is S-Lindelof, then Y is b-closed Lindelof.

(iii) If X is countably S-closed, then Y is countably b-closed compact.

Proof: It can be obtained similarly as the previous Theorem.

Definition 17 *A topological space X is said to be almost regular [17](resp. strongly b-regular) if for any regular closed (resp. b-closed) set $F \subset X$ and any point $x \in X \backslash F$, there exists disjoint open (resp. b-open) sets U and V such that $x \in U$ and $F \subset V$.*

Definition 18 *A function $f : X \to Y$ is called pre-b-closed if the image of each b-closed set of X is a b-closed set in Y.*

Theorem 19 *If a mapping $f : X \to Y$ is pre-b-closed, then for each subset B of Y and a b-open set U of X containing $f^{-1}(B)$, there exists a b-open set V in Y containing B such that $f^{-1}(V) \subset U$.*

Proof: Straightforward.

Theorem 20 *If f is completely b-irresolute b-open from an almost regular space X onto a space Y, then Y is strongly b-regular.*

Proof: Let F be a b-closed set in Y with $y \notin F$. Take $y = f(x)$. Since f is completely b-irresolute, $f^{-1}(F)$ is regular closed and so closed set in X and $x \notin f^{-1}(F)$. By almost regularity of X, there exists disjoint open sets U and V such that $x \in U$ and $f^{-1}(F) \subset V$. We obtain that $y = f(x) \in f(U)$ and $F \subset f(V)$ such that $f(U)$ and $f(V)$ are disjoint b-open sets. Thus, Y is strongly b-regular.

Now, we define the following.

Definition 21 *A topological space X is said to be:*

(i) almost normal [18] if for each closed set A and each regular closed set B such that $A \cap B = \emptyset$, there exist disjoint open sets U and V such that $A \subset U$ and $\subset V$.

(ii) strongly b-normal if for every pair of disjoint b-closed subsets A and B of X, there exist disjoint b-open sets U and V such that $A \subset U$ and $B \subset V$.

Theorem 22 *If $f : X \rightarrow Y$ is completely b-irresolute b-open function from an almost normal space X onto a space Y, then Y is strongly b-normal.*

Proof: Let A and B be two disjoint b-closed subsets in Y. Since f is completely b-irresolute, $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint regular closed and so closed sets in X. By almost normality of X, there exist disjoint open sets U and V such that $f^{-1}(A) \subset U$ and $f^{-1}(B) \subset V$. We obtain that $A \subset f(U)$ and $B \subset f(V)$ such that $f(U)$ and $f(V)$ are disjoint b-open sets. Thus, Y is strongly b-normal.

Definition 23 *A topological space (X, τ) is said to be b-T_1 [6] (resp. r-T_1 [7]) if for each pair of distinct points x and y of X, there exist b-open (resp. regular open) sets U_1 and U_2 such that $x \in U_1$ and $y \in U_2$, $x \notin U_2$ and $y \notin U_1$.*

Theorem 24 *If $f : (X, \tau) \rightarrow (Y, \sigma)$ is completely b-irresolute injective function and Y is b-T_1, then X is r-T_1.*

Proof: Suppose that Y is b-T_1. For any two distinct points x and y of X, there exist b-open sets F_1 and F_2 in Y such that $f(x) \in F_1$, $f(y) \in F_2$, $f(x) \notin F_2$ and $f(y) \notin F_1$. Since f is injective completely b-irresolute function, we have X is r-T_1.

Definition 25 *A topological space (X, τ) is said to be b-T_2 [5] (resp. r-T_2) for each pair of distinct points x and y in X, there exist disjoint b-open (resp. regular open) sets A and B in X such that $x \in A$ and $y \in B$.*

Theorem 26 *If $f : (X, \tau) \to (Y, \sigma)$ is completely b-irresolute injective function and Y is b-T_2, then X is r-T_2.*

Proof: Similar to the proof of Theorem 24.

Theorem 27 *Let Y be a b-T_2 space. Then we have the following*

(i) If $f, g : X \to Y$ are completely b-irresolute functions, then the set $A = \{x \in X : f(x) = g(x)\}$ is δ-closed in X.

(ii) If $f : X \to Y$ is a completely b-irresolute function, then the set $B = \{(x, y) \in X \times X : f(x) = f(y)\}$ is δ-closed in $X \times X$.

Proof: (i). Let $x \notin A$, then $f(x) \neq g(x)$. Since Y is b-T_2, there exist disjoint b-open sets U_1 and U_2 in Y such that $f(x) \in U_1$ and $g(x) \in U_2$. Since f and g are completely b-irresolute, $f^{-1}(U_1)$ and $g^{-1}(U_2)$ are regular open sets. Put $U = f^{-1}(U_1)$ $\cap$ $g^{-1}(U_2)$. Then U is a regular open set containing x and $U \cap A \neq \emptyset$. Hence we have $x \notin \mathrm{Cl}_\delta(A)$. This completes the proof.
(ii). Follows from (i).

3.Completely Weakly b-irresolute functions

Definition 28 *A function $f : (X, \tau) \to (Y, \sigma)$ is said to be completely weakly b-irresolute if for each $x \in X$ and for any b-open set V containing $f(x)$, there exists an open set U containing x such that $f(U) \subset V$.*

It is obvious that every completely b-irresolute function is completely weakly b-irresolute and every completely weakly b-irresolute function is b-irresolute. However, the converses may not be true in general as shown in the following example.

