Acta Universitatis Apulensis No. 74/2023
ISSN: 1582-5329 pp. 85-96
http://www.uab.ro/auajournal/ doi: 10.17114/j.aua.2023.74.06

A GENERAL SEQUENTIAL TOPOLOGICAL HENSTOCK-TYPE INTEGRAL

A.A. Mogbademu, V.O. Iluebe, S.A. Bishop

Abstract. In this paper, we introduce a general Henstock-type integral for Topological valued function via sequential approach and discuss the fundamental properties of this integral.

2010 *Mathematics Subject Classification*: 28B05, 28B10, 28B15, 46G10.

Keywords: Topological Henstock integral, Sequential Henstock integrals, Continuous functions, guage functions.

1. Introduction

The concept of Henstock integral, established to remedy the deficiencies of the Riemann integral was introduced independently in the mid-1950s by R. Henstock and J. Kurzweil respectively. It is a useful generalisation of the Riemann integral and powerful to handle nowhere-continuous functions, extreme oscillatory functions (see [1-14]). While the standard definition of the Henstock integral uses the Riemann sums and $\varepsilon - \delta$ definition, the Sequential Henstock integral which involves the use of sequence of guage functions was introduced. Paxton[12] proved a theorem of a specific definition for Topological Henstock integral which was refined and called the Sequential Topological Henstock integral over a compact subspace. In the last one decade, several studies for varieties of generalized Riemann-type integrals for certain classes of functions have been considered by many researchers in order to improve on the approach of integration, see for example,[1 and 10] and the references therein.

We denote $\mathbb{R}$ and $\mathbb{N}$ as set of real and natural numbers respectively and $\ll$ as much more smaller.

A gauge on $[a, b]$ is a positive real-valued function $\delta : [a, b] \to \mathbb{R}^+$. This gauge is δ-fine if $[u_{i-1}, u_i] \subset [t_i - \delta(t_i), t_i + \delta(t_i)]$.

A sequence of tagged partition P_n of [a,b] is a finite collection of ordered pairs $P_n = \{(u_{(i-1)_n} u_{i_n}), t_{i_n}\}_{i=1}^{m_n}$ where $[u_{i-1}, u_i] \in [a, b]$, $u_{(i-1)_n} \leq t_{i_n} \leq u_{i_n}$ and $a = u_0 < u_{i_1} < ... < u_{m_n} = b$.

2. Basic Definitions

We recall the following definitions (see [5-12]).

Definition 1. *(Henstock integral). A real valued function $f : [a, b] \to \mathbb{R}$ is Henstock integrable to $\alpha \in \mathbb{R}$ on $[a, b]$ if for any $\varepsilon > 0$ there exists a function $\delta(t) > 0$ such that for every $\delta(t) - fine$ partitions $P = \{(u_{i-1}, u_i), t_i\}_{i=1}^{n}$ we have*

$$|\sum_{i=1}^{n} f(t_i)(u_i - u_{i-1}) - \alpha| < \varepsilon,$$

where $(H) \int_{[a,b]} f(t)d(t) = \alpha$ and $[u_{i-1}, u_i] \in [a, b]$ for $u_{i-1} \leq t_i \leq u_i$.

Remark 1. *if $\delta(t) = \delta$ in Definition 1, f is said to be Riemann integrable.*

Definition 2. *(Sequential Henstock Integral). A function $f : [a, b] \to \mathbb{R}$ is Sequential Henstock integrable on $[a, b]$ to $\alpha \in \mathbb{R}$ if there exists a sequence of gauge functions $\{\delta_n(t)\}_{n=1}^{\infty}$ on $[a, b]$ such that for every $\delta_n(t) - fine$ tagged partitions $P_n = \{(u_{(i-1)_n}, u_{i_n}), t_{i_n}\}_{i=1}^{m_n}$, we have*

$$\sum_{i=1}^{n\in\mathbb{N}} f(t_{i_n})(u_{i_n} - u_{(i-1)_n}) \to \alpha asn \to \infty,$$

where $\alpha = \int_{[a,b]} f$.

Remark 2. *If $\delta_n(t) = \delta(t)$ in Definition 2, f is said to be Henstock integrable.*

It is well known that, if a real valued function f is Henstock integrable, then it is Sequential Henstock integrable.

Motivated by results relating to these definitions, we introduce the following new Henstock-type integrals and establish their properties.

