Acta Universitatis Apulensis No. 74/2023
ISSN: 1582-5329 pp. 147-161
http://www.uab.ro/auajournal/ doi: 10.17114/j.aua.2023.74.10

DIFFERENTIAL SUBORDINATION AND SUPERORDINATION RESULTS FOR λ-PSEUDO-STARLIKE AND λ-PSEUDO-CONVEX FUNCTIONS WITH RESPECT TO SYMMETRICAL POINTS DEFINED BY CONVOLUTION STRUCTURE

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Abstract. In this article, we determinate some applications of first order differential subordination and superordination results involving Hadamard product for λ-pseudo-starlike and λ-pseudo-convex functions with respect to symmetrical points defined in the open unit disk U. These results are applied to obtain sandwich results.

2010 *Mathematics Subject Classification*: 30C45, 30C20.

Keywords: Holomorphic functions, Differential subordination, Differential superordination, Hadamard product, λ-Pseudo-starlike functions, λ-Pseudo-convex functions, Symmetrical points.

1. Introduction and Preliminaries

Denote by $\mathcal{H}$ the collection of holomorphic functions in the unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ and assume that $\mathcal{H}[a, n]$ be the subfamily of $\mathcal{H}$ consisting of functions of the form:

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + ... \quad (a \in \mathbb{C},\ n \in \mathbb{N} = \{1, 2, ...\}).$$

Also, let $\mathcal{A}$ be the subfamily of $\mathcal{H}$ consisting of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

A function $f \in \mathcal{A}$ is called starlike with respect to symmetrical points, if (see [10])

$$Re\left\{\frac{zf'(z)}{f(z) - f(-z)}\right\} > 0,\ z \in U.$$

The set of all such functions is denote by S_s^*.

The class of starlike functions with respect to symmetrical points obviously includes the class of convex functions with respect to symmetrical points, C_s the following condition:

$$Re\{\frac{(zf'(z))'}{(f(z)-f(-z))'}\} > 0,\ z \in U.$$

Recently, Babalola [4] defined the family $\mathcal{L}_\lambda$ of λ-pseudo-starlike which are the functions $f \in \mathcal{A}$ such that

$$Re\left\{\frac{z\left(f'(z)\right)^\lambda}{f(z)}\right\} > 0,\ \lambda \geq 1; z \in U.$$

A function $f \in \mathcal{A}$ is called λ-pseudo-starlike with respect to symmetrical points, if

$$Re\left\{\frac{z\left(f'(z)\right)^\lambda}{f(z)-f(-z)}\right\} > 0,\ z \in U.$$

We denote by $\mathcal{L}_{\lambda,s}^*$ the family of all λ-pseudo-starlike functions with respect to symmetrical points.

For the functions $f \in \mathcal{A}$ given by (1) and $g \in \mathcal{A}$ defined by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n,$$

we define the Hadamard product (or convolution) $f * g$ of the functions f and g (as usual) by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n = (g * f)(z).$$

Now we recall the principle of subordination between analytic functions, let the functions f and g be analytic in U, we say that the function f is subordinate to g, if there exists a Schwarz function w analytic in U with $w(0) = 0$ and $|w(z)| < 1$ $(z \in U)$ such that $f(z) = g(w(z))$. This subordination is indicated by $f \prec g$ or $f(z) \prec g(z)$ $(z \in U)$. Furthermore, if the function g is univalent in U, then we have the following equivalent (see [8]), $f(z) \prec g(z) \iff f(0) = g(0)\ and\ f(U) \subset g(U)$.

Let $k, h \in \mathcal{H}$ and $\psi(r, s; z) : C^2 \times U \to C$. If k and $\psi(k(z), zk'(z), z^2k''(z); z)$ are univalent functions in U and if k satisfies the first-order differential superordination

$$h(z) \prec \psi(k(z), zk'(z); z), \tag{2}$$

then k is called a solution of the differential superordination (2). (If f is subordinate to g, then g is superordinate to f). An analytic function q is called a subordinate of (2), if $q \prec k$ for all the functions k satisfying (2). An univalent subordinat $\breve{q}$ that satisfies $q \prec \breve{q}$ for all the subordinants q of (2) is called the best subordinant.

Very recently many authors have obtained sandwich results for certain classes of analytic functions, such as Attiya and Yassen [3], Seoudy [11], Wanas and Srivastava [16], Lupas and Catas [7] and others (see, for example, [1, 2, 6, 9, 12, 13, 14, 15, 17]).

The main object of the present work is to find sufficient condition for certain normalized analytic functions f in U such that $(f * \Psi)(z) \neq 0$ and f to satisfy

$$q_1(z) \prec \left(\frac{2z \left((f*\Phi)'(z) \right)^{\lambda}}{(f*\Psi)(z) - (f*\Psi)(-z)} \right)^{\gamma} \prec q_2(z)$$

and

$$q_1(z) \prec \left(\frac{2\left(\left(z\left(f*\Phi \right)'(z) \right)' \right)^{\lambda}}{\left((f*\Psi)(z) - (f*\Psi)(-z) \right)'} \right)^{\gamma} \prec q_2(z),$$

where q_1 and q_2 are given univalent functions in U with $q_1(0) = q_2(0) = 1$ and $\Phi(z) = z + \sum_{n=2}^{\infty} r_n z^n$, $\Psi(z) = z + \sum_{n=2}^{\infty} e_n z^n$ are analytic functions in U with $r_n \geq 0, e_n \geq 0$.

