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# RESULTS FOR A NEW SUBCLASS OF ANALYTIC FUNCTIONS CONNECTED WITH OPOOLA DIFFERENTIAL OPERATOR AND GEGENBAUER POLYNOMIALS

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Abstract. In this paper, the authors introduce and investigate a subclass of analytic functions $E^m_{(\beta,\lambda,\varphi,\gamma)}(H_\sigma, \alpha, \delta, \mu, \ell)$, satisfying certain subordinate condition associated with Gegenbauer polynomials. For the subclass introduced, we derive initial coefficients $|a_2|$ and $|a_3|$ and also obtain the classical Fekete-Szegë problem for functions in the subclass. Finally, our findings show relevant connections between our results and those in some earlier known investigations and obviously improve them.

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## 1. Introduction and Preliminaries

The usual class of normalized analytic functions is denoted by $A$ and it consists of the functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in U, \tag{1}$$

for which the normalization condition is $f(0) = f'(0) - 1 = 0$. Also, the class of functions defined in (1) that are univalent in $U$ is denoted by $S$ such that $S \subset A$. Suppose we are given the function $f(z)$ of the form (1) and $g(z) = z + \sum_{n=2}^{\infty} a_n z^n$, for which $z \in U$, we say that the function $f(z)$ is subordinate to $g(z)$ in $U$ (i.e, $f(z) \prec g(z) \quad z \in U$) if there exists a Schwarz function $\omega$ which is analytic in $U$ with $\omega(0) = 0$ and $|\omega(z)| < 1 \quad (z \in U)$ such that

$$f(z) = g(\omega(z)) \quad (z \in u). \tag{2}$$

In particular, when $g \in S$,

$$f \prec g \quad (z \in U) \Leftrightarrow f(0) = g(0) \quad and \quad f(U) \subset g(U).$$

This is known as subordination principle and the details can be found in [[12], [14], [21], [37]]. Let $S^*$ and $K$ denote the class of starlike functions $f \in S$ and and the class of convex functions $f \in S$ respectively such that $S^*$ and $K$ satisfies the conditions:

$$Re\left(\frac{zf'(z)}{f(z)}\right) > 0 \quad \equiv Re\left(\frac{zf'(z)}{f(z)}\right) \prec \frac{1+z}{1-z}, \tag{3}$$

and

$$Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0 \equiv Re\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec \frac{1+z}{1-z}. \tag{4}$$

Furthermore, a function $f \in A$ satisfying the conditions

$$\frac{zf'(z)}{f(z)} \prec \frac{1+z}{1-z}$$

and

$$1 + \frac{zf''(z)}{f'(z)} \prec \frac{1+z}{1-z} \tag{5}$$

is said to be starlike of order $\beta$ and convex of order $\beta$ respectively for $0 < \beta \leq 1$.

In geometric functions theory, the arithmetic means of some functions and expressions are frequently used in mathematics. For instance, Mocanu [21] introduced the class of $\alpha$-convex functions $(0 \leq \alpha \leq 1)$ by making use of the arithmetic means as follows:

$$M_\alpha = \left\{ f \in A : \Re\left[(1-\alpha)\left(\frac{zf'(z)}{f(z)}\right) + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)\right] > 0 \quad z \in U \right\}. \tag{6}$$

We note that by varying the value of $\alpha$ in (6), the classes of starlike and convex functions are obtained as some special cases of (6). Thus, the arithmetic bridge between starlikeness and convexity is determined by the class of $\alpha$-convex functions defined in (6). In a similar context, the class of $\mu$-starlike functions $(0 \leq \mu \leq 1)$ consisting of the functions $f \in A$ introduced by Lewandoski et al.[18] that satisfy the inequality

$$\Re\left[\left(\frac{zf'(z)}{f(z)}\right)^{\mu}\left(1 + \frac{zf''(z)}{f'(z)}\right)^{1-\mu}\right] > 0 \quad (z \in U)$$

also determines the arithmetic bridge between starlikeness and convexity.

According to Duren [12], the coefficient problem or inequality is all about the determination of the region of $\mathbb{C}^{n-1}$ occupied by the points $(a_2, \cdots, a_n)$ for all $f \in S$. Attempt to deduce such precise analytic information from the geometric hypothesis of univalence is not only exceedingly challenging but also arduous. However, coefficient problem has been redeveloped in the more distinctive way of making an estimate of $|a_n|$, the modulus of the nth coefficient. Already in 1916, Bieberbach surmised that the *nth* coefficient of a univalent function is less or equal to that of the Koebe function. In fact, his actual conjecture is that $|a_2| \leq 2$ as a simple corollary to the area theorem, which is due to Gromwall. Mathematically speaking, he says: For each function $f \in S, |a_n| \leq n$ for $n = 2, 3, \cdots$ . Unless $f$ is the Koebe function or one of its rotations, strict inequality holds for all $n$. Closely related to coefficient problem is the famous, well known and classical Fekete-Szegö problem (also known as the Fekete-Szegö functional) whose origin is the disproof of the surmise (theory) of Littlewood and Parley on the bound on the coefficient of odd univalent functions. In which case for odd univalent functions, Littlewood and Parley in 1932 proved that for each $n$ the modulus $|c_n|$ is less than an absolute constant $A$ and the true bound is given by $A = 1$. For each $f \in S$, Fekete and Szegö obtained the sharp bound [13]:

$$|a_3 - \xi a_2^2| \leq 1 + 2^{e^{-\frac{2\xi}{(1-\xi)}}}, \quad 0 \leq \xi \leq 1.$$

Expectedly, the Fekete-Szegö problem has continued to receive attention until now, even in the many subclasses of $S$. Consequently, as a result of frantic enquiry, many other functionals have risen after it, each finding application in certain problems of the geometric functions. For example, when $\xi = 1$, it is important to mention a more general problem of this type, which is the Hankel determinant problem.