The following concepts are well known with the case of functions defined in a complete space(see [12])
Let X be a locally compact Hausdorff space with subspace $\Omega \subset X$. We denote the closure of Ω as $\bar{\Omega}$ and the interior as $Int\Omega$. Let A be a family of subsets of X such

that:
i. If $\Omega \in A$, then $\bar{\Omega}$ is compact.

ii. for each $x \in X$, the collection $A(x) = \{A \in A | x \in Int\Omega\}$ is a neighbourhood base at x.

iii. If $A, B \in A$, then $A \cap B \in A$. and there exist disjoint sets $C_1, ..., C_k \in A$ such that $A - B = \bigcup_{i=1}^{k} C_i$.

A gauge (topological) on Ω is a map U assigning to each x a neighbourhood $U(x)$ of x contained in X.

A division (topological) of Ω is a disjoint collection $\{A_1, ..., A_n\} \subset \Omega$ such that that $\bigcup_{i=1}^{n} A_i = A$.

A partition (topological) of Ω is a set $P = \{(A_1, t_1), ..., (A_n, t_n)\}$ such that $\{A_1, ..., A_n\}$ is a division of Ω and $\{x_1, ..., x_n\} \subset \bar{\Omega}$. If U is a gauge on Ω, we say that the partition P is U-fine, if $A_i \subset Ux_i$, for $i = 1, 2, ..., n$.

A volume is a non-negative function such that $\Phi(A) = \sum_{i=1}^{n} \Phi(A_i) = \sum_{i=1}^{n}(v_i - v_{i-1})$.

Note: Volume here can intuitively be defined to represent the "length" of the "interval".

From now on, we use X as a topological space, which is a subset of the real line R, U_n as set of neighbourhood system in X, $\triangle$, a collection of subspace in X, P_n as set of partitions of the non-overlapping subintervals of a compact subspace Ω(Hausdorff) in X

Definition 3. *(Topological Henstock integral). Let X be a locally compact Hausdorff space and let $\Omega \in \triangle$ with $f : \bar{\Omega} \to \mathbb{R}$, then f is Topological Henstock integrable to $\alpha \in \mathbb{R}$ if for any $\varepsilon > 0$ there exists a neighbourhood $U(x) > 0$ such that*

$$|\sum_{i=1}^{n} f(t_i)v(U_i) - \int_{\Omega} f| = |\sigma(f, P) - \int_{\Omega} f| < \varepsilon,$$

for every $U(x)-$ fine partition P of Ω, where $\int_{\Omega} f = \alpha$

This Topological Henstock integral uses the concept of neighbourhood system of a Topological space to define the integral value of the Topological space valued functions.

Definition 4. *(Sequential Topological Henstock integral). Let X be a locally compact Hausdorff space and let $\Omega \in \triangle$ with $f : \bar{\Omega} \to \mathbb{R}$, then f is Sequential Topological Henstock integrable to $\alpha \in \mathbb{R}$ if for any $\varepsilon > 0$, there exists a sequence of neighbourhood $\{U_n(x)\}_{n=1}^{\infty}$ for such that*

$$|\sum_{i=1}^{m_n} f(t_{i_n})(v_{i_n} - v_{(i-1)_n}) - \int_{\Omega} f| = |\sigma(f, P_n) - \int_{\Omega} f| < \varepsilon.$$

For every $U_n(x) - fine$ partition P_n of Ω.

In this paper, we establish the concept of generalized Sequential Topological Henstock Integral and prove its' fundamental properties.

Definition 5. *(General Sequential Topological Henstock integral). Let X be a locally compact Hausdorff space and let $\Omega \in \triangle$ with $F : \bar{\Omega} \times \bar{\Omega} \to \mathbb{R}$, then F is generalized Sequential Topological Henstock integrable to $\alpha \in \mathbb{R}$ if for any $\varepsilon > 0$, there exists a sequence of neighbourhood $\{U_n(x)\}_{n=1}^{\infty}$ such that*

$$|\sum_{i=1}^{m_n} \{F(t_{i_n}, v_{i_n}) - F(t_{i_n}, v_{(i-1)_n})\} - \int_{\Omega} F| = |\sigma(F, P_n) - \int_{\Omega} F| < \varepsilon.$$

For every $U_n(x) - fine$ partition P_n of Ω. We say $\alpha = (GSTH) \int_{\Omega} F$. The set of all functions F which are generalized Sequential Henstock integrable on Ω is denoted by $GSTH(\Omega)$.

