# ON SOME ISOMETRIC SPACES OF $C_0^F$, $C^F$ AND $\ell_\infty^F$

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Abstract. In this article we introduce and investigate the notion of $\Delta_{(r)}$-null, $\Delta_{(r)}$-convergent and $\Delta_{(r)}$-bounded sequences of fuzzy numbers which generalize the notion of null, convergent and bounded sequence of fuzzy numbers.

2000 *Mathematics Subject Classification*: 40A05, 40D25.

## 1. Introduction

The concept of fuzzy sets and fuzzy set operations was first introduced by Zadeh [5] and subsequently several authors have studied various aspects of the theory and applications of fuzzy sets. Bounded and convergent sequences of fuzzy numbers were introduced by Matloka [2] where it was shown that every convergent sequence is bounded. Nanda [3] studied the spaces of bounded and convergent sequence of fuzzy numbers and showed that they are complete metric spaces. Savaş [4] studied the space $m(\Delta)$, which we call the space of $\Delta$-bounded sequence of fuzzy numbers and showed that this is a complete metric space.

Let $w$ denote the space of all real or complex sequences. By $c$, $c_0$ and $\ell_\infty$, we denote the Banach spaces of convergent, null and bounded sequences $x = (x_k)$, respectively normed by

$$\|x\| = \sup_k |x_k|.$$

The notion of difference sequence space was introduced by Kizmaz [1], who studied the difference sequence spaces $\ell_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$.

Tripathy and Esi [6] generalized the above notion as follows:

Let $r$ be a non-negative integer, then for $Z = c_0, c$ and $\ell_\infty$, we have

$$Z(\Delta_r) = \{x = (x_k) \in w : (\Delta_r x_k) \in Z\},$$

where $\Delta_r x = (\Delta_r x_k) = (x_k - x_{k+r})$ and $\Delta_0 x_k = x_k$ for all $k \in N$.

Let $D$ denote the set of all closed bounded intervals $A = [A_1, A_2]$ on the real line $R$. For $A, B \in D$ define

$$A \leq B \text{ iff } A_1 \leq B_1 \text{ and } A_2 \leq B_2,$$

$$h(A, B) = \max(|A_1 - B_1, |A_2 - B_2|).$$

Then $(D, h)$ is a complete metric space. Also $\leq$ is a partial order relation in $D$.

A fuzzy number is a fuzzy subset of the real line $R$ which is bounded, convex and normal. Let $L(R)$ denote the set of all fuzzy numbers which are upper semi continuous and have compact support. In other words, if $X \in L(R)$ then for any $\alpha \in [0, 1]$, $X^\alpha$ is compact where

$$X^\alpha = \begin{cases} t : X(t) \geq \alpha, \text{ if } \alpha \in (0, 1] \\ t : X(t) > 0, \text{ if } \alpha = 0 \end{cases}$$

Define a map $d_1 : L(R) \times L(R) \longrightarrow R$ by

$$d_1(X, Y) = \sup_{0 \leq \alpha \leq 1} h(X^\alpha, Y^\alpha).$$

It is straightforward to see that $d_1$ is a metric on $L(R)$. Infact $(L(R), d_1)$ is a complete metric space.

For $X, Y \in L(R)$ define

$$X \leq Y \text{ iff } X^\alpha \leq Y^\alpha \text{ for any } \alpha \in [0, 1].$$

A subset $E$ of $L(R)$ is said to be bounded above if there exists a fuzzy number $M$, called an upper bound of $E$, such that $X \leq M$ for every $X \in E$. $M$ is called the least upper bound or supremum of $E$ if $M$ is an upper bound and $M$ is the smallest of all upper bounds. A lower bound and the greatest lower bound or infimum are defined similarly. $E$ is said to be bounded if it is both bounded above and bounded below.

We now state the following definitions (see [2, 3]):

A sequence $X = (X_k)$ of fuzzy numbers is a function $X$ from the set $N$ of all positive integers into $L(R)$. The fuzzy number $X_k$ denotes the value of the function at $k \in N$ and is called the $k$-th term or general term of the sequence.