Gegenbauer polynomials (or ultraspherical polynomials) $C_n^\sigma(x)$ are orthogonal polynomials on the interval $[-1, 1]$ with respect to the weight function $(1 - x^2)^{\sigma - \frac{1}{2}}$ in mathematics. They generalize Legendre polynomials and Chebyshev polynomials, and are special cases of Jacobi polynomials [38]:

$$C_n^\sigma(x) = \frac{(2\sigma)_n}{(\sigma + \frac{1}{2})_n} P_n^{(\sigma - \frac{1}{2}, \sigma - \frac{1}{2})}\{x\}.$$

They are named after Leopold Gegenbauer and can be defined in terms of their generating function [36]:

$$\frac{1}{(1 - 2xt - t^2)^n} = \sum_{n=0}^{\infty} C_n^{(\sigma)}(x)\, t^n. \tag{7}$$

Another form of definition for the Gegenbauer polynomial of degree $n$ is

$$C_n^\sigma(x) = \frac{1}{n}\Big[2x(n + \sigma - 1)\, C_{n-1}^\sigma(x) - (n + 2\sigma - 2)\, C_{n-1}^\sigma(x)\Big] \tag{8}$$

with the initial values

$$C_0^{\sigma}(x) = 1, \quad C_1^{\sigma}(x) = 2\sigma x \quad and \quad C_2^{\sigma}(x) = 2\sigma(1+\sigma)x^2 - \sigma. \tag{9}$$

Gegenbauer polynomials are particular solutions of the Gegenbauer differential equation [38]:

$$(1-x^2)y'' - (2\sigma+1)xy' + n(n+2\sigma)y = 0.$$

When $\sigma = \frac{1}{2}$, the equation reduces to the Legendre equation, and the Gegenbauer polynomials reduces to the Legendre polynomials. When $\sigma = 1$, the equation reduces to the Chebyshev differential equation, and the Gegenbauer polynomials reduces to the Chebyshev polynomials of the second kind [9, 41]. It is worth mentioning that many studies have been conducted on different classes defined by many authors that are associated with Chebyshev polynomials and leading to various results ([6],[8],[16],[24],[25],[26],[29],[33]). For further information on orthogonal polynomials generally, interested readers are referred to [10]. Various results which has connection with Gegenbauer polynomials have appeared in the Literature. They include but not limited to ([4], [17], [28]).

**Definition 1** (Class $F_{(\beta,\lambda)}(H,\alpha,\delta,\mu)$)**.** *[15] Let* $0 \le \alpha \le 1$, $1 \le \delta \le 2$, $0 \le \mu \le 1$, $0 \le \beta \le \lambda \le 1$, *and* $t \in (\frac{1}{2}, 1]$. *We say that* $f \in \mathcal{A}$ *of the form (1) belong to* $f \in F_{(\beta,\lambda)}(H,\alpha,\delta,\mu)$ *if*

$$\alpha\left[\frac{zG'(z)}{G(z)}\right]^{\delta} + (1-\alpha)\left[\frac{zG'(z)}{G(z)}\right]^{\mu}\left[1+\frac{zG''(z)}{G'(z)}\right]^{1-\mu} \prec H(z,t) = \frac{1}{1-2tz+z^2} \tag{10}$$

*where* $z \in U$ *and* $G(z) = \lambda\beta z^2 f''(z) + (\lambda-\beta)f'(z) + (1-\lambda+\beta)f(z)$.

**Definition 2** (Opoola Differential Operator)**.** *[22] Let $f(z)$ defined in (1) be in $A$. Then*

$$D^m(\varphi,\gamma,\ell)f(z) = z + \sum_{n=2}^{\infty}[1+(n+\varphi-\gamma-1)\ell]^m\, a_n\, z^n \tag{11}$$

*is the Opoola differential operator. We note that in (11),* $0 \le \varphi \le \gamma, \ell \ge 0$ *and* $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. *It should also be noted that*

1. *When* $\varphi = \gamma$, *and* $\ell = 1$, $D^m(\varphi,\gamma,\ell)f(z)$ *is the Salagean differential operator [34].*
2. *When* $\varphi = \gamma$, *then* $D^m(\varphi,\gamma,\ell)f(z)$ *is the Al-Oboudi differential operator [3].*

Now, in view of (1) and by making use of (7) and (11), we define a new subclass of analytic functions in $U$ with the following subordination condition:

**Definition 3** (Class $E^m_{(\beta,\lambda,\varphi,\gamma)}(H_\phi,\alpha,\delta,\mu,\ell)$). *Let $0 \le \alpha \le 1$, $1 \le \delta \le 2$, $0 \le \mu \le 1$, $0 \le \beta \le \lambda \le 1$, $0 \le \varphi \le \gamma$, $\ell \ge 0$, $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $t \in [\frac{1}{2}, 1]$ and for non-zero real constant $\phi$. We say that $f \in \mathcal{A}$ of the form (1) belong to $E^m_{(\beta,\lambda,\varphi,\gamma)}(H_\phi,\alpha,\delta,\mu,\ell)$ if*

$$\alpha\left[\frac{zK'(z)}{K(z)}\right]^\delta + (1-\alpha)\left[\frac{zK'(z)}{K(z)}\right]^\mu \left[1+\frac{zK''(z)}{K'(z)}\right]^{1-\mu} \prec H_\sigma(z,t) \tag{12}$$

*where $z \in U$, $K(z) = \lambda\beta\, z^2F''(z) + (\lambda-\beta)\, z\, F'(z) + (1-\lambda+\beta)F(z), F(z) = D^m(\varphi,\gamma,\ell)f(z)$ and $H_\sigma(z,t) = \dfrac{1}{(1-2tz+z^2)^\sigma}$.*

## 2. Main Results

### 2.1. Coefficient bounds for the function class $E^m_{(\beta,\lambda,\varphi,\gamma)}(H_\sigma,\alpha,\delta,\mu,\ell)$

**Theorem 1** (Coefficient Estimates). *Let the function $f(z)$ given by (1) be in the class $E^m_{(\beta,\lambda,\varphi,\gamma)}(H_\phi,\alpha,\delta,\mu,\ell)$. Then*

$$|a_2| \le \frac{2\sigma t}{[\alpha\delta+(1-\alpha)(2-\mu)](2\lambda\beta+\lambda-\beta+1)[1+(1+\varphi-\gamma)\ell]^m} \tag{13}$$

