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ON CAPUTO NONLOCAL FRACTIONAL NEUTRAL INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract. This paper deals with the theoretical results for solutions of a nonlocal fractional neutral integro-differential problem with boundary integral conditions. We prove the existence and uniqueness results using the Krasnoselskii's and Banach fixed point theorems. Based on the results obtained, sufficient conditions are provided that ensure the generalized results. Finally, we give an example to illustrate our results.

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1. Introduction

In recent years, fractional differential equations have attracted the attention of many authors because of the numerous applications in various branches of science and engineering, in particular, fluid mechanics, image and signal processing, electromagnetic theory, potential theory, fractals theory, biology, control theory, viscoelasticity, and so on [1, 2, 26, 27, 28, 29]. From the mathematical point of view, a number of researchers working on fractional calculus conduct their research in the field of applications of different fractional operators and various structures of BVPs in modeling abstract and real-world phenomena, but the discussion related to the fractional derivatives is an old problem and continues to receive many kinds of feedback. The physical aspect of the fractional derivative is now proved in many investigations. As we know, fractional-order derivatives have many advantages in comparison to the first-order derivatives. For example, one of the most simple examples in which the fractional derivative has a significant impact can be observed in diffusion processes [22, 23, 30].

Integral boundary conditions have various applications in applied fields such as blood flow problems, chemical engineering, thermoelasticity, underground water flow, population dynamics, etc. For a detailed description of the integral boundary conditions, we refer the reader to some recent papers [6, 7, 8, 13, 14, 19, 20, 21, 24] and the references therein. On the other hand, we know that the delay arises naturally in systems due to the transmission of signal or the mechanical transmission. Moreover, the study of fractional order problems involving various types of delay (finite, infinite and state dependant) considered in Banach spaces has been receiving attention, see [4, 5, 9, 10, 11, 12, 15, 25] and references cited in these articles.

Wang et al. [31] discussed some sufficient conditions for the existence of the solutions of the following impulsive fractional differential equations

$$\begin{aligned}
&D_t^\alpha x(t) = f(t), \quad t \in J' := J \setminus \{t_1, ...t_m\}, J := [0,1], \alpha \in (1,2) \\
&\Delta x(t_k) = Q_k\left(x\left(t_k^-\right)\right), \quad k = 1, 2, \ldots, m, \\
&\Delta x'(t_k) = I_k\left(x\left(t_k^-\right)\right), \quad k = 1, 2, \ldots, m, \\
&x(0) + x'(0) = 0, \\
&x(1) + x'(1) = 0.
\end{aligned}$$

Authors in [31], discussed the existence of the solutions of boundary value problem for impulsive differential equations with Caputo fractional derivative

$$\begin{aligned}
&D_t^\alpha x(t) = f(t, x(t)), \quad t \in J' := J \setminus \{t_1, ...t_m\}, J := [0,1], \alpha \in (1,2) \\
&\Delta x(t_k) = Q_k \in \mathbb{R}, \quad k = 1, 2, \ldots, m, \\
&\Delta x'(t_k) = I_k \in \mathbb{R}, \quad k = 1, 2, \ldots, m, \\
&x(0) = 0, x'(1) = 0,
\end{aligned}$$

In [14] authors have established the existence and uniqueness of a solution for the following system

$$\begin{aligned}
&D_t^\alpha x(t) = f\left(t, x_t, Bx(t)\right), \quad t \in J = [0,T], t \neq t_k \\
&\Delta x(t_k) = Q_k\left(x\left(t_k^-\right)\right), \quad k = 1, 2, \ldots, m, \\
&\Delta x'(t_k) = I_k\left(x\left(t_k^-\right)\right), \quad k = 1, 2, \ldots, m, \\
&x(t) = \phi(t), \quad t \in (-\infty, 0] \\
&ax'(0) + bx'(T) = \int_0^T q(x(s))ds,
\end{aligned}$$

the results are proved by using the contraction and Krasnoselkii's fixed point theorems. This paper is motivated from some recent papers treating the boundary value problems for impulsive fractional differential equations [7, 14, 31].

In this paper, we examine the existence and uniqueness results of fractional neutral Volterra-Fredholm integro-differential equation of the form

$$D_t^\alpha\left[x(t)+\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g\left(s, x_{\rho\left(s, x_s\right)}\right) d s\right]=f\left(t, x_{\rho\left(t, x_t\right)}, B(x)(t), A(x)(t)\right), \quad (1)$$

$$\Delta x\left(t_k\right)=I_k\left(x\left(t_k^{-}\right)\right), \quad \Delta x^{\prime}\left(t_k\right)=Q_k\left(x\left(t_k^{-}\right)\right), t \neq t_k,\ k=1,2, \ldots, m, \quad (2)$$

$$x(t)=\phi(t),\ t \in[-d, 0], \quad (3)$$

$$a x^{\prime}(0)+b x^{\prime}(T)=\int_0^T q(x(s)) d s,\ a+b \neq 0, b \neq 0, \quad (4)$$

where x' denotes the derivative of x with respect to $t \in J=[0,T]$, and $D_t^\alpha, \alpha \in (1,2)$ is Caputo's derivative. Let X be a Banach space and $PC_t := PC([-d,t];X), d > 0, 0 \leq t \leq T < \infty$, be a Banach space of all such functions $\phi : [-d,t] \to X$, which are continuous everywhere except for a finite number of points $t_i, i = 1,2,\ldots,m$, at which $\phi\left(t_i^+\right)$ and $\phi\left(t_i^-\right)$exists and $\phi\left(t_i\right) = \phi\left(t_i^-\right)$, endowed with the norm $\|\phi\|_t = \sup_{-d\leq s\leq t} \|\phi(s)\|_X, \phi \in PC_t$, where $\|\cdot\|_X$ is the norm in X.