A sequence $X = (X_k)$ of fuzzy numbers is said to be convergent to the fuzzy number $X_0$, written as $\lim_k X_k = X_0$, if for every $\varepsilon > 0$, there exists $n_0 \in N$ such that

$$d_1(X_k, X_0) < \varepsilon \text{ for } k > n_0.$$

The set of convergent sequences is denoted by $c^F$. $X = (X_K)$ of fuzzy numbers is said to be a Cauchy sequence if for every $\varepsilon > 0$, there exists $n_0 \in N$ such that

$$d_1(X_k, X_l) < \varepsilon \text{ for } k, l > n_0.$$

A sequence $X = (X_k)$ of fuzzy numbers is said to be bounded if the set $\{X_k : k \in N\}$ of fuzzy numbers is bounded and the set of bounded sequences is denoted by $\ell_\infty^F$.

Let $r$ be a non-negative integer. Then we define the following new definitions:

A sequence $X = (X_k)$ of fuzzy numbers is said to be $\Delta_{(r)}$-convergent to the fuzzy number $X_0$, written as $\lim_k \Delta_{(r)} X_k = X_0$, if for every $\varepsilon > 0$, there exists $n_0 \in N$ such that

$$d_1(\Delta_{(r)} X_k, X_0) < \varepsilon \text{ for } k > n_0,$$

where $(\Delta_{(r)} X_k) = (X_k - X_{k-r})$ and $\Delta_{(0)} X_k = X_k$ for all $k \in N$.

In this expansion it is important to note that we take $X_{k-r} = \bar{0}$, for non-positive values of $k - r$.

Let $c^F(\Delta_{(r)})$ denote the set of all $\Delta_{(r)}$-convergent sequences of fuzzy numbers.

In particular if $X_0 = \bar{0}$, in the above definition, we say $X = (X_k)$ to be $\Delta_{(r)}$-null sequence of fuzzy numbers and we denote the set of all $\Delta_{(r)}$-null sequences of fuzzy numbers by $c_0^F(\Delta_{(r)})$.

A sequence $X = (X_k)$ of fuzzy numbers is said to be $\Delta_{(r)}$-bounded if the set $\{\Delta_{(r)} X_k : k \in N\}$ of fuzzy numbers is bounded.

Let $\ell_\infty^F(\Delta_{(r)})$ denote the set of all $\Delta_{(r)}$-bounded sequences of fuzzy numbers.

Similarly we can define the sets $c_0^F(\Delta_r)$, $c^F(\Delta_r)$ and $\ell_\infty^F(\Delta_r)$ of $\Delta_r$-null, $\Delta_r$-convergent and $\Delta_r$-bounded sequences of fuzzy numbers, where $(\Delta_r X_k) = (X_k - X_{k+r})$ and $\Delta_0 X_k = X_k$ for all $k \in N$.

It is obvious that for any sequence $X = (X_k)$, $X \in Z(\Delta_r)$ if and only if $X \in Z(\Delta_{(r)})$, for $Z = c_0^F, c^F$ and $\ell_\infty^F$. One may find it interesting to see the differences between the difference operator $\Delta_r$ and the new difference operator $\Delta_{(r)}$ through the Theorem 1 and Theorem 2 of next section.

Taking $r = 0$, in the above definitions of spaces we get the spaces $c_0^F, c^F$ and $\ell_\infty^F$.

## 2. Main Results

In this section we investigate the main results of this article.