*and*

$$|a_3| \le \frac{1}{2[\alpha\delta+(1-\alpha)(3-2\mu)][2(3\lambda\beta+\lambda-\beta)+1)][1+(2+\varphi-\gamma)\ell]^m}$$

$$\times\left\{\frac{(1+\phi)[\alpha\delta+(1-\alpha)(2-\mu)]^2-\phi\Big[[\alpha\delta(\delta-3)-(1-\alpha)](\mu^2+5\mu-8)\Big]}{2[\alpha\delta+(1-\alpha)](2-\mu)]^2}\,4t^2-1\right\}. \tag{14}$$

*Proof.* Let the function $f(z)$ given by (1) be in the class $E^m_{(\beta,\lambda,\varphi,\gamma)}(H_\sigma,\alpha,\delta,\mu,\ell)$. From (12), we have

$$\alpha\left[\frac{zK'(z)}{K(z)}\right]^\delta+(1-\alpha)\left[\frac{zK'(z)}{K(z)}\right]^\mu\left[1+\frac{zK''(z)}{K'(z)}\right]^{1-\mu} = 1+C_1^\sigma(t)w(z)+C_2^\sigma(t)w^2(z)+\cdots \tag{15}$$

for some analytic functions

$$w(z) = d_1z+d_2z^2+d_3z^3+\cdots \qquad (z\in U), \tag{16}$$

such that $w(0) = 0$, $|w(z)| < 1$ $(z \in U)$.
For such functions, it is well known that

$$|d_j| \leq 1 \quad (j \in \mathbb{N}) \tag{17}$$

and for all $v \in \mathbb{C}$

$$|d_2 - vd_1^2| \leq max\{1, |v|\}. \tag{18}$$

Therefore from (15) and (16) we have

$$\begin{aligned} &\alpha\left[\frac{zK'(z)}{K(z)}\right]^{\delta} + (1-\alpha)\left[\frac{zK'(z)}{K(z)}\right]^{\mu} \left[1 + \frac{zK''(z)}{K'(z)}\right]^{1-\mu} \\ &= 1 + C_1^{\sigma}(t)\, d_1\, z + [C_1^{\sigma}(t)\, d_2 + C_2^{\sigma}(t)\, d_1^2]\, z^2 + \cdots \end{aligned} \tag{19}$$

where $K(z) = \lambda\beta\, z^2 F''(z) + (\lambda - \beta)\, z\, F'(z) + (1 - \lambda + \beta)F(z)$.
From the LHS of (19) we have the following:

$$\begin{aligned} \alpha\left[\frac{z\,K'(z)}{K(z)}\right]^{\delta} &= \alpha + \alpha\delta(2\lambda\beta + \lambda - \beta + 1)[1 + (1 + \varphi - \gamma)\ell]^m\, a_2\, z + \\ &\Big[2\,\alpha\delta(6\lambda\beta + 2\lambda - 2\beta + 1)[1 + (2 + \varphi - \gamma)\ell]^m\, a_3 \\ &+ \frac{\alpha\delta(\delta - 3)}{2}(2\lambda\beta + \lambda - \beta + 1)^2[1 + (1 + \varphi - \gamma)\ell]^{2m}\, a_2^2\Big]\, z^2 + \cdots, \end{aligned} \tag{20}$$

$$\begin{aligned} \left[\frac{z\,K'(z)}{K(z)}\right]^{\mu} &= 1 + \mu(2\lambda\beta + \lambda - \beta + 1)\, a_2[1 + (1 + \varphi - \gamma)\ell]^m\, z \\ &+ \Big[2\,\mu(6\lambda\beta + 2\lambda - 2\beta + 1)[1 + (2 + \varphi - \gamma)\ell]^m\, a_3 \\ &+ \frac{\mu(\mu - 3)}{2}(2\lambda\beta + \lambda - \beta + 1)^2[1 + (1 + \varphi - \gamma)\ell]^{2m}\, a_2^2\Big]\, z^2 + \cdots \end{aligned} \tag{21}$$

and

$$\begin{aligned} \left[1 + \frac{z\,K''(z)}{K'(z)}\right]^{1-\mu} &= 1 + 2(1 - \mu)(2\lambda\beta + \lambda - \beta + 1)[1 + (1 + \varphi - \gamma)\ell]^m\, a_2\, z \\ &+ 2(1 - \mu)\Big[3(6\lambda\beta + 2\lambda - 2\beta + 1)[1 + (2 + \varphi - \gamma)\ell]^m\, a_3 \\ &+ (\mu - 2)(2\lambda\beta + \lambda - \beta + 1)^2[1 + (1 + \varphi - \gamma)\ell]^{2m}\, a_2^2\Big]\, z^2 + \cdots. \end{aligned} \tag{22}$$

Such that

$$
\begin{aligned}
&(1-\alpha)\left[\frac{z\,G'(z)}{G(z)}\right]^{\mu}\left[1+\frac{z\,G''(z)}{G'(z)}\right]^{1-\mu} = \\
&(1-\alpha)\Big[1+(2-\mu)(2\lambda\beta+\lambda-\beta+1)[1+(1+\varphi-\gamma)\ell]^m\,a_2\,z \\
&+\Big[2(3-2\mu)(6\lambda\beta+2\lambda-2\beta+1)[1+(2+\varphi-\gamma)\ell]^m\,a_3 \\
&+\frac{\mu^2+5\mu-8)}{2}(2\lambda\beta+\lambda-\beta+1)^2[1+(1+\varphi-\gamma)\ell]^{2m}\,a_2^2\Big]z^2+\cdots\Big]
\end{aligned}
\tag{23}
$$