The functions $f : J \times PC_0 \times X \to X, g : J \times PC_0 \to X$, and $q : X \to X$ are given continuous functions where $PC_0 = PC([-d,0],X)$ and for any $x \in PC_T = PC([-d,T],X), t \in J$, we denote by x_t the element of PC_0 defined by $x_t(\theta) = x(t+\theta), \theta \in [-d,0]$. In the impulsive conditions for $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T, Q_k, I_k \in C(X,X), (k = 1,2,\ldots,m)$, are continuous and bounded functions. We have $\Delta x\left(t_k\right) = x\left(t_k^+\right) - x\left(t_k^-\right)$and $\Delta x'\left(t_k\right) = x'\left(t_k^+\right) - x'\left(t_k^-\right)$. The terms $Bx(t)$ and $Ax(t)$ are given by

$$B x(t)=\int_0^t K(t, s) x(s) d s,\ A x(t)=\int_0^T H(t, s) x(s) d s, \quad (5)$$

where $K, H \in C\left(D, \mathbb{R}^+\right)$, be the set of all positive functions which are continuous on $D = \left\{(t,s) \in \mathbb{R}^2 : 0 \leq s \leq t < T\right\}$ and

$$B^*=\sup _{t \in[0, t]} \int_0^t K(t, s) d s<\infty, \quad A^*=\sup _{t \in[0, t]} \int_0^T H(t, s) d s<\infty .$$

To the best of our knowledge, there is no work available in literature on impulsive neutral fractional Volterra-Fredholm integro-differential equation with state dependent delay and with an integral boundary condition. In this article, we first establish a general framework to find a solution to system (1)-(4) and then by using classical fixed point theorems we proved the existence and uniqueness results.

2. Preliminaries

Let us recall some basic definitions of fractional calculus [17, 18, 26, 27, 28, 29].

Definition 1. *Caputo's derivative of order α for a function $f : [0, \infty) \to \mathbb{R}$ is defined as*

$$D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s) ds = J^{n-\alpha} f^{(n)}(t) \tag{6}$$

for $n-1 \le \alpha < n, n \in N$. If $0 \le \alpha < 1$, then

$$D_t^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} f^{(1)}(s) ds \tag{7}$$

Definition 2. *The Riemann-Liouville fractional integral operator for order $\alpha > 0$, of a function $f : \mathbb{R}^+ \to \mathbb{R}$ and $f \in L^1(\mathbb{R}^+, X)$ is defined by*

$$J_t^0 f(t) = f(t), J_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, \quad \alpha > 0, t > 0 \tag{8}$$

where $\Gamma(\cdot)$ is the Euler gamma function.

Lemma 1. *([31]) For $\alpha > 0$, the general solution of fractional differential equations $D_t^{\alpha} x(t) = 0$ is given by $x(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3 + \cdots + c_{n-1} t^{n-1}$ where $c_i \in \mathbb{R}, i = 0, 1, \ldots, n-1, n = [\alpha] + 1$ and $[\alpha]$ represent the integral part of the real number α.*

Lemma 2. *([16], Lemma 2.6). Let $\alpha \in (1, 2), c \in \mathbb{R}$ and $h : J \to \mathbb{R}$ be continuous function. A function $x \in C(J, \mathbb{R})$ is a solution of the following fractional integral equation*

$$x(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s) ds - \int_0^w \frac{(w-s)^{\alpha-1}}{\Gamma(\alpha)} h(s) ds + x_0 - c(t-w), \tag{9}$$

if and only if x is a solution of the following fractional Cauchy problem

$$D_t^{\alpha} x(t) = h(t), \quad t \in J, x(w) = x_0, w \ge 0. \tag{10}$$

As a consequence of Lemma 1 and Lemma 2 we have the following result.

Lemma 3. *Let $\alpha \in (1, 2)$ and $f : J \times PC_0 \times X \to X$ be continuously differentiable function. A piecewise continuous differential function $x(t) : (-d, T] \to X$ is a solution of system (1)-(4) if and only if x satisfied the integral equation*

$$x(t)=\begin{cases}\phi(t), & t\in[-d,0]\\ \phi(0)-\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}g\left(s,x_{\rho(s,x_s)}\right)ds+\frac{bt}{a+b}\left\{\frac{1}{b}\int_0^T q(x(s))ds\right. & \\ -\sum_{i=1}^k Q_i\left(x\left(t_i^-\right)\right)+\int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}g\left(s,x_{\rho(s,x_s)}\right)ds & \\ \left.-\int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}f\left(s,x_{\rho(s,x_s)},B(x)(s),A(x)(s)\right)ds\right\} & \\ +\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f\left(s,x_{\rho(s,x_s)},B(x)(s),A(x)(s)\right)ds, & t\in[0,t_1]\\ \cdots & \\ \phi(0)+\sum_{i=1}^k I_i\left(x\left(t_i^-\right)\right)+\sum_{i=1}^k(t-t_i)Q_i\left(x\left(t_i^-\right)\right) & \\ -\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}g\left(s,x_{\rho(s,x_s)}\right)ds+\frac{bt}{a+b}\left\{\frac{1}{b}\int_0^T q(x(s))ds\right. & \\ -\sum_{i=1}^k Q_i\left(x\left(t_i^-\right)\right)+\int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}g\left(s,x_{\rho(s,x_s)}\right)ds & \\ \left.-\int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}f\left(s,x_{\rho(s,x_s)},B(x)(s),A(x)(s)\right)ds\right\} & \\ +\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f\left(s,x_{\rho(s,x_s)},B(x)(s),A(x)(s)\right)ds, & t\in(t_k,t_{k+1}]\end{cases} \tag{11}$$