Remark 3. *A special case of Definition 5 is discussed as follows:*
(i) Setting $F(t_{i_n}, v_{i_n}) = f(t_{i_n})u_{i_n}$ where $f : \bar{\Omega} \to \mathbb{R}$ and $t_{i_n}, v_{i_n} \in \Omega$ with $U_n(x) > 0$, we obtain the Topological Sequential Henstock integral.
(ii) Considering $F(t_i, v_i) = f(t_i)v_i$ where $f : \bar{\Omega} \to \mathbb{R}$ and $t_{i_n}, v_{i_n} \in \Omega$ with $U_n(x) \equiv U(x)$, we obtain the Topological Henstock integral for the function f. See [12]

3. Main Results

The fundamental properties of a generalized Topological Henstock Integral via sequence approach on classical interval is established in this section.

Theorem 1. *(Uniqueness) If $F \in GTSH(\Omega)$, then there is a unique integral value $\alpha \in \mathbb{R}$ such that for any $\varepsilon > 0$, there is a sequence of neighbourhood $\{U_n(x)\}_{n=1}^{\infty}$ on Ω which satisfies*

$$|\sum_{i=1}^{m_n \in \mathbb{N}} \{F(t_{i_n}, v_{i_n}) - F(t_{i_n}, v_{(i-1)_n})\} - \alpha| < \varepsilon. \tag{1}$$

for any $U_n(x)$-fine partition P_n of Ω.

Proof. Suppose $\alpha_1 = (GTSH)\int_\Omega F$ and $\alpha_2 = (GTSH)\int_\Omega F$ with $\alpha_1 \neq \alpha_2$. For any $\varepsilon > 0$, there is a $\{U_n^1(x)\}_{n=1}^{\infty}$ and $\{U_n^2(t)\}_{n=1}^{\infty}$ on Ω such that for each $U_n^1(x)$-fine tagged partitions P_n^1 of Ω and for each $U_n^2(x)$-fine tagged partitions P_n^2 of Ω, we have

$$|\sum_{i=1}^{m_n \in \mathbb{N}} \{F(t_{i_n}, v_{i_n}) - F(t_{i_n}, v_{(i-1)_n})\} - \alpha_1| < \frac{\varepsilon}{2},$$

and

$$|\sum_{i=1}^{m_n \in \mathbb{N}} \{F(t_{i_n}, v_{i_n}) - F(t_{i_n}, v_{(i-1)_n})\} - \alpha_2| < \frac{\varepsilon}{2}$$

respectively.

Define a positive function $U_n(x)$ on Ω by $U_n(x) = \min\{U_n^1(x), U_n^2(x)\}$. Let $P_n = \{P_n^1 \cup P_n^2\}$ be any $U_n(x)$-fine tagged partition of Ω. Then by triangular inequality, we have

$$\begin{aligned} |\alpha_1 - \alpha_2| &= |\sum_{i=1}^{m_n \in \mathbb{N}} \{F(t_{i_n}, v_{i_n}) - F(t_{i_n}, v_{(i-1)_n})\} - \alpha_1 \\ &\quad + \sum_{i=1}^{m_n \in \mathbb{N}} \{F(t_{i_n}, v_{i_n}) - F(t_{i_n}, v_{(i-1)_n})\} - \alpha_2| \\ &\leq |\sum_{i=1}^{m_n \in \mathbb{N}} \{F(t_{i_n}, v_{i_n}) - F(t_{i_n}, v_{(i-1)_n})\} - \alpha_1| \\ &\quad + |\sum_{i=1}^{m_n \in \mathbb{N}} \{F(t_{i_n}, v_{i_n}) - F(t_{i_n}, v_{(i-1)_n})\} - \alpha_2| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

which is a contradiction. Thus $\alpha_1 = \alpha_2$. This completes the proof.

