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BETWEEN $\alpha - I-$ OPEN SETS AND $S.P^* - I-$ OPEN SETS

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Abstract. The purpose of this research is to introduce a class of strong $\alpha^* - I-$ open sets, which is strictly positioned between the class of all $\alpha - I-$ open and class all $S.P^* - I-$ open and $S.S^* - I-$ open subsets of X. Connections with other classes of sets are provided. Furthermore, we defined the strong $\alpha^* - I-$ interior and strong $\alpha^* - I-$ closure operators and demonstrated their different characteristics using the newly introduced idea.

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1. Introduction and Preliminaries

Kuratowski pioneered the study of ideal topological spaces [19]. Janković and Hamlett [16] conducted the research in a local and methodical manner, including some new findings, enhancements to previously published findings, and applications. Hatir and Noiri [13] introduced the idea of $\alpha - I-$ open, $semi - I-$ open, and $\beta - I-$ open sets in ideal topological spaces. Ekici recently introduced the concepts of $\beta^* - I-$ open and $pre^* - I-$ open sets [7]. Aqeel and Bin Kuddah (see [2],[3]) presented the concepts of $S.S^* - I-$ open sets and $S.P^* - I-$ open sets in 2019. In this work we define the concepts of strong $\alpha^* - I-$ open sets and strong $\alpha^* - I-$ closed sets. Several traits and qualities are investigated.

(X, τ) (just X) is used throughout this research to represent a topological space on which no separation axiom is assumed unless clearly mentioned. The closure and interior of a subset A in a topological space X are given by $cl(A)$ and $int(A)$, respectively.

Definition 1. *[19] An ideal I on X is defined as a nonempty collection of subsets of X satisfying the following two conditions:*

1. *If $A \in I$ and $B \subset A$, then $B \in I$,*
2. *If $A \in I$ and $B \in I$, then $A \cup B \in I$.*

(X, τ, I) denote of an ideal topological space which means a topological space (X, τ) with an ideal I on X.

Definition 2. *[24] For a space (X, τ, I) and a subset A of X, $A^*(I, \tau) = \{x \in X : U \cap A \notin I, \text{for each } U \in \tau(X)\}$ where $\tau(X) = \{U \in \tau : x \in U\}$ is called the local function of A with respect to I and τ . We simply write A^* instead of $A^*(I, \tau)$ in case there is no chance for confusion.*

Definition 3. *[16] $cl^*(A) = A \cup A^*$ defines a Kuratowski closure operator for a topology τ^* (also denoted by τ^* when there is no chance for confusion finer than τ).*

Among the results published in [17, 13, 2, 5, 7, 3, 21, 1, 14, 9, 12, 10, 18, 15, 11] we mention the following results in the form of definition 1.4.

Definition 4. *A subset A of an ideal topological space (X, τ, I) is called:*

1. *$I-$ open, if $A \subset int(A^*)$,*
2. *$semi - I-$ open, if $A \subset cl^*(int(A))$,*
3. *strong $semi^* - I-$ open, if $A \subset cl^*(int^*(A))$,*
4. *$pre - I-$ open, if $A \subset int(cl^*(A))$,*
5. *$pre^* - I-$ open, if $A \subset int^*(cl(A))$,*
6. *strong $pre^* - I-$ open, if $A \subset int^*(cl^*(A))$,*
7. *$\alpha-$ open, if $A \subset int(cl(int(A)))$,*
8. *$\alpha - I-$ open, if $A \subset int(cl^*(int(A)))$,*
9. *$\beta-$ open, if $A \subset cl(int(cl(A)))$,*
10. *$\beta - I-$ open, if $A \subset cl(int(cl^*(A)))$,*
11. *$\beta^* - I-$ open, if $A \subset cl(int^*(cl(A)))$,*
12. *strong $\beta - I-$ open, if $A \subset cl^*(int(cl^*(A)))$,*
13. *$b - I-$ open, if $A \subset cl^*(int(A)) \cup int(cl^*(A))$,*
14. *weakly $semi - I-$ open, if $A \subset cl^*(int(cl(A)))$,*

15. *weakly pre* $-I-$ *open, if* $A \subset {}_s cl(int(cl^*(A)))$,
16. f_I- *set, if* $A \subset (int(A))^*$,
17. $I_\beta-$ *set, if* $int(A) = cl(int(cl^*(A)))$,
18. *almost strong* $I-$ *open, if* $A \subset cl^*(int(A^*))$,
19. $*-$ *perfect, if* $A = A^*$,
20. $S - I-$ *set, if* $int(A) = cl^*(int(A))$,
21. *strong* $S_{\beta I}-$ *set, if* $int(A) = cl^*(int(cl^*(A)))$.

Definition 5. *[6] In an ideal topological space* (X, τ, I), I *is said to be codence if* $\tau \cap I = \phi$.

Lemma 1. *[16] Let* (X, τ, I) *be an ideal space, where* I *is codence, then the following hold:*

1. $cl(A) = cl^*(A)$, *for every* $*-$ *open set* A,
2. $int(A) = int^*(A)$, *for every* $*-$ *closed set* A.

We mention the results presented in [8, 4, 2, 23, 13] in the form of lemma 1.7.

Lemma 2. *For a subset* A *of an ideal topological space* (X, τ, I), *the following are hold:*

1. $PIint(A) = A \cap int(cl^*(A))$,
2. $S.P^*Icl(A) = A \cup cl^*(int^*(A))$,
3. $S.P^*Iint(A) = A \cap int^*(cl^*(A))$,
4. $S.S^*Icl(A) = A \cup int^*(cl^*(A))$,
5. $S.S^*Iint(A) = A \cap cl^*(int^*(A))$,
6. $wsIint(A) = A \cap cl^*(int(cl(A)))$,
7. $wsIcl(A) = A \cup int^*(cl(int(A)))$,
8. $\beta Icl(A) = A \cup int(cl(int^*(A)))$.

Lemma 3. *[24] For two subsets,* A *and* B *of a space* (X, τ, I), *the following are hold:*

1. *If $A \subset B$, then $A^* \subset B^*$,*
2. *If $U \in \tau$, then $U \cap A^* \subset (U \cap A)^*$.*

Lemma 4. *[22] Let (X, τ, I) be an ideal space and A be a $*-$ dense in itself subset of X. Then $A^* = cl(A^*) = cl(A) = cl^*(A)$.*

Corollary 5. *[20] For each $A \subset (X, \tau, I)$ we have :*
$(\cup cl^*(A_\alpha) : \alpha \in \bigtriangledown) \subset cl^*(\cup A_\alpha : \alpha \in \bigtriangledown))$.