Proof. If $t\in[0,t_1]$, then

$$\begin{aligned}&D_t^\alpha\left[x(t)+\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}g\left(s,x_{\rho(s,x_s)}\right)ds\right]=f\left(t,x_{\rho(t,x_t)},B(x)(t),A(x)(s)\right),\\ &x(t)=\phi(t),t\in[-d,0].\end{aligned} \tag{12}$$

Taking the Riemann-Liouville fractional integral of (12) and using the Lemma (2), we have

$$\begin{aligned}&x(t)+\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}g\left(s,x_{\rho(s,x_s)}\right)ds \\ &=a_0+b_0t+\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f\left(s,x_{\rho(s,x_s)},B(x)(s),A(x)(s)\right)ds,\end{aligned} \tag{13}$$

using the initial condition, we get $a_0=\phi(0)$, then (13) becomes

$$\begin{aligned}&x(t)+\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}g\left(s,x_{\rho(s,x_s)}\right)ds \\ &=\phi(0)+b_0t+\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f\left(s,x_{\rho(s,x_s)},B(x)(s),A(x)(s)\right)ds.\end{aligned} \tag{14}$$

Similarly, if $t\in(t_1,t_2]$, then

$$D_t^{\alpha}\left[x(t)+\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g\left(s, x_{\rho(s,x_s)}\right) ds\right] = f\left(t, x_{\rho(t,x_t)}, B(x)(t), A(x)(t)\right), \quad (15)$$

$$x\left(t_1^{+}\right) = x\left(t_1^{-}\right) + I_1\left(x\left(t_1^{-}\right)\right), \quad (16)$$

$$x'\left(t_1^{+}\right) = x'\left(t_1^{-}\right) + Q_1\left(x\left(t_1^{-}\right)\right). \quad (17)$$

Again apply the Riemann-Liouville fractional integral operator on (15) and using the lemma 2, we obtain

$$\begin{aligned} &x(t)+\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g\left(s, x_{\rho(s,x_s)}\right) ds \\ &= a_1 + b_1 t + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f\left(s, x_{\rho(s,x_s)}, B(x)(s), A(x)(s)\right) ds, \end{aligned} \quad (18)$$

rewrite (18) as

$$\begin{aligned} &x\left(t_1^{+}\right)+\int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} g\left(s, x_{\rho(s,x_s)}\right) ds \\ &= a_1 + b_1 t_1 + \int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} f\left(s, x_{\rho(s,x_s)}, B(x)(s), A(x)(s)\right) ds, \end{aligned} \quad (19)$$

due to impulsive condition (16) and the fact that $x(t_1) = x\left(t_1^{-}\right)$, we may write (19) as

$$\begin{aligned} &x(t_1) + I_1\left(x\left(t_1^{-}\right)\right)+\int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} g\left(s, x_{\rho(s,x_s)}\right) ds \\ &= a_1 + b_1 t_1 + \int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} f\left(s, x_{\rho(s,x_s)}, B(x)(s), A(x)(s)\right) ds. \end{aligned} \quad (20)$$

Now from (14), we have

$$\begin{aligned} &x(t_1)+\int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} g\left(s, x_{\rho(s,x_s)}\right) ds \\ &= \phi(0) + b_0 t_1 + \int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} f\left(s, x_{\rho(s,x_s)}, B(x)(s), A(x)(s)\right) ds. \end{aligned} \quad (21)$$

From (20) and (21), we get $a_1 = \phi(0) + b_0 t_1 - b_1 t_1 + I_1\left(x\left(t_1^{-}\right)\right)$, hence (19) can be written as

$$\begin{aligned} x(t)+\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g\left(s, x_{\rho(s,x_s)}\right) ds &= \phi(0) + b_0 t_1 + b_1 (t - t_1) + I_1\left(x\left(t_1^{-}\right)\right) \\ + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f\left(s, x_{\rho(s,x_s)}, B(x)(s), A(x)(s)\right) ds. \end{aligned} \quad (22)$$

On differentiating (18) with respect to t at $t = t_1$, and incorporate second impulsive condition (17), we obtain

$$\begin{aligned} &x'\left(t_1^-\right) + Q_1\left(x\left(t_1^-\right)\right) + \int_0^{t_1} \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} g\left(s, x_{\rho(s,x_s)}\right) ds \\ &= b_1 + \int_0^{t_1} \frac{(t_1-s)^{\alpha-2}}{\Gamma(\alpha-1)} f\left(s, x_{\rho(s,x_s)}, B(x)(s), A(x)(s)\right) ds \end{aligned} \tag{23}$$

Now differentiating (14), with respect to t at $t = t_1$, we get

$$\begin{aligned} &x'\left(t_1\right) + \int_0^{t_1} \frac{(t_1-s)^{\alpha-2}}{\Gamma(\alpha-1)} g\left(s, x_{\rho(s,x_s)}\right) ds \\ &= b_0 + \int_0^{t_1} \frac{(t_1-s)^{\alpha-2}}{\Gamma(\alpha-1)} f\left(s, x_{\rho(s,x_s)}, B(x)(s), A(x)(s)\right) ds. \end{aligned} \tag{24}$$

From (23) and (24), we obtain $b_1 = b_0 + Q_1\left(x\left(t_1^-\right)\right)$. Thus, (22) become

$$\begin{aligned} &x(t) + \int_0^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g\left(s, x_{\rho(s,x_s)}\right) ds \\ &= \phi(0) + b_0 t + I_1\left(x\left(t_1^-\right)\right) + \left(t-t_1\right) Q_1\left(x\left(t_1^-\right)\right) \\ &\quad + \int_0^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f\left(s, x_{\rho(s,x_s)}, B(x)(s), A(x)(s)\right) ds. \end{aligned} \tag{25}$$