Example 29 *Let $X = \{a, b, c\}$ with topologies $\tau = \{\varnothing, \{a, b\}, \{a, c\}, X\}$, $\sigma = \{\varnothing, \{a\}, \{a, b\}, X\}$ and $\gamma = \{\varnothing, \{a\}, \{a, c\}, X\}$. Clearly the identity function $f : (X, \tau) \to (X, \sigma)$ is completely weakly b-irresolute but not completely b-irresolute. Also the identity function $f : (X, \sigma) \to (X, \gamma)$ is b-irresolute but not completely weakly b-irresolute.*

Remark 30 *From the above arguments, we obtain the following implication diagrams:*

completely b-irresolute	$\to$	*completely weakly b-irresolute*	$\to$	*b-irresolute*

Theorem 31 *Let $f : (X, \tau) \to (Y, \sigma)$ be a function, the following statements are equivalent:*

(i) *f is completely weakly b-irresolute;*

(ii) *for each $x \in X$ and each b-open set V of Y containing $f(x)$, there exists an open set U of X containing x such that $f(U) \subset V$;*

(iii) $f(\mathrm{Cl}(A)) \subset b\,\mathrm{Cl}(f(A))$ *for every subset A of X;*

(iv) $\mathrm{Cl}(f^{-1}(B)) \subset f^{-1}(b\,\mathrm{Cl}(B))$ *for every subset B of Y;*

(v) *for each b-closed set V in Y, $f^{-1}(V)$ is closed in X;*

(vi) $f^{-1}(b\,\mathrm{Int}(B)) \subset \mathrm{Int}(f^{-1}(B))$ *for every subset B of Y.*

Proof: Clear.

Theorem 32 *If $f_i : X_i \to Y_i (i{=}1,\ 2)$ are completely weakly b-irresolute functions, then $f_i : X_1 \times X_2 \to Y_1 \times Y_2$ is completely weakly b-irresolute.*

Proof: Follows from Lemma 5.9 of [15].

Theorem 33 *Let $f : (X, \tau) \to (Y, \sigma)$ be a function. If the graph $g : X \to X \times Y$ of f is completely weakly b-irresolute, then so is f.*

Proof: Let V be a b-open subset of Y. Then $f^{-1}(V) = g^{-1}(X \times V)$. Since g is completely weakly b-irresolute and $X \times V$ is b-open in $X \times Y$, $f^{-1}(V)$ is open in X and so, f is completely weakly b-irresolute.

Theorem 34 *Let $f : (X, \tau) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \eta)$ be any two functions. Then*

(i) *If f is completely weakly b-irresolute and g is b-irresolute, then $g \circ f : (X, \tau) \to (Z, \eta)$ is completely weakly b-irresolute.*

(ii) *If f is completely continuous and g is completely weakly b-irresolute, then $g \circ f$ is completely b-irresolute.*

(iii) *If f is strongly continuous and g is completely b-irresolute, then $g \circ f$ is completely b-irresolute.*

(iv) *If f and g are completely b-irresolute, then $g \circ f$ is completely b-irresolute.*

(v) *If f is completely b-irresolute and g is completely weakly b-irresolute, then $g \circ f$ is completely b-irresolute.*

(vi) *If f is completely weakly b-irresolute and g is b-continuous, then $g \circ f$ is continuous.*

(vii) *If f is b-continuous and g is completely weakly b-irresolute, then $g \circ f$ is b-irresolute.*

(viii) *If f is continuous and g is completely weakly b-irresolute, then $g \circ f$ is completely weakly b-irresolute.*

Proof: Follows from their respective definitions.

Recall that a function $f : (X, \tau) \to (Y, \sigma)$ is said to be almost open if $f^{-1}(V)$ is regular open in X for every open set V of Y.

Theorem 35 *If $f : (X, \tau) \to (Y, \sigma)$ is almost open and $g : (Y, \sigma) \to (Z, \gamma)$ is any function such that $g \circ f : (X, \tau) \to (Z, \gamma)$ is completely b-irresolute, then g is completely weakly b-irresolute.*

Proof: Let V be a b-open set in (Z, γ). Since $g \circ f$ is completely b-irresolute, $(g \circ f)^{-1} = f^{-1}(g^{-1}(V))$ is regular open in (X, τ). Since f is almost open surjection, $f(f^{-1}(g^{-1}(V))) = g^{-1}(V)$ is open in Y. Therefore, g is completely weakly b-irresolute.

Theorem 36 *If $f : (X, \tau) \to (Y, \sigma)$ is open surjection and $g : (Y, \sigma) \to (Z, \gamma)$ is any function such that $g \circ f : (X, \tau) \to (Z, \gamma)$ is completely weakly b-irresolute, then g is completely weakly b-irresolute.*

Proof: Similar to proof of Theorem 35.

Definition 37 *(i): A filterbase Ω is said to be Ω-convergent to a point x in X if for any $U \in BO(X, x)$, there exists $B \in \Omega$ such that $B \subset U$.*
(ii): A filterbase Ω is said to be convergent to a point x in X if for any open set U of X containing x, there exists $B \in \Omega$ such that $B \subset U$.

Theorem 38 *If a function $f : (X, \tau) \to (Y, \sigma)$ is completely weakly b-irresolute, then for each point $x \in X$ and each filterbase b in X converging to x, the filterbase $f(\Omega)$ is b-convergent to $f(x)$.*

Proof: Let $x \in X$ and Ω be any filterbase in X converging to x. Since f is completely weakly b-irresolute, then for any b-open set V of (Y, σ) containing $f(x)$, there exists an open set U of X containing x such that $f(U) \subset V$. Since Ω is converging to x, there exists $B \in \Omega$ such that $B \subset U$. This means that $f(B) \subset V$ and hence the filterbase $f(\Omega)$ is b-convergent to $f(x)$.

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