Theorem 2. *If* $F_1, F_2 \in GSTH(\Omega)$ *and* $\lambda_1, \lambda_2 \in \mathbb{R}$, *then* $(\lambda_1 F_1 + \lambda_2 F_2) \in GSTH(\Omega)$ *and*

$$(GSTH)\int_\Omega (\lambda_1 F_1 + \lambda_2 F_2) = \lambda_1 (GSTH)\int_\Omega F_1 + \lambda_2 (GSTH)\int_\Omega F_2 \tag{2}$$

Proof. Let $\alpha_1 = \int_\Omega F_1$ and $\alpha_2 = \int_\Omega F_2$. Choose $\varepsilon > 0$. There is $\varepsilon^{'} > 0$ such that $(\lambda_1 + \lambda_2)\frac{\varepsilon^{'}}{2} \leq \varepsilon$. Then for any $\varepsilon > 0$, there exists a sequence of neighbourhood $\{U_n(x)\}_{n=1}^{\infty}$ on Ω such that for any $U_n^1(x)$-fine tagged partition P_n^1, we have

$$|\sum_{i=1}^{m_n \in \mathbb{N}} \{F_1(t_{i_n}, v_{i_n}) - F_1(t_{i_n}, v_{(i-1)_n})\} - \alpha_1| < \frac{\varepsilon^{'}}{2}.$$

Similarly, for any $\varepsilon > 0$, there exists a sequence of neighbourhoods $\{U_n(x)\}_{n=1}^{\infty}$ on Ω such that for any $U_n^2(x)$-fine tagged partition P_n^2, we have

$$|\sum_{i=1}^{m_n \in \mathbb{N}} \{F_2(t_{i_n}, v_{i_n}) - F_2(t_{i_n}, v_{(i-1)_n})\} - \alpha_2| < \frac{\varepsilon^{'}}{2}.$$

Define a sequence positive neighbourhoods $U_n(x)$ on Ω by $U_n(x) = \min\{U_n^1(x), U_n^2(x)\}$. Therefore for any $U_n(x)$-fine tagged partition P_n of Ω, we have

$$|\sum_{i=1}^{m_n \in \mathbb{N}} \{(\lambda_1 F_1 + \lambda_2 F_2)(t_{i_n}, v_{i_n}) - (\lambda_1 F_1 + \lambda_2 F_2)(t_{i_n}, v_{(i-1)_n})\} - (\lambda_1 \alpha_1 + \lambda_2 \alpha_2)|$$

$$\begin{aligned}
&\leq |\sum_{i=1}^{m_n \in \mathbb{N}} \lambda_1 \{F_1(t_{i_n}, v_{i_n}) - F_1(t_{i_n}, v_{(i-1)_n})\} - \lambda_1 \alpha_1| \\
&\quad + |\sum_{i=1}^{m_n \in \mathbb{N}} \lambda_2 \{F_2(t_{i_n}, v_{i_n}) - F_2(t_{i_n}, v_{(i-1)_n})\} - \lambda_2 \alpha_2| \\
&< \lambda_1 \frac{\varepsilon^{'}}{2} + \lambda_2 \frac{\varepsilon^{'}}{2} \\
&= (\lambda_1 + \lambda_2)\frac{\varepsilon^{'}}{2} \\
&\leq \varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, this gives

$$(GSTH)\int_\Omega (\lambda_1 F_1 + \lambda_2 F_2) = \lambda_1 (GSTH)\int_\Omega F_1 + \lambda_2 (GSTH)\int_\Omega F_2. \tag{3}$$

This completes the proof.

Theorem 3. *(Cauchy Criterion) $F \in GSTH(\Omega)$ if and only if for any $\varepsilon > 0$, there exists a sequence of neighbourhood $\{U_n(x)\}_{n=1}^{\infty}$ on $[a, b]$ such that*

$$|\sigma(F, P_n) - \sigma(F, Q_n)| < \varepsilon,$$

for all $U_n(x) - fine$ tagged partitions P_n and Q_n on Ω.

Proof. Suppose $F \in GSTH(\Omega)$ and $\varepsilon > 0$, there exists a $\{U_n(x)\}_{n=1}^{\infty}$ on Ω such that for $P_n \ll U_n(x)$, we have

$$|\sigma(F, P_n) - \alpha| < \frac{\varepsilon}{2}$$

and

$$|\sigma(F, Q_n) - \alpha| < \frac{\varepsilon}{2}$$

for all $U_n(x) - fine$ tagged partitions P_n and Q_n on Ω. Now, if $P_n \ll U_n(x)$ and $Q_n \ll U_n(x)$, then

$$\begin{aligned} |\sigma(F, P_n) - \alpha + \alpha - \sigma(F, Q_n)| &< |\sigma(F, P_n) - \alpha| + |\sigma(F, Q_n) - \alpha| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Conversely, let $\varepsilon > 0$, there exists a $\{U_n(x)\}_{n=1}^{\infty}$ on Ω such that $Q_n \ll U_n(x)$, we have

