On certain differential sandwich theorems involving a generalized Sălăgean operator and Ruscheweyh operator

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Abstract

In the present paper we introduce sufficient conditions for subordination and superordination involving the operator $DR_\lambda^{m,n}$ and also we obtain sandwich-type results.

1 Introduction

Let $\mathcal{H}(U)$ be the class of analytic function in the open unit disc of the complex plane $U = \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{H}(a, n)$ be the subclass of $\mathcal{H}(U)$ consisting of functions of the form $f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots$.

Let $\mathcal{A}_n = \{f \in \mathcal{H}(U) : f(z) = z + a_{n+1} z^{n+1} + \dots, \ z \in U\}$ and $\mathcal{A} = \mathcal{A}_1$.

Denote by $K = \left\{ f \in \mathcal{A} : \operatorname{Re} \frac{z f''(z)}{f'(z)} + 1 > 0, \ z \in U \right\}$, the class of normalized convex functions in U.

Let the functions f and g be analytic in U. We say that the function f is subordinate to g, written $f \prec g$, if there exists a Schwarz function w, analytic in U, with $w(0) = 0$ and $|w(z)| < 1$, for all $z \in U$, such that $f(z) = g(w(z))$, for all $z \in U$. In particular, if the function g is univalent in U, the above subordination is equivalent to $f(0) = g(0)$ and $f(U) \subset g(U)$.

Let $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ and h be an univalent function in U. If p is analytic in U and satisfies the second order differential subordination

$$\psi(p(z), zp'(z), z^2 p''(z); z) \prec h(z), \quad \text{for } z \in U, \tag{1}$$

then p is called a solution of the differential subordination. The univalent function q is called a dominant of the solutions of the differential subordination, or more simply a dominant, if $p \prec q$ for all p satisfying (1). A dominant $\widetilde{q}$ that satisfies $\widetilde{q} \prec q$ for all dominants q of (1) is said to be the best dominant of (1). The best dominant is unique up to a rotation of U.

Let $\psi : \mathbb{C}^2 \times U \to \mathbb{C}$ and h analytic in U. If p and $\psi\left(p(z), zp'(z), z^2 p''(z); z\right)$ are univalent and if p satisfies the second order differential superordination

$$h(z) \prec \psi(p(z), zp'(z), z^2 p''(z); z), \qquad z \in U, \tag{2}$$

then p is a solution of the differential superordination (2) (if f is subordinate to F, then F is called to be superordinate to f). An analytic function q is called a subordinant if $q \prec p$ for all p satisfying (2). An univalent subordinant $\widetilde{q}$ that satisfies $q \prec \widetilde{q}$ for all subordinants q of (2) is said to be the best subordinant.

Miller and Mocanu [17] obtained conditions h, q and ψ for which the following implication holds

$$h(z) \prec \psi(p(z), zp'(z), z^2 p''(z); z) \Rightarrow q(z) \prec p(z).$$

For two functions $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$ and $g(z) = z + \sum_{j=2}^{\infty} b_j z^j$ analytic in the open unit disc U, the Hadamard product (or convolution product) of $f(z)$ and $g(z)$, written as $(f * g)(z)$, is defined by

$$f(z) * g(z) = (f * g)(z) = z + \sum_{j=2}^{\infty} a_j b_j z^j.$$

Definition 1. *(Al Oboudi [7]) For $f \in \mathcal{A}$, $\lambda \geq 0$ and $n \in \mathbb{N}$, the operator D_λ^m is defined by $D_\lambda^m : \mathcal{A} \to \mathcal{A}$,*

$$
\begin{aligned}
D_\lambda^0 f(z) &= f(z) \\
D_\lambda^1 f(z) &= (1-\lambda) f(z) + \lambda z f'(z) = D_\lambda f(z) \\
&\quad \cdots \\
D_\lambda^m f(z) &= (1-\lambda) D_\lambda^{m-1} f(z) + \lambda z \left(D_\lambda^m f(z)\right)' = D_\lambda \left(D_\lambda^{m-1} f(z)\right), \text{ for } z \in U.
\end{aligned}
$$

Remark 1. *If $f \in \mathcal{A}$ and $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$, then $D_\lambda^m f(z) = z + \sum_{j=2}^{\infty} \left[1 + (j-1)\lambda\right]^m a_j z^j$, for $z \in U$.*

Remark 2. *For $\lambda = 1$ in the above definition we obtain the Sălăgean differential operator [20].*

Definition 2. *(Ruscheweyh [19]) For $f \in \mathcal{A}$ and $n \in \mathbb{N}$, the operator R^n is defined by $R^n : \mathcal{A} \to \mathcal{A}$,*

$$
\begin{aligned}
R^0 f(z) &= f(z) \\
R^1 f(z) &= z f'(z) \\
&\cdots \\
(n+1) R^{n+1} f(z) &= z \left(R^n f(z) \right)' + n R^n f(z), \quad z \in U.
\end{aligned}
$$

Remark 3. *If $f \in \mathcal{A}$, $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$, then*

$$
R^n f(z) = z + \sum_{j=2}^{\infty} \frac{(n+j-1)!}{n!(j-1)!} a_j z^j \ for \ z \in U
$$

.