**Theorem 1.** *$c_0^F(\Delta_{(r)})$, $c^F(\Delta_{(r)})$ and $\ell_\infty^F(\Delta_{(r)})$ are complete metric spaces with the metric $d$ defined by*

$$d(X,Y) = \sup_k d_1(\Delta_{(r)}X_k, \Delta_{(r)}Y_k) \tag{1}$$

*Proof.* We give the proof only for the space $c^F(\Delta_{(r)})$ and for the other spaces it will follow on applying similar arguments. It is easy to see that $d$ is a metric on $c^F(\Delta_{(r)})$. To prove completeness, let $(X^i)$ be a Cauchy sequence in $c^F(\Delta_{(r)})$, where $X^i = (X_k^i)= (X_1^i, X_2^i, \dots)$ for each $i \in N$. Then for a given $\varepsilon > 0$, there exists a positive integer $n_0$ such that

$$d(X^i, X^j) < \varepsilon \text{ for all } i, j \geq n_0.$$

Then using (1), we have

$$\sup_k d_1(\Delta_{(r)}X_k^i, \Delta_{(r)}X_k^j) < \varepsilon \text{ for all } i, j \geq n_0.$$

It follows that

$$d_1(\Delta_{(r)}X_k^i, \Delta_{(r)}X_k^j) < \varepsilon \text{ for all } i, j \geq n_0 \text{ and } k \in N.$$

This implies that $(\Delta_{(r)}X_k^i)$ is a Cauchy sequence in $L(R)$ for all $k \geq 1$. But $L(R)$ is complete and so $(\Delta_{(r)}X_k^i)$ is convergent in $L(R)$ for all $k \geq 1$.
Let $\lim_{i\to\infty} \Delta_{(r)}X_k^i = Z_k$, say for each $k \geq 1$. Considering $k = 1, 2, \dots, r, \dots$, we can easily conclude that $\lim_{i\to\infty} X_k^i = X_k$, exists for each $k \geq 1$.
Now one can find that

$$\lim_{j\to\infty} d_1(\Delta_{(r)}X_k^i, \Delta_{(r)}X_k^j) < \varepsilon \text{ for all } i \geq n_0 \text{ and } k \in N.$$

Hence

$$d_1(\Delta_{(r)}X_k^i, \Delta_{(r)}X_k) < \varepsilon \text{ for all } i \geq n_0 \text{ and } k \in N.$$

This implies that

$$d(X^i, X) < \varepsilon \text{ for all } i \geq n_0.$$

$$i.e., \; X^i \to X \text{ as } i \to \infty, \text{ where } X = (X_k).$$

Now we can easily show that $X = (X_k) \in c^F(\Delta_{(r)})$.
This completes the proof.

**Theorem 2.** *$c_0^F(\Delta_r)$, $c^F(\Delta_r)$ and $\ell_\infty^F(\Delta_r)$ are complete metric spaces with the metric $d'$ defined by*

$$d'(X,Y) = \sum_{k=1}^{r} d_1(X_k, Y_k) + \sup_k d_1(\Delta_r X_k, \Delta_r Y_k)$$

*Proof.* Proof is similar to that of above Theorem.

**Remark.** It is obvious that the matrices $d$ and $d'$ are equivalent.

**Theorem 3.** $(i)$ *The metric spaces $c_0^F(\Delta_{(r)})$, $c^F(\Delta_{(r)})$ and $\ell_\infty^F(\Delta_{(r)})$ are isometric with the metric spaces $c_0^F$, $c^F$ and $\ell_\infty^F$.*
$(ii)$ *The metric spaces $c_0^F(\Delta_r)$, $c^F(\Delta_r)$ and $\ell_\infty^F(\Delta_r)$ are isometric with the metric spaces $c_0^F$, $c^F$ and $\ell_\infty^F$.*

*Proof.* $(i)$ Let us define a mapping $f$ from $Z(\Delta_{(r)})$ into $Z$, for $Z = c_0^F, c^F$ and $\ell_\infty^F$ as follows:

$$fX = (\Delta_{(r)} X_k), \text{ for every } X \in Z(\Delta_{(r)})$$

Then clearly $f$ is one-one, on-to and

$$d(X,Y) = \rho(f(X), f(Y)),$$

where $\rho$ is the metric on $Z$, which can be obtained from (1) by taking $r = 0$.
This completes the proof.

$(ii)$ Proof is similar to that of part (i) in view of above remark and the fact that for any sequence $X = (X_k)$, $X \in Z(\Delta_r)$ if and only if $X \in Z(\Delta_{(r)})$, for $Z = c_0^F, c^F$ and $\ell_\infty^F$.

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