Theorem 6. *[20] For two subsets, A and B of a space (X, τ, I), the following properties are hold:*

1. *If $A \subseteq B$, then $cl^*(A) \subseteq cl^*(B)$,*
2. *$cl^*(cl^*(A)) \subseteq cl^*(A)$,*
3. *$cl^*(A \cap B) \subseteq cl^*(A) \cap cl^*(B)$,*
4. *$cl^*(A \cup B) = cl^*(A) \cup cl^*(B)$,*
5. *$A \subseteq cl^*(A) \subseteq cl(A)$.*

Lemma 7. *[25] Let A and B be subsets of (X, τ, I) and $int^*(A)$ denote the interior of A with respect to τ^*, the following properties are hold:*

1. *If $A \subseteq B$, then $int^*(A) \subseteq int^*(B)$,*
2. *If A is an open in (X, τ, I), then $A = int(A)$ and $A = int^*(A)$,*
3. *$int(A) \subseteq int^*(A) \subseteq A$,*
4. *$int^*(A \cap B) = int^*(A) \cap int^*(B)$,*
5. *$int^*(A) \cup int^*(B) \subset int^*(A \cup B)$.*

2. Strong $\alpha^* - I-$ Open Sets and Strong $\alpha^* - I-$ Closed Sets

Motivated by the definition 4 of [3,6,8] we aim here at defining new type of sets are strong $\alpha^* - I-$ open set ,strong $\alpha^* - I-$ closed set and at investigating several of their properties and relationships to other sets .

Definition 6. *Given a space (X, τ, I) and $A \subset X$, A is called strong $\alpha^* - I -$open set (briefly $S.\alpha^* - I-$ open) if $A \subset int^*(cl^*(int^*(A)))$. We denote by*

$$S.\alpha^* IO(X) = \{A \subset X : A \subset int^*(cl^*(int^*(A)))\}$$

.

Definition 7. *A subset A of a space (X, τ, I) is said to be strong $\alpha^* - I-$ Cclosed set (briefly $S.\alpha^* - I-$ closed) if its complement is a $S.\alpha^* - I-$ open set. We denote that all $S.\alpha^* - I-$ closed sets by $S.\alpha^*IC(X)$.*

The following diagram holds for any subset A of a space (X, τ, I).

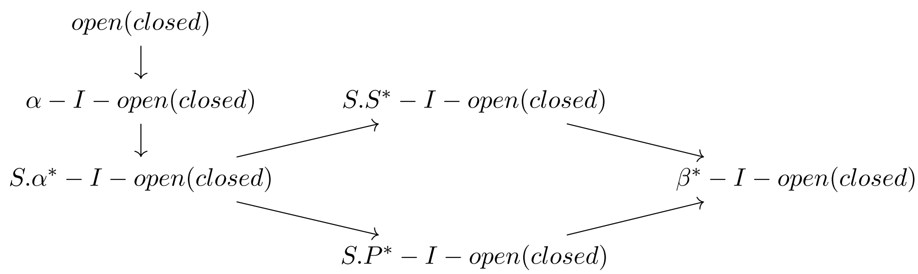

Figure 1: The implication between some generalizations of open(resp.closed) sets.

Remark 1. *The convers of the implication in diagram 1 are not true in general as shown in the following examples.*

Example 1. *Let $X = \{a, b, c\}$, $\tau = \{\phi, X, \{a\}\}$ and $I = \{\phi, \{a\}, \{c\}, \{a, c\}\}$. Then if we take*

1. *$A = \{c\}$ is a $\beta^* - I-$ open set, but $A = \{c\} \notin S.\alpha^*IO(X)$,*
2. *$A = \{b\} \in S.\alpha^*IO(X)$, but $A = \{b\} \notin \tau$.*

Example 2. *Let $X = \{a, b, c, d\}$, $\tau = \{\phi, X, \{d\}, \{a, c\}, \{a, c, d\}\}$ and $I = \{\phi, \{c\}, \{d\}, \{c, d\}\}$. we notice that $A = \{a, b\} \in S.\alpha^*IO(X)$, but A is not $\alpha - I-$ open set.*

Example 3. *Let $X = \{a, b, c, d\}$, $\tau = \{\phi, X, \{a\}, \{b, c\}, \{a, b, c\}\}$ and $I = \{\phi, \{a\}, \{d\}, \{a, d\}\}$. Then $A = \{c, d\} \in SP^*IO(X)$, but $A = \{c, d\} \notin S.\alpha^*IO(X)$.*

Example 4. *Let $X = \{a, b, c\}$, $\tau = \{\phi, X, \{a\}, \{c\}, \{a, c\}\}$ and $I = \{\phi, \{b\}\}$. Then $A = \{a, b\} \in SS^*IO(X)$, but $A = \{a, b\} \notin S.\alpha^*IO(X)$.*

Remark 2. *The strong $\alpha^* - I-$ open sets and $I-$ open sets are independent notions, we show that from the next example.*

Example 5. *From example 3 we obtain*

1. *$A = \{a\} \in S.\alpha^*IO(X)$ while $A \notin IO(X)$,*
2. *$A = \{c\} \in IO(X)$, but $A \notin S.\alpha^*IO(X)$.*

Remark 3. *The strong $\alpha^* - I-$ closed sets and pre$-$ closed sets are independent notions, we show that from the next example.*

Example 6. *From example 3 we obtain*

1. $A = \{b\} \in PC(X)$ *while* $A \notin S.\alpha^* IC(X)$,
2. $A = \{a\} \notin PC(X)$, *but* $A \in S.\alpha^* IC(X)$.