Similarly, for $t \in (t_2, t_3]$, we can write the solution of the problem as

$$\begin{aligned} &x(t) + \int_0^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g\left(s, x_{\rho(s,x_s)}\right) ds \\ &= \phi(0) + b_0 t + I_1\left(x\left(t_1^-\right)\right) + I_2\left(x\left(t_2^-\right)\right) + \left(t-t_1\right) Q_1\left(x\left(t_1^-\right)\right) \\ &\quad + \left(t-t_2\right) Q_2\left(x\left(t_2^-\right)\right) + \int_0^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f\left(s, x_{\rho(s,x_s)}, B(x)(s), A(x)(s)\right) ds. \end{aligned}$$

In general, if $t \in (t_k, t_{k+1}]$, then we have the result

$$\begin{aligned} &x(t) + \int_0^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g\left(s, x_{\rho(s,x_s)}\right) ds \\ &= \phi(0) + b_0 t + \sum_{i=1}^{k} I_i\left(x\left(t_i^-\right)\right) + \sum_{i=1}^{k} \left(t-t_i\right) Q_i\left(x\left(t_i^-\right)\right) \\ &\quad + \int_0^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f\left(s, x_{\rho(s,x_s)}, B(x)(s), A(x)(s)\right) ds. \end{aligned} \tag{26}$$

Finally, we use the integral boundary condition $ax'(0) + bx'(T) = \int_0^T q(x(s))ds$, where $x'(0)$ calculated from (14) and $x'(T)$ from (25). On simplifying, we get the following value of the constant b_0,

$$b_0 = \frac{b}{a+b}\left\{\frac{1}{b}\int_0^T q(x(s))ds - \sum_{i=1}^{m} Q_i\left(x\left(t_i^-\right)\right) + \int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} g\left(s, x_{\rho(s,x_s)}\right) ds \right.$$
$$\left. - \int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} f\left(s, x_{\rho(s,x_s)}, B(x)(s), A(x)(s)\right) ds \right\}.$$

On summarizing, we obtain the desired integral equation (11). Conversely, assuming that x satisfies (11), by a direct computation, it follows that the solution given in (11) satisfies system (1)-(4). This completes the proof of the lemma.

3. Existence result

The function $\rho : J \times PC_0 \to [-d, T]$ is continuous and $\phi(0) \in PC_0$. Let the function t $\to \varphi_t$ be well defined and continuous from the set $\Re(\rho^-) = \{\rho(s,\psi) : (s,\psi) \in [0,T] \times PC_0\}$ into PC_0. Further, we introduce the following assumptions to establish our results.

(H1) There exist positive constants $L_{f1}, L_{f2}, L_{f3}, L_q$ and L_g, such that

$$\|f(t,\psi,x,z) - f(t,\chi,y,v)\|_X \leq L_{f1}\|\psi-\chi\|_{PC_0} + L_{f2}\|x-y\|_X + L_{f3}\|z-v\|_X$$
$$\|g(t,\psi) - g(t,\chi)\|_X \leq L_g\|\psi-\chi\|_{PC_0}, t \in J, \forall\psi,\chi \in PC_0, \forall x,y,z,v \in X$$
$$\|q(x) - q(y)\|_X \leq L_q\|x-y\|_X, \forall x,y \in X.$$

(H2) There exist positive constants L_Q, L_I, such that

$$\|Q_k(x) - Q_k(y)\|_X \leq L_Q\|x-y\|_X, \quad \|I_k(x) - I_k(y)\|_X \leq L_I\|x-y\|_X.$$

(H3) The functions Q_k, I_k, q are bounded continuous and there exist positive constants C_1, C_2, C_3, such that

$$\|Q_k(x)\|_X \leq C_1, \quad \|I_k(x)\|_X \leq C_2, \quad \|q(x)\|_X \leq C_3, \quad \forall x \in X.$$

Our first result is based on the Banach contraction theorem.

Theorem 4. *Let the assumptions (H1)-(H2) are satisfied with*

$$\begin{aligned}\triangle := \Bigg\{ & m\left(L_I + TL_Q\right) + \frac{T^\alpha L_g}{\Gamma(\alpha+1)} + \frac{bT}{a+b}\left(\frac{TL_q}{b} + mL_Q\right.\\ & \left. + \frac{T^{\alpha-1}L_g}{\Gamma(\alpha)} + \frac{T^{\alpha-1}\left(L_{f1} + L_{f2}B^* + L_{f3}A^*\right)}{\Gamma(\alpha)}\right)\\ & + \frac{T^\alpha\left(L_{f1} + L_{f2}B^* + L_{f3}A^*\right)}{\Gamma(\alpha+1)}\Bigg\} < 1\end{aligned}$$

Then (1)-(4) *has a unique solution.*

Proof. We transform problem (1)-(4) into a fixed point problem. Consider the operator $P : PC_T \to PC_T$ defined by

$$Px(t) = \begin{cases} \phi(t), & t \in [-d, 0],\\ \phi(0) - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g\left(s, x_{\rho(s,x_s)}\right) ds + \frac{bt}{a+b}\left\{\frac{1}{b}\int_0^T q(x(s))ds\right. & \\ - \sum_{i=1}^m Q_i\left(x\left(t_i^-\right)\right) + \int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} g\left(s, x_{\rho(s,x_s)}\right) ds & \\ \left. - \int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} f\left(s, x_{\rho(s,x_s)}, B(x)(s), A(x)(s)\right) ds\right\} & \\ + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f\left(s, x_{\rho(s,x_s)}, B(x)(s), A(x)(s)\right) ds, & t \in [0, t_1]\\ \cdots & \\ \phi(0) + \sum_{i=1}^k I_i\left(x\left(t_i^-\right)\right) + \sum_{i=1}^k (t - t_i) Q_i\left(x\left(t_i^-\right)\right) & \\ - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g\left(s, x_{\rho(s,x_s)}\right) ds + \frac{bt}{a+b}\left\{\frac{1}{b}\int_0^T q(x(s))ds - \sum_{i=1}^m Q_i\left(x\left(t_i^-\right)\right)\right. & \\ + \int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} g\left(s, x_{\rho(s,x_s)}\right) ds & \\ \left. - \int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} f\left(s, x_{\rho(s,x_s)}, B(x)(s), A(x)(s)\right) ds\right\} & \\ + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f\left(s, x_{\rho(s,x_s)}, B(x)(s), A(x)(s)\right) ds, & t \in \left(t_k, t_{k+1}\right].\end{cases}$$