$$|\sigma(F, P_n) - \sigma(F, Q_n)| < \frac{1}{n}.$$

We now construct a Cauchy sequence of generalized Henstock sums which converges to a number denoted by α. Without loss of generality, we may assume that $\{U_n(x)\}_{n=1}^{\infty}$ is a decreasing sequence for all $x \in \Omega$. Thus, for any $k > n$, P_k is $U_n(x)$ - fine and letting $n \to \infty$, then

$$|\sigma(F, P_n) - \sigma(F, Q_n)| < \frac{1}{n}$$

is a Cauchy Sequence, Hence

$$\{\sigma(F, P_n)\}_{n=1}^{\infty} \to \alpha$$

as $n \to \infty$ for any $\varepsilon > 0$ and for all $n \geq N$, we have

$$|\sigma(F, P_n) - \alpha| < \frac{1}{n}.$$

Let $\varepsilon > 0$, there exists a $\{U_n(x)\}_{n=1}^{\infty}$ on $[a, b]$ where $\frac{1}{N} < \frac{\varepsilon}{2}$ and for $P_n \ll U_n(x)$ and $Q_n \ll U_n(x)$, we have

$$\begin{aligned} |\sigma(F, Q_n) - \alpha| &< |\sigma(F, Q_n) - \sigma(F, P_n)| + |\sigma(F, Q_n) - \alpha| \\ &< \frac{1}{n} + \frac{1}{n} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Thus, $F \in GTSH(\Omega)$ and $\int_\Omega F = \alpha$.

The following new definition is necessary for the proof of Theorem 5.

Definition 6. *Let X be a locally compact Hausdorff space and let $\Omega \in \triangle$ with a function $F : \bar{\Omega} \times \bar{\Omega} \to \mathbb{R}$. Suppose the interval is divided into i subintervals of equal width $\triangle x = (v_{i_n} - v_{(i-1)_n})$ and from each interval, choose a point $t_{i_n} \in [v_{(i-1)_n}, v_{i_n}]$, the definite integral of the Topological spaced valued function $F(x)$ on Ω for $x \in \Omega$. i.e.*

$$\int_\Omega F(x)dx = lim_{n\to\infty} \sum_{i=1}^{m_n} \{F(t_{i_n}, v_{i_n}) - F(t_{i_n}, v_{(i-1)_n})\} \tag{4}$$

is called the generalized definite integral. The limiting value of the sum of the integral function provides a necessary and sufficient condition for the existence of the integral value of the function.

Theorem 4. *If $F \in GSTH(\Pi)$ and $F \in GSTH(\Phi)$, then $F \in GSTH(\Omega)$ and*

$$(GSTH)\int_\Omega F = (GSTH)\int_\Pi F + (GSTH)\int_\Phi F.$$

i.e

$$\begin{aligned}\sum_{i=1}^{m_n}\{F(t_{i_n}, v_{i_n}) - F(t_{i_n}, v_{(i-1)_n})\} &= \sum_{i=1}^{m_k}\{F(t_{i_n}, v_{i_n}) - F(t_{i_n}, v_{(i-1)_n})\} \\ &\quad +(\sum_{i=1}^{m_{n-k}}\{F(t_{i_n}, v_{i_n}) - F(t_{i_n}, v_{(i-1)_n})\}.\end{aligned}$$

Proof. Since $F \in GSTH(\Pi)$, Let $\varepsilon > 0$ be arbitrary and $\alpha_1 = (GSTH)\int_\Pi F$ then there exists a $\{U_n^1(x)\}_{n=1}^\infty$ on Π such that for $P_n^1 \ll U_n^1(x)$, we have

$$|\sum_{i=1}^{m_k}\{F(t_{i_n}, v_{i_n}) - F(t_{i_n}, v_{(i-1)_n})\} - \alpha_1|.$$

Similarly,
Since $F \in GSTH(\Phi)$, Let $\varepsilon > 0$ be arbitrary and $\alpha_2 = (GSTH)\int_\Phi F$ then there exists a $\{U_n^2(t)\}_{n=1}^\infty$ on Φ such that for $P_n^2 \ll U_n^2(x)$, we have

$$|\sum_{i=1}^{m_{n-k}}\{F(t_{i_n}, v_{i_n}) - F(t_{i_n}, v_{(i-1)_n})\} - \alpha_2|.$$