To prove our main results, we will require the following definition and lemmas.

Definition 1. *[3] Denote by Q the set of all functions f that are analytic and injective on $\bar{U}\backslash E(f)$, where*

$$E(f) = \left\{ \zeta \in \partial U : \lim_{z \to \zeta} f(z) = \infty \right\}$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial U \backslash E(f)$.

Lemma 1. *[3] Let q be univalent in the unite disk U and let θ and ϕ be analytic in a domain D containing $q(U)$ with $\phi(w) \neq 0$ when $w \in q(U)$. set $Q(z) = zq'(z)\phi(q(z))$ and $h(z) = \theta(q(z)) + Q(z)$. Suppose that*
(1)$Q(z)$ is starlike univalent in U,
(2)$Re\left\{ \frac{zh'(z)}{Q(z)} \right\} > 0$ for $z \in U$.
If k is analytic in U, with $k(0) = q(0)$, $k(U) \subset D$ and

$$\theta(k(z)) + zk'(z)\phi(k(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)), \tag{3}$$

then $k \prec q$ and q is the best dominant of (3).

Lemma 2. *[2] Let q be convex univalent in the unit disk U and let θ and ϕ be analytic in a domain D containing $q(U)$. Suppose that*

(1)Re $\left\{\frac{\theta'(q(z))}{\phi(q(z))}\right\} > 0$ *for* $z \in U$,
(2)$Q(z) = zq'(z)\phi(q(z))$ *is starlike univalent in* U.
If $k \in \mathcal{H}[q(0), 1] \cap Q$, *with* $k(U) \subset D$, $\theta(k(z)) + zk'(z)\phi(k(z))$ *is univalent in* U *and*

$$\theta(q(z)) + zq'(z)\phi(q(z)) \prec \theta(k(z)) + zk'(z)\phi(k(z)), \tag{4}$$

then $q \prec k$ *and* q *is the best subordinant of (4).*

2. Subordination Results

Theorem 3. *Let* $\Phi, \Psi \in \mathcal{A}$, $\alpha, \beta, \tau, \varepsilon, \gamma \in \mathbb{C}$ *such that* $\gamma \neq 0$ *and* q *be convex univalent in* U *with* $q(0) = 1$ *and assume that*

$$Re\left\{1 + \frac{\beta q^2(z) - \tau}{\varepsilon q(z)} + \frac{zq''(z)}{q'(z)}\right\} > 0. \tag{5}$$

If $f \in \mathcal{A}$ *satisfies the differential subordination*

$$\Upsilon_1(f, \Phi, \Psi, \alpha, \beta, \tau, \varepsilon, \gamma, \lambda; z) \prec \alpha + \beta q(z) + \frac{\tau}{q(z)} + \varepsilon\frac{zq'(z)}{q(z)}, \tag{6}$$

where

$$\Upsilon_1(f, \Phi, \Psi, \alpha, \beta, \tau, \varepsilon, \gamma, \lambda; z) = \alpha + \beta\left(\frac{2z\left((f*\Phi)'(z)\right)^{\lambda}}{(f*\Psi)(z) - (f*\Psi)(-z)}\right)^{\gamma}$$
$$+ \tau\left(\frac{(f*\Psi)(z) - (f*\Psi)(-z)}{2z\left((f*\Phi)'(z)\right)^{\lambda}}\right)^{\gamma} + \gamma\varepsilon\left[1 + \frac{\lambda z\,(f*\Phi)''(z)}{(f*\Phi)'(z)} - \frac{z\left((f*\Psi)(z) - (f*\Psi)(-z)\right)'}{(f*\Psi)(z) - (f*\Psi)(-z)}\right], \tag{7}$$

then

$$\left(\frac{2z\left((f*\Phi)'(z)\right)^{\lambda}}{(f*\Psi)(z) - (f*\Psi)(-z)}\right)^{\gamma} \prec q(z)$$

and q *is the best dominant of (6).*

Proof. Let us define

$$k(z) = \left(\frac{2z\left((f*\Phi)'(z)\right)^{\lambda}}{(f*\Psi)(z) - (f*\Psi)(-z)}\right)^{\gamma}, \quad (z \in U). \tag{8}$$

Then the function k is analytic in U and $k(0) = 1$.
By setting

$$\theta(w) = \alpha + \beta w + \frac{\tau}{w} \quad and \quad \phi(w) = \frac{\varepsilon}{w},$$

it can be easily observed that $\theta(w)$ and $\phi(w)$ are analytic in $C\backslash\{0\}$ and that $\phi(w) \neq 0$, $w \in C\backslash\{0\}$. Also, we get

$$Q(z) = zq'(z)\phi(q(z)) = \varepsilon\frac{zq'(z)}{q(z)}$$

and

$$h(z) = \theta(q(z)) + Q(z) = \alpha + \beta q(z) + \frac{\tau}{q(z)} + \varepsilon\frac{zq'(z)}{q(z)}.$$