Thus, the LHS of (19) becomes

$$
\begin{aligned}
&=1+[\alpha\delta+(1-\alpha)(2-\mu)](2\lambda\beta+\lambda-\beta+1)[1+(1+\varphi-\gamma)\ell]^m\,a_2\,z \\
&+\Big[2[\alpha\delta+(1-\alpha)(3-2\mu)][2(3\lambda\beta+2\lambda-2\beta)+1][1+(2+\varphi-\gamma)\ell]^m\,a_3 \\
&+\frac{1}{2}[\alpha\delta(\delta-3)+(\mu^2+5\mu-8)(1-\alpha)](2\lambda\beta+\lambda-\beta+1)^2[1+(1+\varphi-\gamma)\ell]^{2m}\,a_2^2\Big]z^2 \\
&+\cdots
\end{aligned}
\tag{24}
$$

Hence, equating (24) to the right hand side of (19) and comparing the coefficients, we have

$$
[\alpha\delta+(1-\alpha)(2-\mu)](2\lambda\beta+\lambda-\beta+1)[1+(1+\varphi-\gamma)\ell]^m\,a_2=C_1^{\sigma}(t)\,d_1 \tag{25}
$$

and

$$
\begin{aligned}
&\frac{1}{2}[\alpha\delta(\delta-3)+(\mu^2+5\mu-8)(1-\alpha)](2\lambda\beta+\lambda-\beta+1)^2[1+(1+\varphi-\gamma)\ell]^{2m}\,a_2^2 \\
&+2[\alpha\delta+(1-\alpha)(3-2\mu)][2(3\lambda\beta+2\lambda-2\beta)+1][1+(2+\varphi-\gamma)\ell]^m\,a_3 \\
&=C_1^{\sigma}(t)\,d_2+C_2^{\sigma}(t)\,d_1^2.
\end{aligned}
\tag{26}
$$

From (9), (17) and (25) we have

$$
a_2=\frac{C_1^{\sigma}(t)\,d_1}{[\alpha\delta+(1-\alpha)(2-\mu)](2\lambda\beta+\lambda-\beta+1)[1+(1+\varphi-\gamma)\ell]^m} \tag{27}
$$

$$
|a_2|\le\frac{2\sigma t}{[\alpha\delta+(1-\alpha)(2-\mu)](2\lambda\beta+\lambda-\beta+1)[1+(1+\varphi-\gamma)\ell]^m} \tag{28}
$$

By using (25), we can rewrite the equality (26) as follows

$$\begin{aligned} &2[\alpha\delta+(1-\alpha)(3-2\mu)][2(3\lambda\beta+2\lambda-2\beta)+1][1+(2+\varphi-\gamma)\ell]^m\, a_3 \\ &= C_1^\sigma(t)\, d_2 + C_2^\sigma(t)\, d_1^2 - \frac{1}{2}[\alpha\delta(\delta-3)+(1-\alpha)(\mu^2+5\mu-8)] \\ &\times (2\lambda\beta+\lambda-\beta+1)^2[1+(1+\varphi-\gamma)\ell]^{2m} \\ &\times \Big[\frac{C_1^\sigma(t)\, d_1}{[\alpha\delta+(1-\alpha)(2-\mu)](2\lambda\beta+\lambda-\beta+1)[1+(1+\varphi-\gamma)\ell]^m}\Big]^2. \end{aligned} \tag{29}$$

If we consider (9) and (17) in (29), we obtain

$$2[\alpha\delta+(1-\alpha)(3-2\mu)][2(3\lambda\beta+2\lambda-2\beta)+1][1+(2+\varphi-\gamma)\ell]^m\, a_3 = \ 2\sigma t$$
$$\left\{d_2 - \frac{1}{2\sigma t}\left(1 - \frac{(1+\phi)[\alpha\delta+(1-\alpha)(2-\mu)]^2 - \phi[\alpha\delta(\delta-3)+(1-\alpha)](\mu^2+5\mu-8)}{2[\alpha\delta+(1-\alpha)(2-\mu)]^2}4t^2\right)\sigma d_1^2\right\} \tag{30}$$

So that

$$a_3 = \frac{2\sigma t}{2[\alpha\delta+(1-\alpha)(3-2\mu)][2(3\lambda\beta+2\lambda-2\beta)+1][1+(1+\varphi-\gamma)\ell]^m}$$
$$\times\left\{d_2 - \frac{1}{2t}\left(1 - \frac{(1+\phi)[\alpha\delta+(1-\alpha)(2-\mu)]^2 - \phi[\alpha\delta(\delta-3)+(1-\alpha)](\mu^2+5\mu-8)}{2[\alpha\delta+(1-\alpha)(2-\mu)]^2}4t^2\right)d_1^2\right\}. \tag{31}$$

Using (17) in (31) we have

$$|a_3| \le \frac{2\sigma t}{2[\alpha\delta+(1-\alpha)(3-2\mu)][2(3\lambda\beta+2\lambda-2\beta)+1][1+(1+\varphi-\gamma)\ell]^m}$$
$$\times\, max\left\{1, \frac{1}{2t}\left|1 - \frac{(1+\sigma)[\alpha\delta+(1-\alpha)(2-\mu)]^2 - \sigma[\alpha\delta(\delta-3)+(1-\alpha)](\mu^2+5\mu-8)}{2[\alpha\delta+(1-\alpha)(2-\mu)]^2}4t^2\right|\right\}. \tag{32}$$

By using Mathematica (version 8.0), we find that

$$\frac{(1+\sigma)[\alpha\delta+(1-\alpha)(2-\mu)]^2 - \sigma[\alpha\delta(\delta-3)+(1-\alpha)](\mu^2+5\mu-8)}{2[\alpha\delta+(1-\alpha)(2-\mu)]^2} \ge 1 \tag{33}$$

for $0 \le \alpha \le 1, 1 \le \delta \le 2, 0 \le \mu \le 1$ and any real constant $\sigma$.