Theorem 8. *Let (X, τ, I) be a space, then B is a $S.\alpha^* - I-$ open set if and only if there exists a $S.\alpha^* - I-$ open set A such that $A \subset B \subset int^*(cl^*(A))$.*

Proof. Let B be a $S.\alpha^* - I-$ open, then $B \subset int^*(cl^*(int^*(B)))$, we put $A = int^*(B)$ which is $*-$ open hence A is a $S.\alpha^* - I-$ open and

$$\begin{aligned} A &= int^*(B) \\ &\subset B \\ &\subset int^*(cl^*(int^*(B))) \\ &= int^*(cl^*(A)). \end{aligned}$$

conversely, if A is a $S.\alpha^* - I-$ open set such that $A \subset B \subset int^*(cl^*(A))$, then $A \subset int^*(cl^*(int^*(A)))$ and $int^*(A) \subset int^*(B)$. Hence

$$\begin{aligned} B &\subset int^*(cl^*(A)) \\ &\subset int^*(cl^*(int^*(cl^*(int^*(A))))) \\ &\subset int^*(cl^*(cl^*(int^*(A)))) \\ &\subset int^*(cl^*(int^*(A))) \\ &\subset int^*(cl^*(int^*(B))). \end{aligned}$$

which shows that B is a $S.\alpha^* - I-$ open set.

Corollary 9. *A subset B of a space (X, τ, I) is a $S.\alpha^* - I-$ open set if and only if there exists a $*-$ open set A such that $A \subset B \subset cl^*(int^*(A))$.*

Proof. Comes directly from theorem 8.

Theorem 10. *Let (X, τ, I) be a space then, A is a $S.\alpha^* - I-$ open set if and only if A is both $S.P^* - I-$ open and $S.S^* - I-$ open set.*

Proof. Necessity, this is obvious.
Sufficiency, Let A be a $S.P^* - I-$ open set and $S.S^* - I-$ open set, then we have

$$\begin{aligned} A &\subset int^*(cl^*(A)) \\ &\subset int^*(cl^*(cl^*(int^*(A)))) \\ &\subset int^*(cl^*(int^*(A))). \end{aligned}$$

Hence A is a $S.\alpha^* - I-$ open set.

Theorem 11. *A subset A of a space (X, τ, I) is said to be a $S.\alpha^* - I-$ closed set if and only if $cl^*(int^*(cl^*(A))) \subset A$.*

Proof. Let A be a $S.\alpha^* - I-$ closed set of (X, τ, I), then $(X - A)$ is a $S.\alpha^* - I-$ open set and hence $(X - A) \subset int^*(cl^*(int^*(X - A))) = X - cl^*(int^*(cl^*(A)))$.
Therefore, we obtain $cl^*(int^*(cl^*(A)) \subset A$.
Conversely, let $cl^*(int^*(cl^*(A))) \subset A$, then $(X - A) \subset int^*(cl^*(int^*(X - A)))$ and hence $(X - A)$ is a $S.\alpha^* - I-$ open set. Therefore, A is a $S.\alpha^* - I-$ closed.

Theorem 12. *Let (X, τ, I) be a space where I be codense, then A is a $S.\alpha^* - I-$ closed if and only if $cl^*(int(cl^*(A))) \subset A$.*

Proof. Let A be a $S.\alpha^* - I-$ closed set of X, then
$A \supset cl^*(int^*(cl^*(A))) = cl^*(int(cl^*(A)))$.
Conversely, let A be any subset of X such that $A \supset cl^*(int(cl^*(A)))$.
This implies that $A = cl^*(int^*(cl^*(A)))$, i.e., A is a $S.\alpha^* - I-$ closed set.

Theorem 13. *A subset A of a space (X, τ, I) is a $S.\alpha^* - I-$ closed if and only if there exists a $S.\alpha^* - I-$ closed set B such that $B \supset A \supset cl^*(int^*(B))$.*

Proof. Let A be a $S.\alpha^* - I-$ closed set of a space (X, τ, I), then $A \supset cl^*(int^*(cl^*(A)))$. We put $B = cl^*(A)$ be a $* - closed\ set$. i.e, B is a $S.\alpha^* - I - closed$ and

$$\begin{aligned} B &= cl^*(A) \\ &\supset A \\ &\supset cl^*(int^*(cl^*(A))) \\ &\supset cl^*(int^*(B)). \end{aligned}$$

Conversely, if B is a $S.\alpha^* - I-$ closed set such that $B \supset A \supset cl^*(int^*(B))$, then
$B \supset cl^*(int^*(cl^*(B)))$ and $cl^*(B) \supset cl^*(A)$. Since

$$\begin{aligned} B \supset A &\supset cl^*(int^*(B)) \\ &\supset cl^*(int^*(cl^*(int^*(cl^*(B))))) \\ &\supset cl^*(int^*(int^*(cl^*(B)))) \\ &= cl^*(int^*(cl^*(B))) \\ &\supset cl^*(int^*(cl^*(A))). \end{aligned}$$

Hence A is a $S.\alpha^* - I-$ closed set.

Corollary 14. *a subset A of a space (X, τ, I) is a $S.\alpha^* - I-$ closed set if and only if there exists a $*-$ closed set B such that $B \supset A \supset cl^*(int^*(B))$.*

Proof. Comes directly from theorem 13.

The following Theorms,Corollaries and remarks introduce properties of $S.\alpha^* - I-$ open set and $S.\alpha^* - I-$ closed set and their relation with some other sets.

Remark 4. *The strong $\alpha^* - I-$ open sets and $b - I-$ open sets are independent notions, we show that from the next examples.*

Example 7. *From example 4 if we take $A = \{a, b\}$, then we get A is a $b - I-$ open, but it is not $S.\alpha^* - I-$ open.*

Example 8. *From example 1 if we take $A = \{b\}$, then we get A is not $b - I-$ open, but it is a $S.\alpha^* - I-$ open.*

Corollary 15. *Let (X, τ, I) be a space. If A is a $S.\alpha^* - I-$ open set, then $cl^*(A)$ is a $S.S^* - I-$ open set.*

Proof. Let A be a $S.\alpha^* - I-$ open. Then $A \subset int^*(cl^*(int^*(A)))$ and

$$\begin{array}{rcl} cl^*(A) & \subset & cl^*(int^*(cl^*(int^*(A)))) \\ & \subset & cl^*(cl^*(int^*(cl^*(A)))) \\ & \subset & cl^*(int^*(cl^*(A))). \end{array}$$

This implies that $cl^*(A)$ is a $S.S^* - I-$ open.

Corollary 16. *Let (X, τ, I) be a space. If A is a $S.\alpha^* - I-$ open, then $int^*(A)$ is a $S.P^* - I-$ open set.*

Proof. Let A be a $S.\alpha^* - I-$ open, then $A \subset int^*(cl^*(int^*(A)))$ and
$int^*(A) \subset int^*(int^*(cl^*(int^*(A)))) \subset int^*(cl^*(int^*(A)))$.
This implies that $int^*(A)$ is a $S.P^* - I-$ open.