Let $x, x^* \in PC_T$ and $t \in [0, t_1]$. Then

$$
\begin{aligned}
&\|P(x)-P(x^*)\|_X \\
&\leq \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left\| g\left(s, x_{\rho(s,x_s)}\right) - g\left(s, x^*_{\rho(s,x^*_s)}\right)\right\|_X ds \\
&+ \frac{bt}{a+b}\left\{ \frac{1}{b}\int_0^T \|q(x(s)) - q(x^*(s))\|_X ds + \sum_{i=1}^m \left\|Q_i\left(x\left(t_i^-\right)\right) - Q_i\left(x^*\left(t_i^-\right)\right)\right\|_X \right. \\
&+ \int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} \left\| g\left(s, x_{\rho(s,x_s)}\right) - g\left(s, x^*_{\rho(s,x^*_s)}\right)\right\|_X ds \\
&\left.+ \int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} \left\| f\left(s, x_{\rho(s,x_s)}, B(x)(s), A(x)(s)\right) - f(s, x^*_{\rho(s,x^*_s)}, B\left(x^*\right)(s), A\left(x^*\right)(s))\right\|_X ds \right\} \\
&+ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left\| f\left(s, x_{\rho(s,x_s)}, B(x)(s), A(x)(s)\right) - f(s, x^*_{\rho(s,x^*_s)}, B\left(x^*\right)(s), A\left(x^*\right)(s))\right\|_X ds \\
&\leq \left\{ \frac{T^\alpha}{\Gamma(\alpha+1)} L_g + \frac{bT}{a+b}\left(\frac{T}{b}L_q + mL_Q + \frac{T^{\alpha-1}}{\Gamma(\alpha)}L_g \right.\right. \\
&\left.\left. + \frac{T^{\alpha-1}}{\Gamma(\alpha)}\left(L_{f1} + L_{f2}B^* + L_{f3}A^*\right)\right) + \frac{T^\alpha}{\Gamma(\alpha+1)}\left(L_{f1} + L_{f2}B^* + L_{f3}A^*\right)\right\} \|x - x^*\|_{PC_T}
\end{aligned}
$$

In a similar way for $t \in (t_k, t_{k+1}]$, we have

$$
\begin{aligned}
&\|P(x)-P(x^*)\|_X \\
&\leq \sum_{i=1}^k \left\|I_i\left(x\left(t_i^-\right)\right) - I_i\left(x^*\left(t_i^-\right)\right)\right\|_X + \sum_{i=1}^k (t-t_i)\left\|Q_i\left(x\left(t_i^-\right)\right) - Q_i\left(x^*\left(t_i^-\right)\right)\right\|_X \\
&\times \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left\| g\left(s, x_{\rho(s,x_s)}\right) - g\left(s, x^*_{\rho(s,x^*_s)}\right)\right\|_X ds \\
&+ \frac{bt}{a+b}\left\{ \frac{1}{b}\int_0^T \|q(x(s)) - q(x^*(s))\|_X ds + \sum_{i=1}^m \left\|Q_i\left(x\left(t_i^-\right)\right) - Q_i\left(x^*\left(t_i^-\right)\right)\right\|_X \right. \\
&+ \int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} \left\| g\left(s, x_{\rho(s,x_s)}\right) - g\left(s, x^*_{\rho(s,x^*_s)}\right)\right\|_X ds \\
&+ \int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} \left\| f\left(s, x_{\rho(s,x_s)}, B(x)(s), A(x)(s)\right)\right. \\
&\left.\left. - f\left(s, x^*_{\rho(s,x^*_s)}, B\left(x^*\right)(s), A\left(x^*\right)(s)\right)\right\|_X ds \right\} \\
&+ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left\| f(s, x_{\rho(s,x_s)}, B(x)(s), A(x)(s))\right.
\end{aligned}
$$

$$
\begin{aligned}
&- f(s, x^*_{\rho(s,x^*_s)}, B\left(x^*\right)(s), A\left(x^*\right)(s))\Big\|_X ds \\
&\leq \Bigg\{ mL_I + mTL_Q + \frac{T^\alpha}{\Gamma(\alpha+1)}L_g + \frac{bT}{a+b}\left(\frac{T}{b}L_q + mL_Q + \frac{T^{\alpha-1}}{\Gamma(\alpha)}L_g\right. \\
&\left. + \frac{T^{\alpha-1}}{\Gamma(\alpha)}\left(L_{f1} + L_{f2}B^* + L_{f3}A^*\right)\right) \\
&+ \frac{T^\alpha}{\Gamma(\alpha+1)}\left(L_{f1} + L_{f2}B^* + L_{f3}A^*\right)\Bigg\} \|x - x^*\|_{PC_T} \\
&\leq \Delta \|x - x^*\|_{PC_T}.
\end{aligned}
$$

Since $\Delta < 1$, implies that the map P is a contraction map and therefore has a unique fixed point $x \in PC_T$, hence system (1)-(4) has a unique solution on the interval $[-d, T]$. This completes the proof of the theorem.

Our second result is based on Krasnoselkii's fixed point theorem.

Theorem 5. *Let B be a closed convex and nonempty subset of a Banach space X. Let P and Q be two operators such that*
(i) $Px + Qy \in B$, whenever $x, y \in B$;
(ii) P is compact and continuous;
(iii) Q is a contraction mapping.
Then there exists $z \in B$ such that $z = Pz + Qz$.