In light of the hypothesis of Theorem 3, we see that $Q(z)$ is starlike univalent in U and

$$Re\left\{\frac{zh'(z)}{Q(z)}\right\} = Re\left\{1 + \frac{\beta q^2(z) - \tau}{\varepsilon q(z)} + \frac{zq''(z)}{q'(z)}\right\} > 0.$$

A simple computation using (8) gives

$$\frac{zk'(z)}{k(z)} = \gamma\left[1 + \frac{\lambda z\,(f*\Phi)''(z)}{(f*\Phi)'(z)} - \frac{z\,((f*\Psi)(z) - (f*\Psi)(-z))'}{(f*\Psi)(z) - (f*\Psi)(-z)}\right].$$

Also, we find that

$$\alpha + \beta k(z) + \frac{\tau}{k(z)} + \varepsilon\frac{zk'(z)}{k(z)} = \Upsilon_1(f, \Phi, \Psi, \alpha, \beta, \tau, \varepsilon, \gamma, \lambda; z), \tag{9}$$

where $\Upsilon_1(f, \Phi, \Psi, \alpha, \beta, \tau, \varepsilon, \gamma, \lambda; z)$ is given by (7).
By using (9) in (6), we deduce that

$$\alpha + \beta k(z) + \frac{\tau}{k(z)} + \varepsilon\frac{zk'(z)}{k(z)} \prec \alpha + \beta q(z) + \frac{\tau}{q(z)} + \varepsilon\frac{zq'(z)}{q(z)}.$$

Hence by an application of Lemma 1, we have $p(z) \prec q(z)$. By using (8), we obtain the result which we needed.

By fixing $\Phi(z) = \Psi(z) = \frac{z}{1-z}$ in Theorem 3, we obtain the following Corollary:

Corollary 4. *Let $\alpha, \beta, \tau, \varepsilon, \gamma \in \mathbb{C}$ such that $\gamma \neq 0$ and q be convex univalent in U with $q(0) = 1$ and assume that (5) holds true. If $f \in \mathcal{A}$ satisfies the differential subordination*

$$\Upsilon_2(f, \alpha, \beta, \tau, \varepsilon, \gamma, \lambda; z) \prec \alpha + \beta q(z) + \frac{\tau}{q(z)} + \varepsilon\frac{zq'(z)}{q(z)}, \tag{10}$$

where

$$\Upsilon_2(f,\alpha,\beta,\tau,\varepsilon,\gamma,\lambda;z)=\alpha+\beta\left(\frac{2z\left(f'(z)\right)^{\lambda}}{f(z)-f(-z)}\right)^{\gamma}+\tau\left(\frac{f(z)-f(-z)}{2z\left(f'(z)\right)^{\lambda}}\right)^{\gamma}$$
$$+\gamma\varepsilon\left[1+\frac{\lambda zf''(z)}{f'(z)}-\frac{z\left(f(z)-f(-z)\right)'}{f(z)-f(-z)}\right], \tag{11}$$

then

$$\left(\frac{2z\left(f'(z)\right)^{\lambda}}{f(z)-f(-z)}\right)^{\gamma}\prec q(z)$$

and q is the best dominant of (10).

By taking $\lambda=1$ in Theorem 3, we obtain the following corollary:

Corollary 5. *Let $\Phi,\Psi\in\mathcal{A}$, $\alpha,\beta,\tau,\varepsilon,\gamma\in\mathbb{C}$ such that $\gamma\neq 0$ and q be convex univalent in U with $q(0)=1$ and assume that (5) holds true. If $f\in\mathcal{A}$ satisfies the differential subordination*

$$\Upsilon_3(f,\Phi,\Psi,\alpha,\beta,\tau,\varepsilon,\gamma;z)\prec\alpha+\beta q(z)+\frac{\tau}{q(z)}+\varepsilon\frac{zq'(z)}{q(z)}, \tag{12}$$

where

$$\Upsilon_3(f,\Phi,\Psi,\alpha,\beta,\tau,\varepsilon,\gamma;z)=\alpha+\beta\left(\frac{2z\left(f*\Phi\right)'(z)}{(f*\Psi)(z)-(f*\Psi)(-z)}\right)^{\gamma}$$
$$+\tau\left(\frac{(f*\Psi)(z)-(f*\Psi)(-z)}{2z\left(f*\Phi\right)'(z)}\right)^{\gamma}+\gamma\varepsilon\left[1+\frac{z\left(f*\Phi\right)''(z)}{\left(f*\Phi\right)'(z)}-\frac{z\left((f*\Psi)(z)-(f*\Psi)(-z)\right)'}{(f*\Psi)(z)-(f*\Psi)(-z)}\right], \tag{13}$$

then

$$\left(\frac{2z\left(f*\Phi\right)'(z)}{(f*\Psi)(z)-(f*\Psi)(-z)}\right)^{\gamma}\prec q(z)$$

and q is the best dominant of (12).