The following theorem shows that the union of $S.\alpha^* - I-$ open sets gives a $S.\alpha^* - I-$ open set, while the intersection of a $S.\alpha^* - I-$ open set and an open set gives a $S.\alpha^* - I-$ open set.

Theorem 17. *Let (X, τ, I) be a space, A and B are subsets of X. the following are hold:*

1. *If $U \in S.\alpha^* IO(X, \tau)$, for each $\gamma \in \Delta$, then $\bigcup\{U_\gamma : \gamma \in \Delta\} \in S.\alpha^* IO(X, \tau)$ and If $U \in S.\alpha^* IC(X, \tau)$, for each $\gamma \in \Delta$, then $\bigcap\{U_\gamma : \gamma \in \Delta\} \in S.\alpha^* IC(X, \tau)$,*
2. *If $A \in S.\alpha^* IO(X, \tau)$, and $B \in \tau$, then $A \cap B \in S.\alpha^* IO(X, \tau)$ and If $A \in S.\alpha^* IC(X, \tau)$ and $B \in \tau^c$, then $A \cup B \in S.\alpha^* IC(X, \tau)$,*
3. *If $A \in S.\alpha^* IO(X)$ and B is a $S.\beta - I-$ open set, then $A \cup B$ is a $\beta^* - I-$ open set and If $A \in S.\alpha^* IC(X)$ and B is a $S.\beta - I-$ closed set, then $A \cap B$ is a $\beta^* - I-$ closed set.*

Proof. We only need to prove the case of opennes.

1. Since $U_\gamma \in S\alpha^* IO(X, \tau)$, we have $U_\gamma \subset itn^*(cl^*(int^*(U_\gamma)))$, for each $\gamma \in \Delta$. Then we obtain
$$\begin{array}{rcl} \bigcup_{\gamma \in \Delta} U_\gamma & \subset & \bigcup_{\gamma \in \Delta} int^*(cl^*(int^*(U_\gamma))) \\ & \subset & int^*(\bigcup_{\gamma \in \Delta} cl^*(int^*(U_\gamma))) \\ & = & int^*(cl^*(\bigcup_{\gamma \in \Delta} int^*(U_\gamma))) \\ & \subset & int^*(cl^*(int^*(\bigcup_{\gamma \in \Delta} U_\gamma))) \end{array}$$
This shows that $\bigcup_{\gamma \in \Delta} U_\gamma \in S.\alpha^* IO(X, \tau)$.

2. Let $A \in S.\alpha^* IO(X, \tau)$ and $B \in \tau$. Then $A \subset int^*(cl^*(int^*(A)))$ and $B = int(B) \subset int^*(B)$. Thus, we obtain

$$\begin{aligned} A \cap B &\subset int^*(cl^*(int^*(A))) \cap int^*(B) \\ &\subset int^*(cl^*(int^*(A)) \cap B) \\ &= int^*(((int^*(A))^* \cup int^*(A)) \cap B) \\ &= int^*(((int^*(A))^* \cap B) \cup (int^*(A) \cap B)) \\ &\subset int^*((int^*(A \cap B))^* \cup (int^*(A) \cap B)) \\ &\subset int^*((int^*(A \cap B))^* \cup int^*(A \cap B)) \\ &= int^*(cl^*(int^*(A \cap B))). \end{aligned}$$

Hence $A \cap B$ is a $S.\alpha^* - I-$ open.

3. Let A is a $S.\alpha^* - I-$ open set, then $A \subset int^*(cl^*(int^*(A)))$, B is a $S.\beta - I-$ open, then $B \subset cl^*(int(cl^*(B)))$. Now

$$\begin{aligned} A \cup B &\subset int^*(cl^*(int^*(A))) \cup cl^*(int(cl^*(B))) \\ &\subset cl^*(int^*(cl^*(A))) \cup cl^*(int^*(cl^*(B))) \\ &\subset cl(int^*(cl(A))) \cup cl(int^*(cl(B))) \\ &= cl(int^*(cl(A)) \cup int^*(cl(B))) \\ &\subset cl(int^*(cl(A) \cup cl(B))) \\ &= cl(int^*(cl(A \cup B))). \end{aligned}$$

Hence $A \cup B$ is a $\beta^* - I-$ open set.

Theorem 18. *Let (X, τ, I) be a space, where I is codense then the following hold:*

1. *Every $S.\alpha^* - I-$ open set is a $\beta - I-$ open set,*
2. *Every $S.\alpha^* - I-$ open set is a $pre - I-$ open set,*
3. *Every $S.\alpha^* - I-$ open set is a weakly $semi - I-$ open set.*

Proof. 1. Let A is a $S.\alpha^* - I-$ open set, then

$$\begin{aligned} A &\subset int^*(cl^*(int^*(A))) \\ &\subset cl^*(int^*(cl^*(A))) \\ &= cl^*(int(cl^*(A))) \\ &\subset cl(int(cl^*(A))). \end{aligned}$$

Hence A is a $\beta - I-$ open set.

2. Let A is a $S.\alpha^* - I-$ open set, then

$$\begin{aligned} A &\subset int^*(cl^*(int^*(A))) \\ &\subset int^*(cl^*(cl^*(A))) \\ &= int^*(cl^*(A)) \\ &= int(cl^*(A)). \end{aligned}$$

Hence A is a $pre - I-$ open set.

3. Let A is a $S.\alpha^* - I-$ open set, then

$$\begin{aligned} A &\subset int^*(cl^*(int^*(A))) \\ &\subset cl^*(int^*(cl^*(A))) \\ &= cl^*(int(cl^*(A))) \\ &\subset cl^*(int(cl(A))). \end{aligned}$$

Hence A is a weakly $semi - I-$ open set.

Remark 5. *The reverse of theorem 18 is not true in general as shown in the following example.*

Example 9. *Let $X = \{a, b, c, d\}$, $\tau = \{\phi, X, \{c\}, \{a, b, d\}\}$ and $I = \{\phi, \{a\}\}$, then we get*

1. *$A = \{b\}$ is a $\beta - I-$ open set, but $A \notin S.\alpha^* IO(X)$,*
2. *$A = \{a, b\} \in PIO(X)$ set, but $A \notin S.\alpha^* IO(X)$,*
3. *$A = \{c, d\}$ is a weakly $semi - I-$ open set, but it $A \notin S.\alpha^* IO(X)$.*

Theorem 19. *Let (X, τ, I) be a space and $A \subset X$ be a $*-$ closed set. Then A is a $S.\alpha^* - I-$ open set if and only if A is a $S.P^* - I-$ open set.*

Proof. let A be a $S.\alpha^* - I-$ open set, then A is a $S.P^* - I-$ open set.
conversely, let A be a $S.P^* - I-$ open set, then $A \subset int^*(cl^*(A))$.
Since A is a $*-$ closed set, then $int^*(cl^*(A)) = int^*(A)$.
Now $A = int^*(A) \subset int^*(cl^*(int^*(A)))$. Which shows A is a $S.\alpha^* - I-$ open set.