Theorem 6. *Let the function f, g be continuous for every $t \in [0, T]$, and satisfy the assumptions (H1)-(H3) with*

$$
\begin{aligned}
\Delta := &\Bigg\{ \frac{T^\alpha}{\Gamma(\alpha+1)}L_g + \frac{bT}{a+b}\left(\frac{T}{b}L_q + \frac{T^{\alpha-1}}{\Gamma(\alpha)}L_g + \frac{T^{\alpha-1}}{\Gamma(\alpha)}\left(L_{f1} + L_{f2}B^* + L_{f3}A^*\right)\right) \\
&+ \frac{T^\alpha}{\Gamma(\alpha+1)}\left(L_{f1} + L_{f2}B^* + L_{f3}A^*\right)\Bigg\} < 1
\end{aligned}
$$

Then system (1)-(4) *has at least one solution on* $[-d, T]$.

Proof. Choose

$$
\begin{aligned}
r \geq &\Bigg[\|\phi(0)\| + mL_I r + mTL_Q r + \frac{T^\alpha}{\Gamma(\alpha+1)}L_g r + \frac{bT}{a+b}\left(\frac{T}{b}L_q r + mL_Q r\right. \\
&\left. + \frac{T^{\alpha-1}}{\Gamma(\alpha)}L_g r + \frac{T^{\alpha-1}}{\Gamma(\alpha)}\left(L_{f1} r + L_{f2}B^* r + L_{f3}A^* r\right)\right) \\
&+ \frac{T^\alpha}{\Gamma(\alpha+1)}\left(L_{f1} r + L_{f2}B^* r + L_{f3}A^* r\right)\Bigg]
\end{aligned}
$$

Define $PC_T^r = \{x \in PC_T : \|x\|_{PC_T} \leq r\}$, then PC_T^r is a bounded, closed convex subset in PC_T. Consider the operators $N : PC_T^r \to PC_T^r$ and $P : PC_T^r \to PC_T^r$ for $t \in J_k = (t_k, t_{k+1}]$, defined by

$$\begin{aligned} N(x) =& \phi(0) + \sum_{i=1}^{k} I_i\left(x\left(t_i^-\right)\right) + \sum_{i=1}^{k}\left(t-t_i\right) Q_i\left(x\left(t_i^-\right)\right) - \frac{bt}{a+b}\sum_{i=1}^{m} Q_i\left(x\left(t_i^-\right)\right) \\ P(x) =& \frac{bt}{a+b}\left\{\frac{1}{b}\int_0^T q(x(s))ds + \int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} g\left(s, x_{\rho(s,x_s)}\right) ds \right. \\ & \left. - \int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} f\left(s, x_{\rho(s,x_s)}, B(x)(s), A(x)(s)\right) ds\right\} \\ & - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g\left(s, x_{\rho(s,x_s)}\right) ds \\ & + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f\left(s, x_{\rho(s,x_s)}, B(x)(s), A(x)(s)\right) ds. \end{aligned} \tag{27}$$

We complete the proof in the following steps:

Step 1. Let $x, x^* \in PC_T^r$ then,

$$\begin{aligned} \|N(x) + P(x^*)\|_X \leq& \|\phi(0)\|_X + \sum_{i=1}^{k}\left\|I_i\left(x\left(t_i^-\right)\right)\right\|_X + \sum_{i=1}^{k}\left(t-t_i\right)\left\|_i\left(x\left(t_i^-\right)\right)\right\|_X \\ & + \frac{bt}{a+b}\sum_{i=1}^{m}\left\|Q_i\left(x\left(t_i^-\right)\right)\right\|_X + \frac{bt}{a+b}\left\{\frac{1}{b}\int_0^T \left\|q\left(x^*(s)\right)\right\|_X ds \right. \\ & + \int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}\left\|g\left(s, x^*_{\rho(s,x^*_s)}\right)\right\|_X ds \\ & \left. + \int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}\left\|f\left(s, x^*_{\rho(s,x^*_s)}, B\left(x^*\right)(s), A\left(x^*\right)(s)\right)\right\|_X ds\right\} \\ & + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\left\|g\left(s, x^*_{\rho(s,x^*_s)}\right)\right\|_X ds \\ & + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\left\|f\left(s, x^*_{\rho(s,x^*_s)}, B\left(x^*\right)(s), A\left(x^*\right)(s)\right)\right\|_X ds \\ \leq& \left[\phi(0)\| + mC_2 + mTC_1 + \frac{T^\alpha}{\Gamma(\alpha+1)}L_g r + \frac{bT}{a+b}\left(\frac{T}{b}C_3 \right.\right. \\ & \left. + mC_1 + \frac{T^{\alpha-1}}{\Gamma(\alpha)}L_g r + \frac{T^{\alpha-1}}{\Gamma(\alpha)}\left(L_{f1}r + L_{f2}B^* r + L_{f3}A^* r\right)\right) \\ & \left. + \frac{T^\alpha}{\Gamma(\alpha+1)}\left(L_{f1}r + L_{f2}B^* r + L_{f3}A^* r\right)\right] \leq r. \end{aligned}$$

Which shows that PC_T^r is closed with respect to both the maps.
Step 2. N is continuous. Let $x_n \to x$ be sequence in PC_T^r, then for each $t \in J_k$

$$\begin{aligned}\|N(x_n) - N(x)\|_X \leq & \sum_{i=1}^{k} \left\|I_i\left(x_n\left(t_i^-\right)\right) - I_i\left(x\left(t_i^-\right)\right)\right\|_X \\ & + \sum_{i=1}^{k} (t-t_i) \left\|Q_i\left(x_n\left(t_i^-\right)\right) - Q_i\left(x\left(t_i^-\right)\right)\right\|_X \\ & + \frac{bt}{a+b} \sum_{i=1}^{m} \left\|Q_i\left(x_n\left(t_i^-\right)\right) - Q_i\left(x\left(t_i^-\right)\right)\right\|_X .\end{aligned}$$