We define a neighbourhood $U_n(x) = \min\{U_n^1(x), U_n^2(x)\}$ and $U_n(x)$-fine tagged partition $P_n = \{P_n^1 \cup P_n^2\}$ in order to force a point c to be a tag of each $P_n \ll U_n(x)$. Using the right-left procedure, we split each partition P_n at the tag c so that it becomes a partition point of each P_n

$$U_n(x) = \begin{cases} min\{U_n^1(x), \frac{1}{2}(c-x)\}, & \text{if } x \in \Pi \\ min\{U_n^1(x), \delta_n^2(c)\}, & \text{if } x = c \\ min\{U_n^2(x), \frac{1}{2}(x-c)\}, & \text{if } x \in \Phi \end{cases} \tag{5}$$

Let $P_n \ll U_n(x)$ for. Let $P_n^1 \in \Pi$ consisting $P_n \cap \Pi$ and $P_n^2 \in \Phi$ consisting $P_n \cap \Phi$. Then the right-left procedures provides that

$$\begin{aligned} \sum_{i=1}^{m_n}\{F(t_{i_n}, v_{i_n}) - F(t_{i_n}, v_{(i-1)_n})\} &= \sum_{i=1}^{m_k}\{F(t_{i_n}, v_{i_n}) - F(t_{i_n}, v_{(i-1)_n})\} \\ &+ \sum_{i=1}^{m_{n-k}}\{F(t_{i_n}, v_{i_n}) - F(t_{i_n}, v_{(i-1)_n})\}. \end{aligned}$$

Given $\varepsilon > 0$, there exists a $\{U_n(x)\}_{n=1}^{\infty}$ such that for $P_n \ll U_n(x)$, we have

$$|\sum_{i=1}^{m_n \in \mathbb{N}}\{F(t_{i_n}, v_{i_n}) - F(t_{i_n}, v_{(i-1)_n})\} - (\alpha_1 + \alpha_2)|$$

$$\begin{aligned} &= |\sum_{i=1}^{m_k}\{F(t_{i_n}, v_{i_n}) - F(t_{i_n}, v_{(i-1)_n})\} + \sum_{i=1}^{m_{n-k}}\{F(t_{i_n}, v_{i_n}) - F(t_{i_n}, u_{(i-1)_n})\} - (\alpha_1 + \alpha_2)| \\ &\leq |\sum_{i=1}^{m_k}\{F(t_{i_n}, v_{i_n}) - F(t_{i_n}, v_{(i-1)_n})\} - \alpha_1| + |\sum_{i=1}^{m_{n-k}}\{F(t_{i_n}, v_{i_n}) - F(t_{i_n}, v_{(i-1)_n})\} - \alpha_2)| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Hence by Theorem 2, $F \in GSTH(\Omega)$ and

$$(GSTH)\int_{\Omega} F = (GSTH)\int_{\Pi} F + (GSTH)\int_{\Phi} F.$$

Theorem 5. *Let X be a locally compact Hausdorff space and let $\Omega \in \triangle$. If $F : \bar{\Omega} \times \bar{\Omega} \to \mathbb{R}$ is generalized Sequential Henstock integrable on Ω, then F^2 is also generalized Sequential Topological Henstock integrable on Ω.*

Proof. . Since F is bounded, there exists a positive real number K such that for all $t \in [a,b]$, we have $|F| < K$. Let $\varepsilon > 0$ and choose a positive sequence of neighbourhoods $\{U_n(x)\}_{n=1}^{\infty}$ such that P_n and Q_n be two $U_n(x)$-fine partitions of $[a,b]$. Therefore by Theorem 2.3,

$$|\sigma(F,P_n) - \sigma(F,Q_n)| < \frac{\varepsilon}{nK}.$$

Now

$$\begin{aligned}\sum_{i=1}^{m_n\in\mathbb{N}} \{F^2(t_{i_n}, v_{i_n}) - F^2(t_{i_n}, v_{(i-1)_n})\} &= \sum_{i=1}^{m_n\in\mathbb{N}} \{F(t_{i_n})(F(t_{i_n}, v_{i_n}) - F(t_{i_n}, v_{(i-1)_n}))\} \\ &= \sum_{i=1}^{m_n\in\mathbb{N}} F(t_{i_n}) \sum_{i=1}^{m_n\in\mathbb{N}} \{F(t_{i_n}, v_{i_n}) - F(t_{i_n}, u_{(i-1)_n})\}.\end{aligned}$$