Theorem 20. *Let (X, τ, I) be a space, and $A \subset X$, then the followings hold:*

1. *A is a $S.\alpha^* - I-$ open set, if it is both strong $\beta - I-$ open set and strong $S_{\beta I}-$ set,*
2. *A is a $S.\alpha^* - I-$ open set, if it is both $semi - I-$ open set and $S - I-$ set.*

Proof.
1. Let A be a strong $\beta - I-$ open set, then $A \subset cl^*(int(cl^*(A)))$. Since A is a strong $S_{\beta I}-$ set, then $int(A) = cl^*(int(cl^*(A)))$.
Now $A = int(A) \subset int^*(cl^*(int^*(A)))$. Hence A is a $S.\alpha^* - I$ -open set.
2. Let A be a $semi - I-$ open set, then $A \subset cl^*(int(A))$. Since A is a $S - I-$ set, then $int(A) = cl^*(int(A))$.Now $A = int(A) \subset int^*(cl^*(int^*(A)))$.
Hence A is a $S.\alpha^* - I-$ open set.

Theorem 21. *Let (X, τ, I) be a space. A is a $S.\alpha^* - I-$ open set if it is both $pre^* - I-$ open set and closed set (resp. A is a $S.\alpha^* - I-$ closed set if it is both $pre^* - I-$ closed set and open set).*

Proof. According to the duality of closeness and opennes, we only need to prove the case of $S.\alpha^* - I-$ open.
Let A is a $pre^* - I-$ open set, then $A \subset int^*(cl(A))$. Since A is a closed set,then

$$\begin{aligned} A &\subset int^*(cl(A)) \\ &= int^*(A) \\ &\subset int^*(cl^*(int^*(A))). \end{aligned}$$

Hence A is a $S.\alpha^* - I-$ open set.

Theorem 22. *Let (X, τ, I) be a space and $A \subset X$ be $\alpha-$ open set and $\beta-$ closed set. Then A is a $S.\alpha^* - I-$ open set.*

Proof. Let A is an $\alpha-$ open set, then $A \subset int(cl(int(A)))$, since A is a $\beta-$ closed set, then $A \supset int(cl(int(A))) \Rightarrow A = int(cl(int(A))) \Rightarrow int(A) = int(cl(int(A)))$.
Now $A = int(A) \subset int^*(cl^*(int^*(A)))$. Hence A is a $S.\alpha^* - I-$ open set.

Theorem 23. *Let (X, τ, I) be an ideal topological space, $A \subset X$ and A is a $S.\alpha^* - I-$ open set, then the followings hold:*

1. $S.S^*Icl(A) = int^*(cl^*(A))$,
2. $S.P^*Icl(A) = cl^*(int^*(A))$.

Proof. Let A be a $S.\alpha^* - I-$ open set in X. Then we have:

1. $A \subset int^*(cl^*(int^*(A))) \subset int^*(cl^*(A))$.
   Thus we have $S.S^*Icl(A) = int^*(cl^*(A))$.
2. $A \subset int^*(cl^*(int^*(A))) \subset cl^*(int^*(A))$.
   Hence $S.P^*Icl(A) = cl^*(int^*(A))$.

Remark 6. *The reverse of theorem 23 is not true in general as shown in the following examples.*

Example 10. *From example 3 if $A = \{b\}$, then $S.S^*Icl(A) = int^*(cl^*(A))$, but $A \notin S.\alpha^*IO(X)$.*

Example 11. *From example 4 if $A = \{b, c\}$, then $S.P^*Icl(A) = cl^*(int^*(A))$, but $A \notin S.\alpha^*IO(X)$.*

Theorem 24. *Let (X, τ, I) be a space, $A \subset X$ and A is a $S.\alpha^* - I-$ closed set then the followings hold:*

1. $S.S^*Iint(A) = cl^*(int^*(A))$,
2. $S.P^*Iint(A) = int^*(cl^*(A))$.

Proof. Let A be a $S.\alpha^* - I-$ closed set in X. Then we have

1. $A \supset cl^*(int^*(cl^*(A))) \supset cl^*(int^*(A))$.
   Thus we have $S.S^*Iint(A) = cl^*(int^*(A))$.
2. $A \supset cl^*(int^*(cl^*(A))) \supset int^*(cl^*(A))$.
   Hence $S.P^*Iint(A) = int^*(cl^*(A))$.

Remark 7. *The reverse of theorem 24 is not true in general as shown in the following examples.*

Example 12. *From example 3 if* $A = \{a, b\}$*, then* $S.S^*Iint(A) = cl^*(int^*(A))$*, but* $A \notin S.\alpha^*IC(X)$.

Example 13. *From example 4 if* $A = \{c\}$*, then* $S.P^*Iint(A) = int^*(cl^*(A))$*, but* $A \notin S.\alpha^*IC(X)$.

Theorem 25. *Let* (X, τ, I) *be a space. If* A *is* $*-$ *perfect,and* A *is* $S.\alpha^* - I-$ *open set ,then the following hold:*

1. *A is an $\alpha-$ open set,*
2. *A is an almost strong $I-$ open set,*
3. *A is a $semi - I-$ open set.*

Proof. 1. Let A be a $S.\alpha^* - I-$ open set, then

$$\begin{aligned} A &\subset int^*(cl^*(int^*(A))) \\ &\quad = int(cl^*(int(A)) \\ &\subset int(cl(int(A))). \end{aligned}$$

This implies A is an $\alpha-$ open set.

2. Let A be a $S.\alpha^* - I-$ open set, then

$$\begin{aligned} A &\subset int^*(cl^*(int^*(A))) \\ &\subset cl^*(cl^*(int^*(A))) \\ &= cl^*(int(A)) \\ &= cl^*(int(A^*). \end{aligned}$$

Hence A is an almost strong $I-$ open set.