Consequently, we obtain

$$|a_3| \leq \frac{1}{2[\alpha\delta+(1-\alpha)(3-2\mu)][2(3\lambda\beta+2\lambda-2\beta)+1][1+(1+\varphi-\gamma)\ell]^m}$$
$$\times\left\{\frac{(1+\sigma)[\alpha\delta+(1-\alpha)(2-\mu)]^2-\sigma[\alpha\delta(\delta-3)+(1-\alpha)](\mu^2+5\mu-8)}{2[\alpha\delta+(1-\alpha)(2-\mu)]^2}4t^2\right\}. \tag{34}$$

Taking $\sigma=1$ in Theorem 1, we obtain the following Corollary:

**Corollary 2.** *Let the function f(z) given by (1) be in the class $E^m_{(\beta,\lambda,\varphi,\gamma)}(H_1,\alpha,\delta,\mu,\ell)$. Then*

$$|a_2| \leq \frac{2t}{[\alpha\delta+(1-\alpha)(2-\mu)](2\lambda\beta+\lambda-\beta+1)[1+(1+\varphi-\gamma)\ell]^m}$$

*and*

$$|a_3| \leq \frac{1}{2[\alpha\delta+(1-\alpha)(3-2\mu)][2(3\lambda\beta+\lambda-\beta)+1)][1+(2+\varphi-\gamma)\ell]^m}$$
$$\times\left\{\frac{2[\alpha\delta+(1-\alpha)(2-\mu)]^2-\Big[[\alpha\delta(\delta-3)-(1-\alpha)](\mu^2+5\mu-8)\Big]}{2[\alpha\delta+(1-\alpha)](2-\mu)]^2}\,4t^2-1\right\}.$$

**Remark 1.** *The estimates $|a_2|$ and $|a_3|$ in Corollary 2 are the coefficient estimates that will be obtained if we use the Chebyshev polynomials instead of Gegenbauer polynomials.*

Taking $\sigma=\frac{1}{2}$ in Theorem 1, we obtain the following Corollary:

**Corollary 3.** *Let the function f(z) given by (1) be in the class $E^m_{(\beta,\lambda,\varphi,\gamma)}(H_{\frac{1}{2}},\alpha,\delta,\mu,\ell)$. Then*

$$|a_2| \leq \frac{t}{[\alpha\delta+(1-\alpha)(2-\mu)](2\lambda\beta+\lambda-\beta+1)[1+(1+\varphi-\gamma)\ell]^m}$$

*and*

$$|a_3| \leq \frac{1}{2[\alpha\delta+(1-\alpha)(3-2\mu)][2(3\lambda\beta+\lambda-\beta)+1)][1+(2+\varphi-\gamma)\ell]^m}$$
$$\times\left\{\frac{3[\alpha\delta+(1-\alpha)(2-\mu)]^2-\Big[[\alpha\delta(\delta-3)-(1-\alpha)](\mu^2+5\mu-8)\Big]}{4[\alpha\delta+(1-\alpha)](2-\mu)]^2}\,t^2-1\right\}.$$

**Remark 2.** *The estimates* $|a_2|$ *and* $|a_3|$ *in Corollary 3 are the coefficient estimates that will be obtained if we use the Legendre polynomials instead of Gegenbauer polynomials.*

Taking $\varphi = \gamma$ in Theorem 1, we obtain the following Corollary:

**Corollary 4.** *Let the function f(z) given by (1) be in the class* $E^m_{(\beta,\lambda)}(H_\sigma, \alpha, \delta, \mu, \ell)$*. Then*

$$|a_2| \leq \frac{2\phi t}{(1+\ell)^m[\alpha\delta + (1-\alpha)(2-\mu)](2\lambda\beta + \lambda - \beta + 1)}$$

*and*

$$|a_3| \leq \frac{1}{2(1+2\ell)^m[\alpha\delta + (1-\alpha)(3-2\mu)][2(3\lambda\beta + \lambda - \beta) + 1)]}$$
$$\times \left\{ \frac{(1+\phi)[\alpha\delta + (1-\alpha)(2-\mu)]^2 - \phi\Big[[\alpha\delta(\delta-3) - (1-\alpha)](\mu^2 + 5\mu - 8)\Big]}{2[\alpha\delta + (1-\alpha)](2-\mu)]^2} 4t^2 - 1 \right\}.$$

**Remark 3.** *The estimates* $|a_2|$ *and* $|a_3|$ *in Corollary 4 are the coefficient estimates that will be obtained if we use the Al-Oboudi differential Operator instead of the Opoola differential operator as* $F(z)$ *in this paper.*

Taking $\varphi = \gamma, \ell = 1$ in Theorem 1, we obtain the following Corollary:

**Corollary 5.** *Let the function f(z) given by (1) be in the class* $E^m_{(\beta,\lambda)}(H_\sigma, \alpha, \delta, \mu, 1)$*. Then*

$$|a_2| \leq \frac{2\phi t}{2^m[\alpha\delta + (1-\alpha)(2-\mu)](2\lambda\beta + \lambda - \beta + 1)}$$

*and*

$$|a_3| \leq \frac{1}{2[\alpha\delta + (1-\alpha)(3-2\mu)]3^m[2(3\lambda\beta + \lambda - \beta) + 1)]}$$
$$\times \left\{ \frac{(1+\phi)[\alpha\delta + (1-\alpha)(2-\mu)]^2 - \phi\Big[[\alpha\delta(\delta-3) - (1-\alpha)](\mu^2 + 5\mu - 8)\Big]}{2[\alpha\delta + (1-\alpha)](2-\mu)]^2} 4t^2 - 1 \right\}.$$

**Remark 4.** *The estimates* $|a_2|$ *and* $|a_3|$ *in Corollary 5 are the coefficient estimates that will be obtained if we use the Salagean differential Operator instead of the Opoola differential operator that was used in this paper.*

Taking $\lambda = 0, \beta = 0, m = 0, \sigma = 1$ in Theorem 1, we obtain the following Corollary:

**Corollary 6.** *Let the function f(z) given by (1) be in the class* $E(H_1, \alpha, \delta, \mu)$. *Then*

$$|a_2| \leq \frac{2t}{\alpha\delta + (1-\alpha)(2-\mu)}$$

*and*

$$|a_3| \leq \frac{1}{2[\alpha\delta + (1-\alpha)(3-2\mu)]} \times \left\{ \frac{2[\alpha\delta + (1-\alpha)(2-\mu)]^2 - [\alpha\delta(\delta-3) - (1-\alpha)](\mu^2 + 5\mu - 8)}{2[\alpha\delta + (1-\alpha)](2-\mu)]^2} 4t^2 - 1 \right\}.$$