Theorem 6. *Let $\Phi,\Psi\in\mathcal{A}$, $\alpha,\beta,\tau,\varepsilon,\gamma\in\mathbb{C}$ such that $\gamma\neq 0$ and q be convex univalent in U with $q(0)=1$ and assume that (5) holds true. If $f\in\mathcal{A}$ satisfies the differential subordination*

$$\Upsilon_4(f,\Phi,\Psi,\alpha,\beta,\tau,\varepsilon,\gamma,\lambda;z)\prec\alpha+\beta q(z)+\frac{\tau}{q(z)}+\varepsilon\frac{zq'(z)}{q(z)}, \tag{14}$$

where

$$\Upsilon_4(f,\Phi,\Psi,\alpha,\beta,\tau,\varepsilon,\gamma,\lambda;z)$$
$$=\alpha+\beta\left(\frac{2\left(\left(z\left(f*\Phi\right)'(z)\right)'\right)^{\lambda}}{\left((f*\Psi)(z)-(f*\Psi)(-z)\right)'}\right)^{\gamma}+\tau\left(\frac{\left((f*\Psi)(z)-(f*\Psi)(-z)\right)'}{2\left(\left(z\left(f*\Phi\right)'(z)\right)'\right)^{\lambda}}\right)^{\gamma}$$
$$+\gamma\varepsilon\left[\frac{\lambda z\left(z\left(f*\Phi\right)'(z)\right)''}{\left(z\left(f*\Phi\right)'(z)\right)'}-\frac{z\left((f*\Psi)(z)-(f*\Psi)(-z)\right)''}{\left((f*\Psi)(z)-(f*\Psi)(-z)\right)'}\right], \tag{15}$$

then

$$\left(\frac{2\left(\left(z\left(f*\Phi\right)'(z)\right)'\right)^{\lambda}}{\left((f*\Psi)(z)-(f*\Psi)(-z)\right)'}\right)^{\gamma}\prec q(z)$$

and q is the best dominant of (14).

Proof. Let us define

$$k(z)=\left(\frac{2\left(\left(z\left(f*\Phi\right)'(z)\right)'\right)^{\lambda}}{\left((f*\Psi)(z)-(f*\Psi)(-z)\right)'}\right)^{\gamma}, \quad (z\in U). \tag{16}$$

Then the function k is analytic in U and $k(0)=1$.
After some calculations from (16), we conclude that

$$\alpha+\beta k(z)+\frac{\tau}{k(z)}+\varepsilon\frac{zk'(z)}{k(z)}=\Upsilon_4(f,\Phi,\Psi,\alpha,\beta,\tau,\varepsilon,\gamma,\lambda;z), \tag{17}$$

where $\Upsilon_4(f,\Phi,\Psi,\alpha,\beta,\tau,\varepsilon,\gamma,\lambda;z)$ is given by (15).
In view of (17), the subordination (14), can be written as

$$\alpha+\beta k(z)+\frac{\tau}{k(z)}+\varepsilon\frac{zk'(z)}{k(z)}\prec\alpha+\beta q(z)+\frac{\tau}{q(z)}+\varepsilon\frac{zq'(z)}{q(z)}.$$

By setting $\theta(w)=\alpha+\beta w+\frac{\tau}{w}$ and $\phi(w)=\frac{\varepsilon}{w}$, it is easily observed that $\theta(w)$ and $\phi(w)$ are analytic in $C\backslash\{0\}$ and that $\phi(w)\neq 0$, $w\in C\backslash\{0\}$. Hence the result now follows by an application of Lemma 1.

By fixing $\Phi(z)=\Psi(z)=\frac{z}{1-z}$ in Theorem 6, we obtain the following corollary:

Corollary 7. *Let $\alpha,\beta,\tau,\varepsilon,\gamma\in\mathbb{C}$ such that $\gamma\neq 0$ and q be convex univalent in U with $q(0)=1$ and assume that (5) holds true. If $f\in\mathcal{A}$ satisfies the differential subordination*

$$\Upsilon_5(f,\alpha,\beta,\tau,\varepsilon,\gamma,\lambda;z)\prec\alpha+\beta q(z)+\frac{\tau}{q(z)}+\varepsilon\frac{zq'(z)}{q(z)}, \tag{18}$$

where

$$\Upsilon_5(f,\alpha,\beta,\tau,\varepsilon,\gamma,\lambda;z)=\alpha+\beta\left(\frac{2\left((zf'(z))'\right)^{\lambda}}{(f(z)-(f(-z))'}\right)^{\gamma}+\tau\left(\frac{(f(z)-f(-z))'}{2\left((zf'(z))'\right)^{\lambda}}\right)^{\gamma}$$
$$+\gamma\varepsilon\left[\frac{\lambda z\,(zf'(z))''}{(zf'(z))'}-\frac{z\,(f(z)-f(-z))''}{(f(z)-f(-z))'}\right], \tag{19}$$

then

$$\left(\frac{2\left((zf'(z))'\right)^{\lambda}}{(f(z)-(f(-z))'}\right)^{\gamma}\prec q(z)$$

and q is the best dominant of (18).