Since the functions Q_k and $I_k, k = 1, \ldots, m$, are continuous, hence $\|N(x_n) - N(x)\| \to 0$, as $n \to \infty$. Which implies that the mapping N is continuous on PC_T^r.

$$\begin{aligned}& \|N(x) + P(x^*)\|_X \\ \leq & \|\phi(0)\|_X + \sum_{i=1}^{k} \left\|I_i\left(x\left(t_i^-\right)\right)\right\|_X + \sum_{i=1}^{k} (t-t_i) \left\|Q_i\left(x\left(t_i^-\right)\right)\right\|_X \\ & + \frac{bt}{a+b} \sum_{i=1}^{m} \left\|Q_i\left(x\left(t_i^-\right)\right)\right\|_X + \frac{bt}{a+b} \left\{ \frac{1}{b} \int_0^T \|q(x^*(s))\|_X \, ds \right. \\ & + \int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} \left\| g\left(s, x^*_{\rho(s,x_s^*)}\right)\right\|_X ds \\ & \left. + \int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} \left\| f\left(s, x^*_{\rho(s,x_s^*)}, B(x^*)(s), A(x^*)(s)\right)\right\|_X ds \right\} \\ & + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left\| g\left(s, x^*_{\rho(s,x_s^*)}\right)\right\|_X ds \\ & + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left\| f\left(s, x^*_{\rho(s,x_s^*)}, B(x^*)(s), A(x^*)(s)\right)\right\|_X ds \\ \leq & \left[\|\phi(0)\| + mC_2 + mTC_1 + \frac{T^\alpha}{\Gamma(\alpha+1)} L_g r + \frac{bT}{a+b} \left(\frac{T}{b} C_3 \right. \right. \\ & \left. + mC_1 + \frac{T^{\alpha-1}}{\Gamma(\alpha)} L_g r + \frac{T^{\alpha-1}}{\Gamma(\alpha)} \left(L_{f1} r + L_{f2} B^* r + L_{f3} A^* r\right) \right) \\ & \left. + \frac{T^\alpha}{\Gamma(\alpha+1)} \left(L_{f1} r + L_{f2} B^* r + L_{f3} A^* r\right) \right] \\ \leq & r.\end{aligned}$$

Step 3. The fact that the mapping N is uniformly bounded is a consequence of the following inequality. For each $t \in J_k, k = 0, 1, \ldots, m$ and for each $x \in PC_T^r$, we have

$$\begin{aligned}\|N(x)\|_X \leq & \|\phi(0)\|_X+\sum_{i=1}^{k}\left\|I_i\left(x\left(t_i^-\right)\right)\right\|_X+\sum_{i=1}^{k}\left(t-t_i\right)\left\|Q_i\left(x\left(t_i^-\right)\right)\right\|_X \\ & +\frac{bt}{a+b}\sum_{i=1}^{m}\left\|Q_i\left(x\left(t_i^-\right)\right)\right\|_X \\ \leq & \|\phi(0)\|+mC_2+mTC_1+\frac{bT}{a+b}mC_1.\end{aligned}$$

Step 4. Now, to show that N is equi-continuous, let $l_1, l_2 \in J_k, t_k \leq l_1 < l_2 \leq t_{k+1}$, $k=1,\ldots,m, x \in PC_T^r$, we have

$$\begin{aligned}\|N(x)(l_2)-N(x)(l_1)\|_X \quad \leq \quad & (l_2-l_1)\sum_{i=1}^{k}\left\|Q_i\left(x\left(t_i^-\right)\right)\right\|_X \\ & +\frac{b\left(l_2-l_1\right)}{a+b}\sum_{i=1}^{m}\left\|Q_i\left(x\left(t_i^-\right)\right)\right\|_X .\end{aligned}$$

As $l_2 \to l_1$, then $\|N(x)(l_2)-N(x)(l_1)\| \to 0$ implies that N is an equi-continuous map. Combining the Steps 2 to 4 , together with the Arzela Ascoli's theorem, we conclude that the operator N is compact.

Step 5. Now, we show that P is a contraction mapping. Let $x, x^* \in PC_T^r$ and $t \in J_k, k=1,\ldots,m$, we have

$$\begin{aligned}\|P(x)-P\left(x^*\right)\|_X \leq & \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\left\|g\left(s, x_{\rho(s,x_s)}\right)-g\left(s, x^*_{\rho(s,x^*_s)}\right)\right\|_X ds \\ & +\frac{bt}{a+b}\left\{\frac{1}{b}\int_0^T \|q(x(s))-q\left(x^*(s)\right)\|_X ds\right. \\ & +\int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}\left\|g\left(s, x_{\rho(s,x_s)}\right)-g\left(s, x^*_{\rho(s,x^*_s)}\right)\right\|_X ds \\ & +\int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}\Big\|f\left(s, x_{\rho(s,x_s)}, B(x)(s), A(x)(s)\right) \\ & \left.-f(s, x^*_{\rho(s,x^*_s)}, B\left(x^*\right)(s), A\left(x^*\right)(s))\Big\|_X ds\right\} \\ & +\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\Big\|f\left(s, x_{\rho(s,x_s)}, B(x)(s), A(x)(s)\right) \\ & -f(s, x^*_{\rho(s,x^*_s)}, B\left(x^*\right)(s), A\left(x^*\right)(s))\Big\|_X ds\end{aligned}$$

$$\leq \left\{ \frac{T^{\alpha}}{\Gamma(\alpha+1)} L_g + \frac{bT}{a+b} \left(\frac{T}{b} L_q + \frac{T^{\alpha-1}}{\Gamma(\alpha)} L_g \right.\right.$$
$$\left. + \frac{T^{\alpha-1}}{\Gamma(\alpha)} \left(L_{f1} + L_{f2} B^* + L_{f3} A^* \right) \right)$$
$$\left. + \frac{T^{\alpha}}{\Gamma(\alpha+1)} \left(L_{f1} + L_{f2} B^* + L_{f3} A^* \right) \right\} \|x - x^*\|_{PC_T^r}$$
$$\leq \Delta \|x - x^*\|_{PC_T^r} .$$

As $\Delta < 1$, it implies that P is a contraction map. Thus all the assumptions of the Krasnoselkii's theorem are satisfied. Hence we have that the set PC_T^r has a fixed point which is the solution of system (1)- (4) on $(-d, T]$. This completes the proof of the theorem.