**Remark 5.** *The estimate of* $|a_3|$ *which is obtained in Corollary 6 is better than the corresponding estimate in [40].*

Taking $\alpha = 0, \lambda = 0, \beta = 0, m = 0, \sigma = 1$ in Theorem 1, we obtain the following Corollary:

**Corollary 7.** *Let the function f(z) given by (1) be in the class* $L(\mu, t)$. *Then*

$$|a_2| \leq \frac{2t}{(2-\mu)}$$

*and*

$$|a_3| \leq \frac{(16 - 3\mu + \mu^2)t^2}{(3-2\mu)(2-\mu)^2} - \frac{1}{2(3-2\mu)}.$$

**Remark 6.** *The estimate of* $|a_3|$ *which is obtained in Corollary 7 is better than the corresponding estimate in [2].*

Taking $\alpha = 1 - \eta, \delta = 1, \mu = 0, \beta = 0, m = 0, \sigma = 1$ in Theorem 1, we obtain the following Corollary:

**Corollary 8.** *Let the function f(z) given by (1) be in the class* $G_\lambda^\eta(t)$. *Then*

$$|a_2| \leq \frac{2t}{(1+\eta)(1+\lambda)}$$

*and*

$$|a_3| \leq \frac{1}{2(1+2\eta)(1+2\lambda)} \left[ \frac{4t^2(\eta^2 + 5\eta + 2)}{(1+\eta)^2} - 1 \right].$$

**Remark 7.** *The estimate of* $|a_3|$ *which is obtained in Corollary 8 is better than the corresponding estimate in [5].*

Taking $\alpha = 1-\eta, \delta = 1, \mu = 0, \lambda = 0, \beta = 0, m = 0, \sigma = 1$ in Theorem 1, we obtain the following Corollary:

**Corollary 9.** *Let the function f(z) given by (1) be in the class $K(\eta, t)$. Then*

$$|a_2| \leq \frac{2t}{(1+\eta)}$$

*and*

$$|a_3| \leq \frac{1}{2(1+2\eta)}\left[\frac{4t^2(\eta^2+5\eta+2)}{(1+\eta)^2} - 1\right].$$

**Remark 8.** *The estimate of $|a_3|$ which is obtained in Corollary 9 is better than the corresponding estimate in [1].*

Taking $\alpha = 1, \delta = 1, m = 0, \sigma = 1$ in Theorem 1, we obtain result of [7] the following Corollary:

**Corollary 10.** *Let the function f(z) given by (1) be in the class $N(\lambda, \beta, t)$. Then*

$$|a_2| \leq \frac{2t}{2\lambda\beta + \lambda - \beta + 1}$$

*and*

$$|a_3| \leq \frac{8t^2 - 1}{2[6\lambda\beta + 2\lambda - 2\beta + 1]}.$$

Taking $\mu = 0, \alpha = 0, \lambda = 0, \beta = 0, m = 0, \sigma = 1$ in Theorem 1, we obtain result of
[11] the following Corollary:

**Corollary 11.** *Let the function f(z) given by (1) be in the class $H(t)$. Then*

$$|a_2| \leq t$$

*and*

$$|a_3| \leq \frac{4t^2}{3} - \frac{1}{6}.$$

Taking $\alpha = 0, \delta = 1\mu = 1, \lambda = 1, \beta = 1, m = 0, \sigma = 1$ in Theorem 1, we obtain the following Corollary:

**Corollary 12.** *Let the function f(z) given by (1) be in the class $E_{1,1}(H, 0, 1, 1)$. Then*

$$|a_2| \leq t$$

*and*

$$|a_3| \leq \frac{4t^2}{7} - \frac{1}{14}.$$

## 2.2. Fekete–Szegö inequality for the function class $E^m_{(\beta,\lambda,\varphi,\gamma)}(H_\sigma, \alpha, \delta, \mu, \ell)$.

Fekete-Szegö inequality is one of the famous problem related to coefficients of univalent analytic functions. Several works on it have appeared in Literature ([6, 7, 8], [13], [16], [23], [26], [27], [30, 31, 32], [33], [35]). Just to mention view.

**Theorem 13.** *Let the function $f(z)$ given by (1) be in the class $E^m_{(\beta,\lambda,\varphi,\gamma)}(H_\sigma, \alpha, \delta, \mu, \ell)$. Then, for some $\xi \in \mathbb{R}$,*

$$|a_3 - \xi a_2^2| \leq \begin{cases} \frac{2\sigma t}{M} & \text{for } \xi \in [\xi_1, \xi_2], \\ \frac{2\sigma t}{M}\left|\frac{2\sigma(1+\phi)t^2-\phi}{2\sigma t} - \frac{R\sigma t}{B} - \xi\frac{2\sigma t M}{B(2\lambda\beta+\lambda-\beta+1)^2[1+(1+\varphi-\gamma)\ell]^{2m}}\right| & \text{for } \xi \notin [\xi_1, \xi_2] \end{cases} \tag{35}$$

*where* $\xi_1 = \left\{\frac{2((1+\sigma)B-R)t^2-(1+2t)B}{4\sigma^2t^2M}\right\}(2\lambda\beta + \lambda - \beta + 1)^2[1 + (1 + \varphi - \gamma)\ell]^{2m}$,
$\xi_2 = \left\{\frac{2((1+\sigma)B-R)t^2-(1-2t)B}{4\sigma^2t^2M}\right\}(2\lambda\beta + \lambda - \beta + 1)^2[1 + (1 + \varphi - \gamma)\ell]^{2m}$
*such that*

$$B = [\alpha\delta + (1-\alpha)(2-\mu)]^2,$$
$$M = 2[\alpha\delta + (1-\alpha)(3-2\mu)][2(3\lambda\beta + 2\lambda - 2\beta) + 1][1 + (2 + \varphi - \gamma)\ell]^m,$$
$$R = [\alpha\delta(\delta-3) + (1-\alpha)](\mu^2 + 5\mu - 8)$$