By taking $\lambda=1$ in Theorem 6, we obtain the following corollary:

Corollary 8. *Let $\Phi,\Psi\in\mathcal{A}$, $\alpha,\beta,\tau,\varepsilon,\gamma\in\mathbb{C}$ such that $\gamma\neq 0$ and q be convex univalent in U with $q(0)=1$ and assume that (5) holds true. If $f\in\mathcal{A}$ satisfies the differential subordination*

$$\Upsilon_6(f,\Phi,\Psi,\alpha,\beta,\tau,\varepsilon,\gamma;z)\prec\alpha+\beta q(z)+\frac{\tau}{q(z)}+\varepsilon\frac{zq'(z)}{q(z)}, \tag{20}$$

where

$$\Upsilon_6(f,\Phi,\Psi,\alpha,\beta,\tau,\varepsilon,\gamma;z)=\alpha+\beta\left(\frac{2\left(z\,(f*\Phi)'(z)\right)'}{((f*\Psi)(z)-(f*\Psi)(-z))'}\right)^{\gamma}$$
$$+\tau\left(\frac{((f*\Psi)(z)-(f*\Psi)(-z))'}{2\left(z\,(f*\Phi)'(z)\right)'}\right)^{\gamma}+\gamma\varepsilon\left[\frac{z\left(z\,(f*\Phi)'(z)\right)''}{\left(z\,(f*\Phi)'(z)\right)'}-\frac{z\,((f*\Psi)(z)-(f*\Psi)(-z))''}{((f*\Psi)(z)-(f*\Psi)(-z))'}\right], \tag{21}$$

then

$$\left(\frac{2\left(z\,(f*\Phi)'(z)\right)'}{((f*\Psi)(z)-(f*\Psi)(-z))'}\right)^{\gamma}\prec q(z)$$

and q is the best dominant of (20).

3. Superordination Results

Theorem 9. *Let $\Phi, \Psi \in \mathcal{A}$, $\alpha, \beta, \tau, \varepsilon, \gamma \in \mathbb{C}$ such that $\gamma \neq 0$ and q be convex univalent in U with $q(0) = 1$ and assume that*

$$Re\left\{\frac{\left(\beta q^2(z) - \tau\right) q'(z)}{\varepsilon q(z)}\right\} > 0. \tag{22}$$

Suppose that $f \in \mathcal{A}$, $\left(\frac{2z\left((f*\Phi)'(z)\right)^{\lambda}}{(f*\Psi)(z)-(f*\Psi)(-z)}\right)^{\gamma} \in \mathcal{H}[q(0),1] \cap Q$ and $\Upsilon_1(f, \Phi, \Psi, \alpha, \beta, \tau, \varepsilon, \gamma, \lambda; z)$ as defined by (7) be univalent in U. If

$$\alpha + \beta q(z) + \frac{\tau}{q(z)} + \varepsilon \frac{zq'(z)}{q(z)} \prec \Upsilon_1(f, \Phi, \Psi, \alpha, \beta, \tau, \varepsilon, \gamma, \lambda; z), \tag{23}$$

then

$$q(z) \prec \left(\frac{2z\left((f*\Phi)'(z)\right)^{\lambda}}{(f*\Psi)(z) - (f*\Psi)(-z)}\right)^{\gamma}$$

and q is the best subordinant of (23).

Proof. Let the function k be defined by (8). By a straightforward computation, the superordination (23) becomes

$$\alpha + \beta q(z) + \frac{\tau}{q(z)} + \varepsilon \frac{zq'(z)}{q(z)} \prec \alpha + \beta k(z) + \frac{\tau}{k(z)} + \varepsilon \frac{zk'(z)}{k(z)}.$$

By setting $\theta(w) = \alpha + \beta w + \frac{\tau}{w}$ and $\phi(w) = \frac{\varepsilon}{w}$, it is easily observed that $\theta(w)$ and $\phi(w)$ are analytic in $C\backslash\{0\}$ and that $\phi(w) \neq 0$, $w \in C\backslash\{0\}$. Also, we have

$$Re\left\{\frac{\theta'(q(z))}{\phi(q(z))}\right\} = Re\left\{\frac{\left(\beta q^2(z) - \tau\right) q'(z)}{\varepsilon q(z)}\right\} > 0.$$

Now Theorem 9 follows by applying Lemma 2.

By fixing $\Phi(z) = \Psi(z) = \frac{z}{1-z}$ in Theorem 9, we obtain the following corollary:

Corollary 10. *Let $\alpha, \beta, \tau, \varepsilon, \gamma \in \mathbb{C}$ such that $\gamma \neq 0$ and q be convex univalent in U with $q(0) = 1$ and assume that (22) holds true. Suppose that $f \in \mathcal{A}$, $\left(\frac{2z(f'(z))^{\lambda}}{f(z)-f(-z)}\right)^{\gamma} \in \mathcal{H}[q(0),1] \cap Q$ and $\Upsilon_2(f, \alpha, \beta, \tau, \varepsilon, \gamma, \lambda; z)$ as defined by (11) be univalent in U. If*

$$\alpha + \beta q(z) + \frac{\tau}{q(z)} + \varepsilon \frac{zq'(z)}{q(z)} \prec \Upsilon_2(f, \alpha, \beta, \tau, \varepsilon, \gamma, \lambda; z), \tag{24}$$

then

$$q(z) \prec \left(\frac{2z\left(f'(z)\right)^{\lambda}}{f(z)-f(-z)}\right)^{\gamma}$$

and q is the best subordinant of (24).