4. Illustrative example

Example 1. Consider the following fractional boundary value problem

$$D_t^{\alpha} \left[x(t) + \int_0^t \frac{1}{47} x(t - \sigma(x) ds \right] = \frac{e^t x(t - \sigma(x(t)))}{25 + x^2(t - \sigma(x(t)))}$$
$$+ \int_0^t \cos(t-s) \frac{xe^s}{4+x} ds + \int_0^T \sin(t-s) \frac{xe^s}{6+x} ds, t \in [0,T], t \neq t_i$$
$$\Delta x\left(t_i\right) = \int_{-d}^{t_i} \frac{\gamma_i\left(t_i - s\right) x(s)}{25} ds, \quad \Delta x'\left(t_i\right) = \int_{-d}^{t_i} \frac{\gamma_i\left(t_i - s\right) x(s)}{9} ds$$
$$x(t) = \phi(t), t \in (-d, 0], \quad x'(0) + x'(T) = \int_0^T \sin\left(\frac{1}{4} x(s)\right) ds,$$

where $\gamma_i \in C([0, \infty), X), \sigma \in C(X, [0, \infty)), 0 < t_1 < t_2 < \cdots < t_n < T$. Set $\gamma > 0$, and choose PC^{γ} as

$$PC^{\gamma} = \left\{ \phi \in PC((0, \infty], X) : \lim_{t \to -d} e^{\gamma t} \phi(t) \text{ exist } \right\}$$

with the norm $\|\phi\|_{\gamma} = \sup_{t \in (0,\infty]} e^{\gamma t} |\phi(t)|, \phi \in PC^{\gamma}$. We set

$$\rho(t, \varphi) = t - \sigma(\varphi(0)), \quad (t, \varphi) \in J \times PC^{\gamma}$$
$$f(t, \varphi) = \frac{e^t(\varphi)}{25 + (\varphi)^2}, \quad (t, \varphi) \in J \times PC^{\gamma}$$
$$g(t, \varphi) = \frac{\varphi}{47} ds, \quad \varphi \in PC^{\gamma},$$
$$B(x)(t) = \int_0^t \cos(t-s) \frac{xe^s}{(4+x)} ds, \quad (t, x) \in I \times PC^{\gamma},$$

$$A(x)(t)=\int_0^T \sin(t-s)\frac{xe^s}{(6+x)}ds, \quad (t,x)\in I\times PC^\gamma,$$
$$Q_k\left(x\left(t_k\right)\right)=\int_{-d}^{t_i}\frac{\gamma_i\left(t_i-s\right)x(s)}{25}ds$$
$$I_k\left(x\left(t_k\right)\right)=\int_{-d}^{t_i}\frac{\gamma_i\left(t_i-s\right)x(s)}{9}ds$$

We can see that all the assumptions of Theorem 4 are satisfied with

$$|f(t,\varphi)-f(t,\chi)|\leq e^t\frac{\|\varphi-\chi\|}{25}, \quad \forall t\in J,\varphi,\chi\in PC^\gamma$$
$$|B(x)-B(y)|\leq e^t\frac{\|x-y\|}{4}, \quad \forall t\in J,x,y\in PC^\gamma$$
$$|A(x)-A(y)|\leq e^t\frac{\|x-y\|}{6}, \quad \forall t\in J,x,y\in PC^\gamma$$
$$|g(t,\varphi)-g(t,\chi)|\leq \frac{1}{47}\|\varphi-\chi\|, \quad \forall t\in J,\varphi,\chi\in PC^\gamma,$$
$$\left|Q_k\left(x\left(t_k\right)\right)-Q_k\left(y\left(t_k\right)\right)\right|\leq \gamma^*\frac{1}{25}\|x-y\|, \quad x,y\in X$$
$$\left|I_k\left(x\left(t_k\right)\right)-I_k\left(y\left(t_k\right)\right)\right|\leq \gamma^*\frac{1}{9}\|x-y\|, \quad x,y\in X$$
$$|q(x)-q(y)|\leq \frac{1}{4}\|x-y\|, \quad x,y\in X$$

Further, we observe that

$$\left\{mL_I+mTL_Q+\frac{T^\alpha}{\Gamma(\alpha+1)}L_g+\frac{bT}{a+b}\left(\frac{T}{b}L_q+mL_Q+\frac{T^{\alpha-1}}{\Gamma(\alpha)}L_g\right.\right.$$
$$\left.\left.+\frac{T^{\alpha-1}}{\Gamma(\alpha)}\left(L_{f1}+L_{f2}B^*+L_{f3}A^*\right)\right)+\frac{T^\alpha}{\Gamma(\alpha+1)}\left(L_{f1}+L_{f2}B^*+L_{f3}A^*\right)\right\}<1,$$

when we fix $\gamma^*=\int_{-d}^{t}\gamma_i\left(t_i-s\right)ds<0, 0<t_1<t_2<t_3<1, \alpha=3/2, T=1$.

This implies that there exists a unique solution of the considered problem.

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