*Proof.* Let $f \in E^m_{(\beta,\lambda,\varphi,\gamma)}(H_\phi, \alpha, \delta, \mu, \ell)$ and

$$B = [\alpha\delta + (1-\alpha)(2-\mu)]^2,$$
$$M = 2[\alpha\delta + (1-\alpha)(3-2\mu)][2(3\lambda\beta + 2\lambda - 2\beta) + 1][1 + (2 + \varphi - \gamma)\ell]^m,$$
$$R = [\alpha\delta(\delta-3) + (1-\alpha)](\mu^2 + 5\mu - 8)$$

From (25) and (26) for some $\xi \in \mathbb{R}$, we can easily see that

$$|a_3 - \xi a_2^2| = \frac{C_1^\sigma(t)}{K}\left|d_2 + \left\{\frac{C_2^\sigma(t)}{C_1^\sigma(t)} - \frac{R}{2B}C_1^\sigma(t) - \xi\frac{C_1^\sigma(t)M}{B(2\lambda\beta+\lambda-\beta+1)^2[1+(1+\varphi-\gamma)\ell]^{2m}}\right\}d_1^2\right| \tag{36}$$

Then, in view of (17), we conclude that

$$|a_3 - \xi a_2^2| \leq \frac{C_1^\sigma(t)}{M} max\left\{1, \left|\frac{C_2^\sigma(t)}{C_1^\sigma(t)} - \frac{R}{2B}C_1^\sigma(t) - \xi\frac{C_1^\sigma(t)M}{B(2\lambda\beta+\lambda-\beta+1)^2[1+(1+\varphi-\gamma)\ell]^{2m}}\right|\right\} \tag{37}$$

Finally, by using (9) in (37), we get

$$|a_3 - \xi a_2^2| \leq \frac{2\sigma t}{M} max\left\{1, \left|\frac{2\sigma(1+\phi)t^2-\sigma}{2\sigma t} - \frac{\sigma t R}{B} - \xi \frac{2\sigma t M}{B(2\lambda\beta+\lambda-\beta+1)^2[1+(1+\varphi-\gamma)\ell]^{2m}}\right|\right\} \tag{38}$$

Because $t > 0$, we have

$$\begin{aligned}
&\left\{1, \left|\frac{2\sigma(1+\phi)t^2-\sigma}{2\sigma t} - \frac{\sigma t R}{B}\right.\right.\\
&\left.\left.- \xi \frac{2\sigma t M}{B(2\lambda\beta+\lambda-\beta+1)^2[1+(1+\varphi-\gamma)\ell]^{2m}}\right|\right\} \leq 1\\
&\Leftrightarrow \left\{\frac{2((1+\sigma)B-R)t^2-(1+2t)B}{4\sigma^2t^2M}\right\}(2\lambda\beta+\lambda-\beta+1)^2[1+(1+\varphi-\gamma)\ell]^{2m}\\
&\leq \xi \leq \left\{\frac{2((1+\phi)B-R)t^2-(1-2t)B}{4\phi^2t^2M}\right\}(2\lambda\beta+\lambda-\beta+1)^2[1+(1+\varphi-\gamma)\ell]^{2m}\\
&\Leftrightarrow \xi_1 \leq \xi \leq \xi_2.
\end{aligned} \tag{39}$$

Thus, the proof of the theorem is complete.

Taking $\alpha = 1, \delta = 1, m = 0, \sigma = 1$ in Theorem 13, we obtain result in [7] in the following Corollary:

**Corollary 14.** *Let the function f(z) given by (1) be in the class $N(\lambda, \beta, t)$. Then for some $\xi \in \mathbb{R}$.*

$$|a_3 - \xi a_2^2| \leq \begin{cases} \frac{t}{6\lambda\beta+2\lambda-2\beta+1} & for\ \xi \in [\xi_1, \xi_2], \\ \frac{t}{6\lambda\beta+2\lambda-2\beta+1}\left|\frac{8t^2-1}{2t} - \xi\frac{4t(6\lambda\beta+2\lambda-2\beta+1)}{(2\lambda\beta+\lambda-\beta+1)^2}\right| & for\ \xi \notin [\xi_1, \xi_2] \end{cases}$$

*where*

$$\xi_1 = \frac{(8t^2-2t-1)(2\lambda\beta+\lambda-\beta+1)^2}{8t^2(6\lambda\beta+2\lambda-2\beta+1)}$$

*and*

$$\xi_2 = \frac{(8t^2+2t-1)(2\lambda\beta+\lambda-\beta+1)^2}{8t^2(6\lambda\beta+2\lambda-2\beta+1)}$$

Taking $\lambda = 0, \beta = 0, m = 0, \sigma = 1$ in Theorem 13, we obtain result of [33] in the following Corollary:

**Corollary 15.** *Let the function f(z) given by (1) be in the class $F(H,\alpha,\delta,\mu)$. Then for some $\xi \in \mathbb{C}$,*

$$|a_3 - \xi\, a_2^2| \leq \frac{t}{\alpha\delta + (1-\alpha)(3-2\mu)} \times max\Bigg\{1, \left|2t\left(\frac{2\xi[\alpha\delta+(1-\alpha)(3-2)]}{[\alpha\delta+(1-\alpha)(2-\mu)]^2} - \frac{3\alpha\delta+(1-\alpha)(8-5\mu)-\alpha(\delta^2-\mu^2)-\mu^2}{2[\alpha\delta+(1-\alpha)(2-\mu)]^2}\right) - \frac{4t^2-1}{2t}\right|\Bigg\}$$

Taking $\alpha = 1-\eta, \delta = 1, \mu = 0, \beta = 0, m = 0, \sigma = 1$ in Theorem 13, we obtain result of
[5] the in following Corollary:

**Corollary 16.** *Let the function f(z) given by (1) be in the class $G_\lambda^\eta(t)$. Then for some $\xi \in \mathbb{R}$.*

$$|a_3 - \xi a_2^2| \leq \begin{cases} \frac{t}{(1+2\eta)(1+2\lambda)} & \text{for } \xi \in [\xi_1, \xi_2], \\ \frac{t}{(1+2\eta)(1+2\lambda)}\left|\frac{4t^2-1}{2t} - \frac{2(1+3\eta)t}{(1+\eta)^2} - \xi\frac{4t(1+2\eta)(1+2\lambda)}{(1+\eta)^2(1+\lambda)^2}\right| & \text{for } \xi \notin [\xi_1, \xi_2] \end{cases}$$