By taking $\lambda = 1$ in Theorem 9, we obtain the following corollary:

Corollary 11. *Let $\Phi, \Psi \in \mathcal{A}$, $\alpha, \beta, \tau, \varepsilon, \gamma \in \mathbb{C}$ such that $\gamma \neq 0$ and q be convex univalent in U with $q(0) = 1$ and assume that (22) holds true. Suppose that $f \in \mathcal{A}$, $\left(\frac{2z(f*\Phi)'(z)}{(f*\Psi)(z)-(f*\Psi)(-z)}\right)^{\gamma} \in \mathcal{H}[q(0), 1] \cap Q$ and $\Upsilon_3(f, \Phi, \Psi, \alpha, \beta, \tau, \varepsilon, \gamma; z)$ as defined by (13) be univalent in U. If*

$$\alpha + \beta q(z) + \frac{\tau}{q(z)} + \varepsilon \frac{zq'(z)}{q(z)} \prec \Upsilon_3(f, \Phi, \Psi, \alpha, \beta, \tau, \varepsilon, \gamma; z), \tag{25}$$

then

$$q(z) \prec \left(\frac{2z\left(f*\Phi\right)'(z)}{(f*\Psi)(z)-(f*\Psi)(-z)}\right)^{\gamma}$$

and q is the best subordinant of (25).

Theorem 12. *Let $\Phi, \Psi \in \mathcal{A}$, $\alpha, \beta, \tau, \varepsilon, \gamma \in \mathbb{C}$ such that $\gamma \neq 0$ and q be convex univalent in U with $q(0) = 1$ and assume that (22) holds true. Suppose that $f \in \mathcal{A}$, $\left(\frac{2\left(\left(z(f*\Phi)'(z)\right)'\right)^{\lambda}}{((f*\Psi)(z)-(f*\Psi)(-z))'}\right)^{\gamma} \in \mathcal{H}[q(0), 1] \cap Q$ and $\Upsilon_4(f, \Phi, \Psi, \alpha, \beta, \tau, \varepsilon, \gamma, \lambda; z)$ as defined by (15) be univalent in U. If*

$$\alpha + \beta q(z) + \frac{\tau}{q(z)} + \varepsilon \frac{zq'(z)}{q(z)} \prec \Upsilon_4(f, \Phi, \Psi, \alpha, \beta, \tau, \varepsilon, \gamma, \lambda; z), \tag{26}$$

then

$$q(z) \prec \left(\frac{2\left(\left(z\left(f*\Phi\right)'(z)\right)'\right)^{\lambda}}{((f*\Psi)(z)-(f*\Psi)(-z))'}\right)^{\gamma}$$

and q is the best subordinant of (26).

For the choice of $k(z) = \left(\frac{2\left(\left(z(f*\Phi)'(z)\right)'\right)^{\lambda}}{((f*\Psi)(z)-(f*\Psi)(-z))'}\right)^{\gamma}$, the proof of Theorem 12 is line similar to the proof of Theorem 9 and hence we omit it.

By fixing $\Phi(z) = \Psi(z) = \frac{z}{1-z}$ in Theorem 12, we obtain the following corollary:

Corollary 13. *Let $\alpha, \beta, \tau, \varepsilon, \gamma \in \mathbb{C}$ such that $\gamma \neq 0$ and q be convex univalent in U with $q(0) = 1$ and assume that (22) holds true. Suppose that $f \in \mathcal{A}$, $\left(\frac{2\left((zf'(z))'\right)^{\lambda}}{(f(z)-f(-z))'}\right)^{\gamma} \in \mathcal{H}[q(0), 1] \cap Q$ and $\Upsilon_5(f, \alpha, \beta, \tau, \varepsilon, \gamma, \lambda; z)$ as defined by (19) be univalent in U. If*

$$\alpha + \beta q(z) + \frac{\tau}{q(z)} + \varepsilon \frac{zq'(z)}{q(z)} \prec \Upsilon_5(f, \alpha, \beta, \tau, \varepsilon, \gamma, \lambda; z), \tag{27}$$

then

$$q(z) \prec \left(\frac{2\left((zf'(z))'\right)^{\lambda}}{(f(z) - f(-z))'}\right)^{\gamma}$$

and q is the best subordinant of (27).

By taking $\lambda = 1$ in Theorem 12, we obtain the following corollary:

Corollary 14. *Let $\Phi, \Psi \in \mathcal{A}$, $\alpha, \beta, \tau, \varepsilon, \gamma \in \mathbb{C}$ such that $\gamma \neq 0$ and q be convex univalent in U with $q(0) = 1$ and assume that (22) holds true. Suppose that $f \in \mathcal{A}$, $\left(\frac{2\left(z(f*\Phi)'(z)\right)'}{((f*\Psi)(z)-(f*\Psi)(-z))'}\right)^{\gamma} \in \mathcal{H}[q(0), 1] \cap Q$ and $\Upsilon_6(f, \Phi, \Psi, \alpha, \beta, \tau, \varepsilon, \gamma; z)$ as defined by (21) be univalent in U. If*

$$\alpha + \beta q(z) + \frac{\tau}{q(z)} + \varepsilon \frac{zq'(z)}{q(z)} \prec \Upsilon_6(f, \Phi, \Psi, \alpha, \beta, \tau, \varepsilon, \gamma; z), \tag{28}$$

then

$$q(z) \prec \left(\frac{2\left(z\left(f*\Phi\right)'(z)\right)'}{((f*\Psi)(z) - (f*\Psi)(-z))'}\right)^{\gamma}$$

and q is the best subordinant of (28).