3. Let A be a $S.\alpha^* - I-$ open set, then

$$\begin{aligned} A &\subset int^*(cl^*(int^*(A))) \\ &\subset cl^*(int^*(A)) \\ &= cl^*(int(A)). \end{aligned}$$

This implies A is a $semi - I-$ open set.

Theorem 26. *Let (X, τ, I) be a space and $A \subset A^*$ and A^* is a $S.\alpha^* - I-$ closed set. Then $X - cl^*(A)$ is a $S.\alpha^* - I-$ open set.*

Proof. Given $A \subset A^*$, then $A^* = cl(A) = cl^*(A)$. Also A^* is a $S.\alpha^* - I-$ closed set, $X - A^*$ is $S.\alpha^* - I-$ open set.Therefore, $X - cl^*(A)$ is a $S.\alpha^* - I-$ open set.

Theorem 27. *Let (X, τ, I) be a space. Then $A \cup (X - A^*)$ is a $S.\alpha^* - I-$ closed set if and only if $A^* - A$ is a $S.\alpha^* - I-$ open set .*

Proof. Suppose $A \cup (X - A^*)$ is a $S.\alpha^* - I-$ closed set. Since $X - (A^* - A) = A \cup (X - A^*)$, then $A^* - A$ is a $S.\alpha^* - I-$ open set.
Converse part is obviously true.

Theorem 28. *Let (X, τ, I) be a space and $A \subset X$, then*

1. *If A is a $S.P^* - I-$ closed set and $S.\alpha^* - I-$ open set, then A is a $*-$ open set,*
2. *If A is a f_I- set which is $\alpha-$ open set, then A is a $S.\alpha^* - I-$ open set.*

Proof.
1. Let A is a $S.P^* - I - closed$ set and $S.\alpha^* - I - open$ set, then $cl^*(int^*(A)) \subset A$ and $A \subset int^*(cl^*(int^*(A)))$. Now $A \subset int^*(A) = int^*(A)$. Hence A is a $*-$ open set.
2. Let A is a f_I- set, then $A \subset (int(A))^*$ and so $int(A) \subset (int(A))^*$ and $cl(int(A)) = cl^*(int(A))$. Since A is an $\alpha-$ open set, then

$$\begin{aligned} A &\subset int(cl(int(A))) \\ &= int(cl^*(int(A))) \\ &\subset int^*(cl^*(int^*(A))). \end{aligned}$$

Hence A is a $S.\alpha^* - I-$ open set .

Remark 8. *The converse of the results in theorem 28 are not true in general, as shown by the following examples.*

Example 14. *From example 2 if we take*

1. *$A = \{a\} \in \tau^*$ and $A \in S.\alpha^*IO(X)$, but $A \notin SP^*IC(X)$ set,*
2. *$A = \{a, b\} \in S.\alpha^*IO(X)$, but it is not $\alpha-$ open set or f_I- set.*

Example 15. *From example 3 if we take*

1. *$A = \{a\}$ then $A \in S.\alpha^*IO(X)$ and A is an $\alpha-$ open set, but A is not f_I- set,*
2. *$A = \{b, c, d\}$ then $A \in S.\alpha^*IO(X)$ and A is af_I-set, while A is not $\alpha-$ open .*

Theorem 29. *Let (X, τ, I) be a space. If A is a $*-$ perfect, then every $S.\alpha^* - I-$ open set is a weakly $pre - I-$ open set.*

Proof. Let A be a $S.\alpha^* - I-$ open set, then

$$\begin{aligned} A &\subset int^*(cl^*(int^*(A))) \\ &\subset int^*(cl^*(int^*(A \cup A^*))) \\ &= int(cl^*(int(cl^*(A)))) \\ &\subset int(cl(int(cl^*(A)))) \\ &= {}_scl(int(cl^*(A))). \end{aligned}$$

Hence A is a weakly $pre - I-$ open set.

Theorem 30. *Let (X, τ, I) be a space and $A \subset X$, if A is an $I_\beta-$ set, then every $\beta - I-$ open set is a $S.\alpha^* - I-$ open set.*

Proof. Let A is a $\beta - I-$ open set, then $A \subset cl(int(cl^*(A)))$. Since A is an $I_\beta-$ set, then $cl(int(cl^*(A))) = int(A)$. Hence

$$\begin{aligned} A &\subset cl(int(cl^*(A))) \\ &= int(A) \\ &\subset int^*(cl^*(int^*(A))). \end{aligned}$$

which shows that A is a $S.\alpha^* - I-$ open set.

3. Strong $\alpha^* - I-$ Interior and strong $\alpha^* - I-$ Closure Operators

This section introduces the definitions of Strong $\alpha^* - I-$ Interior and strong $\alpha^* - I-$ Closure Operators and some of their properties.

Definition 8. *The strong $\alpha^* - I-$ interior of a subset A of a space (X, τ, I) denoted by $S.\alpha^* Iint(A)$ is defined by union of all strong $\alpha^* - I-$ open sets of X contained A.*

$$S.\alpha^* Iint(A) = \{\cup B : B \subset A, B \textit{ is an } S.\alpha^* - I - \textit{ open set}\}.$$

The following theorem provides an equivalent definition for definition 8.

Theorem 31. *For a subset A of a space (X, τ, I), $S.\alpha^* Iint(A) = A \cap int^*(cl^*(int^*(A)))$*

Proof. If A is any subset of X, then

$$\begin{aligned} A \cap int^*(cl^*(int^*(A))) &\subset int^*(cl^*(int^*(A))) \\ &= int^*(cl^*(int^*(int^*(A)))) \\ &= int^*(cl^*(int^*(A \cap int^*(A)))) \\ &\subset int^*(cl^*(int^*(A \cap int^*(cl^*(int^*(A)))))). \end{aligned}$$

Hence $A \cap int^*(cl^*(int^*(A)))$ is a $S.\alpha^* - I-$ open set contained in A.
Therefore , $A \cap int^*(cl^*(int^*(A))) \subset S.\alpha^* Iint(A)$.
On other hand, since $S.\alpha^* Iint(A)$ is $S.\alpha^* - I-$ open set, then

$$\begin{aligned} S.\alpha^* Iint(A) &\subset int^*(cl^*(int^*(S.\alpha^* IInt(A)))) \\ &\subset int^*(cl^*(int^*(A))). \end{aligned}$$

,so $S.\alpha^* Iint(A) \subset A \cap int^*(cl^*(int^*(A)))$.
Therefore, $S.\alpha^* Iint(A) = A \cap int^*(cl^*(int^*(A)))$.