Thus,

$$\begin{aligned}|\sigma(F^2,P_n) - \sigma(F^2,Q_n)| &\leq |\sum_{i=1}^{m_n\in\mathbb{N}} F(t_{i_n})|\sigma(F,P_n) - \sigma(F,Q_n)|) \\ &< nK.\frac{\varepsilon}{nK} = \varepsilon.\end{aligned}$$

Hence, by Theorem 3, F^2 is also generalized Sequential Topological Henstock integrable on Ω.

Example 1. *See Example 2.8 in [10] for details. We say* $\alpha = (GHS)\int_{[a,b]} F_i$. *The set of all functions* F_i *for all* $i \in \mathbb{N}$ *which are generalized Sequential Topological Henstock integrable on* $[a,b]$ *is denoted by* $GSTH(\Omega)$.

Acknowledgements The authors are thankful to the editors and the anonymous reviewers for their valuable suggestions and comments that helped to improve this manuscript.

References

[1] J. A. Chartfield, *Equivalence Of Integrals* , Proc. of Amer. Math. Soc. in Math. Vol. 3(1973), 279-285.

[2] D. K. Ganguly, R. Mukherjee, *The Generalised Approximate Perron Integral*, Mathematica Slovaca, 58(1)(2001), 31-42.

[3] R. Gordon, *The Integral of Lebesgues, Denjoy, Perron and Henstock*, Grad. stud. in Math. Amer. Math. Soc. Prov, RI, 4(1994), 1-5.

[4] M. E. Hamid, L. Xu, Z. Gong, *The Henstock-Stieltjes Integral For Set Valued Functions*, Int. J. of Pure and Applied Math. 114(2)(2017), 261-275..

[5] V. O. Iluebe and A. A. Mogbademu, *Equivalence Between Sequential Henstock And Topological Henstock Integral For Interval valued Functions*, J. Mahani Math.Res. 12(2)(2023),267-274.

[6] V. O. Iluebe and A. A. Mogbademu, *Convergence Results for Sequential Henstock Stieltjes Integral in Real Valued Space*, . TJMM, 14(2)(2022), 169-176..

[7] V. O. Iluebe and A. A. Mogbademu, *Sequential Henstock Integral For Interval Valued Functions*, . CJMS. 11(2022), 358-367.

[8] V. O. Iluebe and A. A. Mogbademu, *On ap-Sequential Henstock Integral for Interval valued Functions* Int. J. Nonlinear Anal. Appl. 13(2)(2022), 3095-3103.

[9] V. O. Iluebe and A. A. Mogbademu, *On Sequential Henstock Stieltjes Integral For* $L^p[0,1]$*-Interval Valued Functions*, Bull. Int. Math. Virtual Inst., 12(2022), 369 - 378.

[10] V. O. Iluebe and A. A. Mogbademu, *Generalized Henstock-type integral for Real Valued Functions*, Heliyon Journal Of Mathematics. (2023). To appear.

[11] K. Parmar, *Study of Henstock-Kursweil Integrals*, International Journal of Mathematics Trends and Technology, 19(2)(2001), 130-135.

[12] L. A. Paxton, *Sequential Approach to the Henstock Integral*, Washington State University, arXiv:1609.05454v1 [maths.CA] 18 Sep, (2016), 3-5.

[13] R. Sergio, Jr. Canoy, *On Equivalence of the* ϕ*-Integral and the Henstock Integral for TVS-valued Functions*, Mathematical Analysis. 8(2014), 625-632.

[14] J. L. Ying, *On The Equivalence Of Mcshane And Lebesgue Integrals*, Real Analysis Exchange. 21(2)(1995-96), 767-770.

Adesanmi Alao Mogbademu
Department of Mathematics, Faculty of Science,
University of Lagos,
Lagos, Nigeria.
email: *amogbademu@unilag.edu.ng*

Victor Odalochi Iluebe
Department of Mathematics, Faculty of Science,
University of Lagos,
Lagos, Nigeria.

email: *victorodalochi1960@gmail.com*

Sheila Amina Bishop
Department of Mathematics, Faculty of Science,
University of Lagos,
Lagos, Nigeria.
email: *asbishop@unilag.edu.ng*