4. Sandwich Results

Concluding the results of differential subordination and superordination, we arrive at the following "sandwich results".

Theorem 15. *Let q_1 and q_2 be convex univalent in U with $q_1(0) = q_2(0) = 1$, $\alpha, \beta, \tau, \varepsilon, \gamma \in \mathbb{C}$ such that $\gamma \neq 0$ and let q_2 satisfies (5) and q_1 satisfies (22). For $f, \Phi, \Psi \in \mathcal{A}$, let $\left(\frac{2z\left((f*\Phi)'(z)\right)^{\lambda}}{(f*\Psi)(z)-(f*\Psi)(-z)}\right)^{\gamma} \in \mathcal{H}[1, 1] \cap Q$ and $\Upsilon_1(f, \Phi, \Psi, \alpha, \beta, \tau, \varepsilon, \gamma, \lambda; z)$ as defined by (7) be univalent in U. If*

$$\alpha + \beta q_1(z) + \frac{\tau}{q_1(z)} + \varepsilon \frac{zq_1'(z)}{q_1(z)} \prec \Upsilon_1(f, \Phi, \Psi, \alpha, \beta, \tau, \varepsilon, \gamma, \lambda; z) \prec \alpha + \beta q_2(z) + \frac{\tau}{q_2(z)} + \varepsilon \frac{zq_2'(z)}{q_2(z)},$$

then

$$q_1(z) \prec \left(\frac{2z\left((f*\Phi)'(z)\right)^{\lambda}}{(f*\Psi)(z)-(f*\Psi)(-z)}\right)^{\gamma} \prec q_2(z)$$

and q_1, q_2 *are respectively the best subordinant and the best dominant.*

Theorem 16. *Let* q_1 *and* q_2 *be convex univalent in* U *with* $q_1(0) = q_2(0) = 1$, $\alpha, \beta, \tau, \varepsilon, \gamma \in \mathbb{C}$ *such that* $\gamma \neq 0$ *and let* q_2 *satisfies (5) and* q_1 *satisfies (22). For* $f, \Phi, \Psi \in \mathcal{A}$, *let* $\left(\frac{2\left(\left(z(f*\Phi)'(z)\right)'\right)^{\lambda}}{((f*\Psi)(z)-(f*\Psi)(-z))'}\right)^{\gamma} \in \mathcal{H}[1,1] \cap Q$ *and* $\Upsilon_4(f, \Phi, \Psi, \alpha, \beta, \tau, \varepsilon, \gamma, \lambda; z)$ *as defined by (15) be univalent in* U. *If*

$$\alpha + \beta q_1(z) + \frac{\tau}{q_1(z)} + \varepsilon \frac{zq_1'(z)}{q_1(z)} \prec \Upsilon_4(f, \Phi, \Psi, \alpha, \beta, \tau, \varepsilon, \gamma, \lambda; z) \prec \alpha + \beta q_2(z) + \frac{\tau}{q_2(z)} + \varepsilon \frac{zq_2'(z)}{q_2(z)},$$

then

$$q_1(z) \prec \left(\frac{2\left(\left(z(f*\Phi)'(z)\right)'\right)^{\lambda}}{((f*\Psi)(z)-(f*\Psi)(-z))'}\right)^{\gamma} \prec q_2(z)$$

and q_1, q_2 *are respectively the best subordinant and the best dominant.*

By making use of Corollaries 4 and 10, we obtain the following corollary:

Corollary 17. *Let* q_1 *and* q_2 *be convex univalent in* U *with* $q_1(0) = q_2(0) = 1$, $\alpha, \beta, \tau, \varepsilon, \gamma \in \mathbb{C}$ *such that* $\gamma \neq 0$ *and let* q_2 *satisfies (5) and* q_1 *satisfies (22). For* $f \in \mathcal{A}$, *let* $\left(\frac{2z(f'(z))^{\lambda}}{f(z)-f(-z)}\right)^{\gamma} \in \mathcal{H}[1,1] \cap Q$ *and* $\Upsilon_2(f, \alpha, \beta, \tau, \varepsilon, \gamma, \lambda; z)$ *as defined by (11) be univalent in* U. *If*

$$\alpha + \beta q_1(z) + \frac{\tau}{q_1(z)} + \varepsilon \frac{zq_1'(z)}{q_1(z)} \prec \Upsilon_2(f, \alpha, \beta, \tau, \varepsilon, \gamma, \lambda; z) \prec \alpha + \beta q_2(z) + \frac{\tau}{q_2(z)} + \varepsilon \frac{zq_2'(z)}{q_2(z)},$$

then

$$q_1(z) \prec \left(\frac{2z\left(f'(z)\right)^{\lambda}}{f(z)-f(-z)}\right)^{\gamma} \prec q_2(z)$$

and q_1, q_2 *are respectively the best subordinant and the best dominant.*

By making use of Corollaries 5 and 11, we obtain the following corollary:

Corollary 18. *Let* q_1 *and* q_2 *be convex univalent in* U *with* $q_1(0) = q_2(0) = 1$, $\alpha, \beta, \tau, \varepsilon, \gamma \in \mathbb{C}$ *such that* $\gamma \neq 0$ *and let* q_2 *satisfies (5) and* q_1 *satisfies (22). For*

$f, \Phi, \Psi \in \mathcal{A}$, *let* $\left(\frac{2z(f*\Phi)'(z)}{(f*\Psi)(z)-(f*\Psi)(-z)}\right)^{\gamma} \in \mathcal{H}[1,1] \cap Q$ *and* $\Upsilon_3(f, \Phi, \Psi, \alpha, \beta, \tau, \varepsilon, \gamma; z)$ *as defined by (13) be univalent in* U. *If*

$$\alpha + \beta q_1(z) + \frac{\tau}{q_1(z)} + \varepsilon \frac{z q_1'(z)}{q_1(z)} \prec \Upsilon_3(f, \Phi, \Psi, \alpha, \beta, \tau, \varepsilon, \gamma; z) \prec \alpha + \beta q_2(z) + \frac{\tau}{q_2(z)} + \varepsilon \frac{z q_2'(z)}{q_2(z)},$$

then

$$q_1(z) \prec \left(\frac{2z\,(f*\Phi)'(z)}{(f*\Psi)(z)-(f*\Psi)(-z)}\right)^{\gamma} \prec q_2(z)$$

and q_1, q_2 *are respectively the best subordinant and the best dominant.*

By making use of Corollaries 7 and 13, we obtain the following corollary:

Corollary 19. *Let* q_1 *and* q_2 *be convex univalent in* U *with* $q_1(0) = q_2(0) = 1$, $\alpha, \beta, \tau, \varepsilon, \gamma \in \mathbb{C}$ *such that* $\gamma \neq 0$ *and let* q_2 *satisfies (5) and* q_1 *satisfies (22). For* $f \in \mathcal{A}$, *let* $\left(\frac{2\left((zf'(z))'\right)^{\lambda}}{(f(z)-f(-z))'}\right)^{\gamma} \in \mathcal{H}[1,1] \cap Q$ *and* $\Upsilon_5(f, \alpha, \beta, \tau, \varepsilon, \gamma, \lambda; z)$ *as defined by (19) be univalent in* U. *If*

$$\alpha + \beta q_1(z) + \frac{\tau}{q_1(z)} + \varepsilon \frac{z q_1'(z)}{q_1(z)} \prec \Upsilon_5(f, \alpha, \beta, \tau, \varepsilon, \gamma, \lambda; z) \prec \alpha + \beta q_2(z) + \frac{\tau}{q_2(z)} + \varepsilon \frac{z q_2'(z)}{q_2(z)},$$

then

$$q_1(z) \prec \left(1 + \frac{z^{2-\lambda} f''(z)}{\left(zf'(z)\right)^{1-\lambda}}\right)^{\gamma} \prec q_2(z)$$

and q_1, q_2 *are respectively the best subordinant and the best dominant.*

By making use of Corollaries 8 and 14, we obtain the following corollary:

Corollary 20. *Let* q_1 *and* q_2 *be convex univalent in* U *with* $q_1(0) = q_2(0) = 1$, $\alpha, \beta, \tau, \varepsilon, \gamma \in \mathbb{C}$ *such that* $\gamma \neq 0$ *and let* q_2 *satisfies (5) and* q_1 *satisfies (22). For* $f, \Phi, \Psi \in \mathcal{A}$, *let* $\left(\frac{2\left(z(f*\Phi)'(z)\right)'}{((f*\Psi)(z)-(f*\Psi)(-z))'}\right)^{\gamma} \in \mathcal{H}[1,1] \cap Q$ *and* $\Upsilon_6(f, \Phi, \Psi, \alpha, \beta, \tau, \varepsilon, \gamma; z)$ *as defined by (21) be univalent in* U. *If*

$$\alpha + \beta q_1(z) + \frac{\tau}{q_1(z)} + \varepsilon \frac{z q_1'(z)}{q_1(z)} \prec \Upsilon_6(f, \Phi, \Psi, \alpha, \beta, \tau, \varepsilon, \gamma; z) \prec \alpha + \beta q_2(z) + \frac{\tau}{q_2(z)} + \varepsilon \frac{z q_2'(z)}{q_2(z)},$$

then

$$q_1(z) \prec\prec \left(\frac{2\left(z\,(f*\Phi)'(z)\right)'}{((f*\Psi)(z)-(f*\Psi)(-z))'}\right)^{\gamma} \prec q_2(z)$$

and q_1, q_2 *are respectively the best subordinant and the best dominant.*

5. Conclusions

In this paper, using the convolution structure for λ-pseudo-starlike and λ-pseudo-convex functions with respect to symmetrical points in the open unit disk U and satisfied its specific relationship to give the subordination, superordination, and some sandwich results. For future studies, the subordination and superordination results studied here can inspire investigations where other relationship.

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