*where*

$$\xi_1 = \left\{\frac{4(\eta^2+5\eta+2)t^2 - (1+2t)(1+\eta)^2}{8(1+2\eta)(1+2\eta)t^2}\right\}(1+\lambda)^2$$

*and*

$$\xi_2 = \left\{\frac{4(\eta^2+5\eta+2)t^2 - (1-2t)(1+\eta)^2}{8(1+2\eta)(1+2\eta)t^2}\right\}(1+\lambda)^2$$

Taking $\alpha = 1-\eta, \delta = 1, \mu = 0, \lambda = 0, \beta = 0, m = 0, \phi = 1$ in Theorem 13, we obtain result of
[1] in the following Corollary:

**Corollary 17.** *Let the function f(z) given by (1) be in the class $K(\eta, t)$. Then for some $\xi \in \mathbb{R}$.*

$$|a_3 - \xi a_2^2| \leq \begin{cases} \frac{t}{1+2\eta} & \text{for } \xi \in [\xi_1, \xi_2], \\ \frac{t}{1+2\eta}\left|\frac{4t^2-1}{2t} - \frac{2(1+3\eta)t}{(1+\eta)^2} - \xi\frac{4t(1+2\eta)}{(1+\eta)^2}\right| & \text{for } \xi \notin [\xi_1, \xi_2] \end{cases}$$

*where*

$$\xi_1 = \frac{4(\eta^2+5\eta+2)t^2 - (1+2t)(1+\eta)^2}{8(1+2\eta)t^2}$$

*and*

$$\xi_2 = \frac{4(\eta^2+5\eta+2)t^2 - (1-2t)(1+\eta)^2}{8(1+2\eta)t^2}$$

Taking $\alpha = 1 - \eta, \delta = 1, \mu = 0, \lambda = 0, \beta = 0, m = 0, \phi = 1$ in Theorem 13, we obtain result of
[2] in the following Corollary:

**Corollary 18.** *Let the function f(z) given by (1) be in the class $L(\mu, t)$. Then for some $\xi \in \mathbb{R}$.*

$$|a_3 - \xi a_2^2| \leq \begin{cases} \frac{t}{3-2\mu} & \text{for } \xi \in [\xi_1, \xi_2], \\ \frac{t}{3-2\mu}\left|\frac{4t^2-1}{2t} - \frac{(\mu^2+5\mu-8)t}{(2-\mu)^2} - \xi\frac{4t(3-2\mu)}{(2-\mu)^2}\right| & \text{for } \xi \notin [\xi_1, \xi_2] \end{cases}$$

*where*

$$\xi_1 = \frac{2(\mu^2 - 13\mu + 16)t^2 - (2-\mu)^2(1+2t)}{8(3-2\mu)t^2}$$

*and*

$$\xi_2 = \frac{2(\mu^2 - 13\mu + 16)t^2 - (2-\mu)^2(1-2t)}{8(3-2\mu)t^2}$$

Taking $\alpha = 0, m = 0, \phi = 1$ in Theorem 13, we obtain the following Corollary:

**Corollary 19.** *Let the function f(z) given by (1) be in the class $E_{(\beta,\lambda)}(H, 0, \delta, \mu)$. Then for some $\xi \in \mathbb{R}$,*

$$|a_3 - \xi a_2^2| \leq \begin{cases} \frac{t}{(3-2\mu)(6\lambda\beta+2\lambda-2\beta+1)} & \text{for } \xi \in [\xi_1, \xi_2], \\ \frac{t}{(3-2\mu)(6\lambda\beta+2\lambda-2\beta+1)}\left|\frac{4t^2-1}{2t} - \frac{(\mu^2+5\mu-8)t}{(2-\mu)^2} - \xi\frac{4t(3-2\mu)(6\lambda\beta+2\lambda-2\beta+1)}{(2-\mu)^2(2\lambda\beta+\lambda-\beta+1)^2}\right| & \text{for } \xi \notin [\xi_1, \xi_2] \end{cases}$$

*where*

$$\xi_1 = \left\{\frac{2(\mu^2 - 13\mu + 16)t^2 - (2-\mu)^2(1+2t)}{8(3-2\mu)(6\lambda\beta + 2\lambda - 2\beta + 1)t^2}\right\}(2\lambda\beta + \lambda - \beta + 1)^2$$

*and*

$$\xi_2 = \left\{\frac{2(\mu^2 - 13\mu + 16)t^2 - (2-\mu)^2(1-2t)}{8(3-2\mu)(6\lambda\beta + 2\lambda - 2\beta + 1)t^2}\right\}(2\lambda\beta + \lambda - \beta + 1)^2$$

Taking $\alpha = 0, \lambda = 0, \beta = 0, m = 0, \sigma = 1$ in Theorem 13, we obtain the following Corollary:

**Corollary 20.** *Let the function f(z) given by (1) be in the class $E(H, 0, \delta, \mu)$. Then for some $\xi \in \mathbb{R}$,*

$$|a_3 - \xi a_2^2| \leq \begin{cases} \frac{t}{(3-2\mu)} & \text{for } \xi \in [\xi_1, \xi_2], \\ \frac{t}{(3-2\mu)}\left|\frac{4t^2-1}{2t} - \frac{(\mu^2+5\mu-8)t}{(2-\mu)^2} - \xi\frac{4t(3-2\mu)}{(2-\mu)^2}\right| & \text{for } \xi \notin [\xi_1, \xi_2] \end{cases}$$

*where*

$$\xi_1 = \left\{ \frac{2(\mu^2 - 13\mu + 16)t^2 - (2-\mu)^2(1+2t)}{8(3-2\mu)t^2} \right\}$$

*and*

$$\xi_2 = \left\{ \frac{2(\mu^2 - 13\mu + 16)t^2 - (2-\mu)^2(1-2t)}{8(3-2\mu)t^2} \right\}$$

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