Lemma 32. *Let (X,τ,I) be a space and $A \subset X$, then A is a $S.\alpha^* - I-$ open set if and only if $S.\alpha^* Iint(A) = A$*

Proof. Let A is a $S.\alpha^* - I-$ open set, then $A \subset int^*(cl^*(int^*(A)))$.
Hence

$$\begin{aligned} S.\alpha^* Iint(A) &= A \cap int^*(cl^*(int^*(A))) \\ &= A. \end{aligned}$$

Conversely, since $S.\alpha^* Iint(A) = A \cap int^*(cl^*(int^*(A)))$ and by hypothesis $S.\alpha^* Iint(A) = A$, we get $A \subset int^*(cl^*(int^*(A)))$.
This implies that A is a $S.\alpha^* - I-$ open set.

Definition 9. *The strong $\alpha^* - I-$ closure of a subset A of a space (X,τ,I) denoted by $S.\alpha^* Icl(A)$ is defined by intersection of all strong $\alpha^* - I-$ closed sets of X containing A.*

$$S.\alpha^* Icl(A) = \{\cap B : B \supset A, B \text{ is an } S.\alpha^* - I - \text{closed set}\}.$$

Lemma 33. *Let $A \subset (X,\tau,I)$, then*

1. $X - S.\alpha^* Iint(A) = S.\alpha^* Icl(X - A)$,
2. $X - S.\alpha^* Icl(A) = S.\alpha^* Iint(X - A)$.

Proof. 1. Since $S.\alpha^* Iint(A) = \{\cup B : B \subset A, B \text{ is a } S.\alpha^* - I - \text{ open set}\}$, then

$$\begin{aligned} X - S.\alpha^* Iint(A) &= X - \{\cup B : B \subset A, B \text{ is a } S.\alpha^* - I - \text{ open set}\} \\ &= \{\cap X - B : X - B \supset X - A, X - B \text{ is a } S.\alpha^* - I - \text{ closed set}\} \\ &= \{\cap F : F \supset X - A, F \text{ is a } S.\alpha^* - I - \text{ closed set}\} \\ &= S.\alpha^* Icl(X - A). \end{aligned}$$

2. Since $S.\alpha^* Icl(A) = \{\cap B : B \supset A, B \text{ is a } S.\alpha^* - I - \text{ closed set}\}$, then

$$\begin{aligned} X - S.\alpha^* Icl(A) &= X - \{\cap B : B \supset A, B \text{ is a } S.\alpha^* - I - closed \text{ set}\} \\ &= \{\cup X - B : X - B \subset X - A, X - B \text{ is a } S.\alpha^* - I - \text{ open set}\} \\ &= \{\cup F : F \subset X - A, F \text{ is a } S.\alpha^* - I - \text{ open set}\} \\ &= S.\alpha^* Iint(X - A). \end{aligned}$$

The following theorem provides an equivalent definition for definition 9.

Theorem 34. *For $A \subset (X,\tau,I)$, $S.\alpha^* Icl(A) = A \cup cl^*(int^*(cl^*(A)))$*

Proof. If A is any subset of X, then

$$\begin{aligned} A \cup cl^*(int^*(cl^*(A))) &\supset cl^*(int^*(cl^*(A))) \\ &= cl^*(int^*(cl^*(cl^*(A)))) \\ &= cl^*(int^*(cl^*(A \cup cl^*(A)))) \\ &\supset cl^*(int^*(cl^*(A \cup cl^*(int^*(cl^*(A)))))). \end{aligned}$$

Thus $A \cup cl^*(int^*(cl^*(A)))$ is a $S.\alpha^* - I - closed$ set containing A. Thus
$S.\alpha^* Icl(A) \subset A \cup cl^*(int^*(cl^*(A)))$.
On other hand, since $S.\alpha^* Icl(A)$ is a $S.\alpha^* - I - closed$ set, we have

$$\begin{aligned} S.\alpha^* Icl(A) &\supset cl^*(int^*(cl^*(S.\alpha^* Icl(A)))) \\ &\supset cl^*(int^*(cl^*(A))). \end{aligned}$$

,so $S.\alpha^* Icl(A) \supset A \cup cl^*(int^*(cl^*(A)))$. Therefore, $S.\alpha^* Icl(A) = A \cup cl^*(int^*(cl^*(A)))$.

Theorem 35. *Let $A \subset (X, \tau, I)$, then A is a $S.\alpha^* - I - closed$ set if and only if $S.\alpha^* Icl(A) = A$.*

Proof. Let A is a $S.\alpha^* - I - closed$ set, then $A \supset cl^*(int^*(cl^*(A)))$.
Hence $S.\alpha^* Icl(A) = A \cup cl^*(int^*(cl^*(A))) = A$.
Conversely, since $S.\alpha^* Icl(A) = A \cup cl^*(int^*(cl^*(A)))$ and by hypothesis
$S.\alpha^* Icl(A) = A$, we get $A \supset cl^*(int^*(cl^*(A)))$.
This implies that A is a $S.\alpha^* - I - closed$ set.

Theorem 36. *For $A \subset (X, \tau, I)$, if I is codense, then the following properties hold:*

1. $\beta Icl(A) \subset S.\alpha^* Icl(A)$,
2. $S.\alpha^* Iint(A) \subset pIint(A)$,
3. $S.\alpha^* Iint(A) \subset wsIint(A)$,
4. $wsIcl(A) \subset S.\alpha^* Icl(A)$.

Proof. 1. Since $\beta Icl(A) = A \cup int(cl(int^*(A)))$, then

$$\begin{aligned} \beta Icl(A) &= A \cup int(cl^*(int^*(A))) \\ &= A \cup int^*(cl^*(int^*(A))) \\ &\subset A \cup cl^*(int^*(cl^*(A))). \end{aligned}$$

Hence $\beta Icl(A) \subset S.\alpha^* Icl(A)$.

2. Since $S\alpha^* Iint(A) = A \cap int^*(cl^*(int^*(A)))$, then

$$\begin{aligned} S.\alpha^* IInt(A) &\subset A \cap int^*(cl^*(A)) \\ &= A \cap int(cl^*(A)) \\ &= pIint(A). \end{aligned}$$

Hence $S.\alpha^* Iint(A) \subset pIint(A)$.

3. Since $S.\alpha^* Iint(A) = A \cap int^*(cl^*(int^*(A)))$, then

$$\begin{aligned} S.\alpha^* Iint(A) &\subset A \cap cl^*(int^*(cl^*(A))) \\ &= A \cap cl^*(int(cl^*(A))) \\ &\subset A \cap cl^*(int(cl(A))) \\ &= wsIint(A) \end{aligned}$$

Hence $S.\alpha^* Iint(A) \subset wsIint(A)$.

4. since $wsIcl(A) = A \cup int^*(cl(int(A)))$, then

$$\begin{array}{rcl} wsIcl(A) & \subset & A \cup int^*(cl(int^*(A))) \\ & = & A \cup int^*(cl^*(int^*(A))) \\ & \subset & A \cap cl^*(int^*(cl^*(A))) \\ & = & S.\alpha^*Icl(A) \end{array}$$

Hence $wsIcl(A) \subset S.\alpha^*Icl(A)$.

Theorem 37. *For $A \subset (X, \tau, I)$, the following properties hold.*

1. $cl^*(S.\alpha^*Icl(A)) = cl^*(A)$,
2. $int^*(S.\alpha^*Iint(A)) = int^*(A)$.

Proof. 1. we know that $S.\alpha^*Icl(A) \supset A$, this implies that $cl^*(S.\alpha^*Icl(A)) \supset cl^*(A)$.
On other hand,

$$\begin{array}{rcl} cl^*(S.\alpha^*Icl(A)) & = & cl^*(A \cup cl^*(int^*(cl^*(A)))) \\ & = & cl^*(A) \cup cl^*(cl^*(int^*(cl^*(A)))) \\ & = & cl^*(A) \cup cl^*(int^*(cl^*(A))) \\ & = & cl^*(A \cup int^*(cl^*(A))) \\ & \subset & cl^*(A \cup cl^*(cl^*(A))) \\ & = & cl^*(A \cup cl^*(A)) \\ & = & cl^*(cl^*(A)) \\ & = & cl^*(A). \end{array}$$

This implies that $cl^*(S.\alpha^*Icl(A)) = cl^*(A)$.

2. we know that $S.\alpha^*Iint(A) \subset A$, this implies that $int^*(S.\alpha^*IInt(A)) \subset int^*(A)$.
On other hand,

$$\begin{array}{rcl} int^*(S.\alpha^*Iint(A)) & = & int^*(A \cap int^*(cl^*(int^*(A)))) \\ & = & int^*(A) \cap int^*(int^*(cl^*(int^*(A)))) \\ & = & int^*(A) \cap int^*(cl^*(int^*(A))) \\ & = & int^*(A \cap cl^*(int^*(A))) \\ & \supset & int^*(A \cap int^*(int^*(A))) \\ & = & int^*(A \cap int^*(A)) \\ & = & int^*(int^*(A)) \\ & = & int^*(A). \end{array}$$

This implies that $int^*(S.\alpha^*Iint(A)) = int^*(A)$.

Theorem 38. *For $A \subset X$ of a space (X, τ, I), the following properties are hold.*

1. *If A is a $S.P^* - I-$ open set in X, then $S.\alpha^*Icl(A) = cl^*(int^*(cl^*(A)))$,*

2. *If A is a $S.P^* - I-$ closed set in X, then $S.\alpha^* Iint(A) = int^*(cl^*(int^*(A)))$.*

Proof. 1. Let A is a $S.P^* - I-$ open set, then we have

$$\begin{aligned} A &\subset int^*(cl^*(A)) \\ &\subset cl^*(int^*(cl^*(A))). \end{aligned}$$

This implies that

$$\begin{aligned} S.\alpha^* Icl(A) &= A \cup cl^*(int^*(cl^*(A))) \\ &= cl^*(int^*(int^*(A))). \end{aligned}$$

2. Let A is a $S.P^* - I-$ closed set, then we have

$$\begin{aligned} A &\supset cl^*(int^*(A)) \\ &\supset int^*(cl^*(int^*(A))). \end{aligned}$$

This implies that

$$\begin{aligned} S.\alpha^* IIint(A) &= A \cap int^*(cl^*(int^*(A))) \\ &= int^*(cl^*(int^*(A))). \end{aligned}$$

Remark 9. *The reverse of theorem 38 is not true in general as shown by the following example.*

Example 16. *From example 4 if*

1. *$A = \{a, b\}$, then $S.\alpha^* Icl(A) = cl^*(int^*(cl^*(A)))$, but $A \notin SP^* IO(X)$,*
2. *$A = \{a\}$, then $S.\alpha^* IInt(A) = int^*(cl^*(int^*(A)))$, but $A \notin SP^* IC(X)$.*

Theorem 39. *For $A \subset (X, \tau, I)$, the following properties are hold.*

1. *If A is a $S.\beta - I-$ open set in X, then $S.\alpha^* Icl(A) = cl^*(int^*(cl^*(A)))$,*
2. *If A is a $S.\beta - I-$ closed set in X, then $S.\alpha^* Iint(A) = int^*(cl^*(int^*(A)))$.*

Proof. 1. Let A is a $S.\beta - I-$ open set, then we have

$$\begin{aligned} A &\subset cl^*(int(cl^*(A))) \\ &\subset cl^*(int^*(cl^*(A))). \end{aligned}$$

This implies that

$$\begin{aligned} S.\alpha^* Icl(A) &= A \cup cl^*(int^*(cl^*(A))) \\ &= cl^*(int^*(int^*(A))). \end{aligned}$$

2. Let A is a $S.\beta - I-$ closed set, then we have

$$\begin{aligned} A &\supset int^*(cl(int^*(A))) \\ &\supset int^*(cl^*(int^*(A))). \end{aligned}$$

This implies that

$$\begin{aligned} S.\alpha^* Iint(A) &= A \cap int^*(cl^*(int^*(A))) \\ &= int^*(cl^*(int^*(A))). \end{aligned}$$

Remark 10. *The reverse of theorem 39 is not true in general as shown by the following example.*

Example 17. *From example 1 if*

1. $A = \{b\}$, *then* $S.\alpha^*Icl(A) = cl^*(int^*(cl^*(A)))$, *but* A *is not* $S.\beta - I-$ *open set,*
2. $A = \{a\}$, *then* $S.\alpha^*Iint(A) = int^*(cl^*(int^*(A)))$, *but* A *is not* $S.\beta - I-$ *closed set.*

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