Using the results of Miller and Mocanu [17], [18], Bulboaca [14] considered certain classes of first order differential superordinations. Recently, many authors [15], [22], [21], [16], [23], have used the results of Bulboaca and obtain certain sufficient conditions applying first order differential subordinations and superordinations. In order to prove our subordination and superordination results, we make use of the following:

Definition 3. *[18] Denote by Q the set of all functions f that are analytic and injective on $\overline{U} \backslash E(f)$, where $E(f) = \{\zeta \in \partial U : \lim_{z \to \zeta} f(z) = \infty\}$, and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial U \backslash E(f)$.*

Lemma 1. *[18] Let the function q be univalent in the unit disc U and θ and ϕ be analytic in a domain D containing $q(U)$ with $\phi(w) \neq 0$ when $w \in q(U)$. Set $Q(z) = z q'(z) \phi(q(z))$ and $h(z) = \theta(q(z)) + Q(z)$. Suppose that*
   1. Q is starlike univalent in U and
   2. $Re\left(\frac{z h'(z)}{Q(z)} \right) > 0$ for $z \in U$.
   If p is analytic with $p(0) = q(0)$, $p(U) \subseteq D$ and

$$
\theta(p(z)) + z p'(z) \phi(p(z)) \prec \theta(q(z)) + z q'(z) \phi(q(z)),
$$

then $p(z) \prec q(z)$ and q is the best dominant.

Lemma 2. *[14] Let the function q be convex univalent in the open unit disc U and ν and ϕ be analytic in a domain D containing $q(U)$. Suppose that*
   1. $Re\left(\frac{\nu'(q(z))}{\phi(q(z))} \right) > 0$ for $z \in U$ and
   2. $\psi(z) = z q'(z) \phi(q(z))$ is starlike univalent in U.

If $p(z) \in \mathcal{H}[q(0), 1] \cap Q$, with $p(U) \subseteq D$ and $\nu(p(z)) + zp'(z) \phi(p(z))$ is univalent in U and

$$\nu(q(z)) + zq'(z) \phi(q(z)) \prec \nu(p(z)) + zp'(z) \phi(p(z)),$$

then $q(z) \prec p(z)$ and q is the best subordinant.

2   Main results

Definition 4. *Let $\lambda \geq 0$ and $n, m \in \mathbb{N}$. Denote by $DR_{\lambda}^{m,n} : \mathcal{A} \to \mathcal{A}$ the operator given by the Hadamard product of the generalized Sălăgean operator D_{λ}^m and the Ruscheweyh operator R^n,*

$$DR_{\lambda}^{m,n} f(z) = (D_{\lambda}^m * R^n) f(z),$$

for any $z \in U$ and each nonnegative integers m, n.

Remark 4. *If $f \in \mathcal{A}$ and $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$, then*
$DR_{\lambda}^{m,n} f(z) = z + \sum_{j=2}^{\infty} [1 + (j-1)\lambda]^m \frac{(n+j-1)!}{n!(j-1)!} a_j^2 z^j$, for $z \in U$.
This operator was studied in [12] and [13].

Remark 5. *For $\lambda = 1$, $m = n$, we obtain the Hadamard product SR^n [1] of the Sălăgean operator S^n and Ruscheweyh derivative R^n, which was studied in [2], [3].*

Remark 6. *For $m = n$ we obtain the Hadamard product DR_{λ}^n [4] of the generalized Sălăgean operator D_{λ}^n and Ruscheweyh derivative R^n, which was studied in [5], [6], [8], [9], [10], [11].*

Using simple computation one obtains the next result.

Proposition 1. *[12].For $m, n \in \mathbb{N}$ and $\lambda \geq 0$ we have*

$$DR_{\lambda}^{m+1,n} f(z) = (1 - \lambda) DR_{\lambda}^{m,n} f(z) + \lambda z (DR_{\lambda}^{m,n} f(z))' \qquad (3)$$

and
$$z (DR_{\lambda}^{m,n} f(z))' = (n+1) DR_{\lambda}^{m,n+1} f(z) - n DR_{\lambda}^{m,n} f(z). \qquad (4)$$

First, our purpose is to find sufficient conditions for certain normalized analytic functions f such that
$q_1(z) \prec \frac{z^{\delta} DR_{\lambda}^{m,n+1} f(z)}{(DR_{\lambda}^{m,n} f(z))^{1+\delta}} \prec q_2(z), \quad z \in U, 0 < \delta \leq 1$
where q_1 and q_2 are given univalent functions.

Theorem 1. *Let* $\frac{z^\delta DR_\lambda^{m,n+1} f(z)}{\left(DR_\lambda^{m,n} f(z)\right)^{1+\delta}} \in \mathcal{H}(U)$, $z \in U$, $f \in \mathcal{A}$, $m, n \in \mathbb{N}$, $\lambda \geq 0$, $0 < \delta \leq 1$ *and let the function* $q(z)$ *be convex and univalent in* U *such that* $q(0) = 1$. *Assume that*

$$Re\left(1 + \frac{\alpha}{\beta} q(z) - \frac{zq'(z)}{q(z)} + \frac{zq''(z)}{q'(z)}\right) > 0, \quad z \in U, \tag{5}$$

for $\alpha, \beta \in \mathbb{C}, \beta \neq 0$, $z \in U$, *and*

$$\psi_\lambda^{m,n}(\alpha, \beta, \delta; z) := \alpha \frac{z^\delta DR_\lambda^{m,n+1} f(z)}{\left(DR_\lambda^{m,n} f(z)\right)^{1+\delta}} + \tag{6}$$

$$+\beta \left[\delta(n+1) - 1 + (n+2)\frac{DR_\lambda^{m,n+2} f(z)}{DR_\lambda^{m,n+1} f(z)} - (1+\delta)(n+1)\frac{DR_\lambda^{m,n+1} f(z)}{DR_\lambda^{m,n} f(z)}\right].$$

If q *satisfies the following subordination*

$$\psi_\lambda^{m,n}(\alpha, \beta, \delta; z) \prec \alpha q(z) + \frac{\beta z q'(z)}{q(z)}, \tag{7}$$

for $\alpha, \beta \in \mathbb{C}$, $\beta \neq 0$ *then*

$$\frac{z^\delta DR_\lambda^{m,n+1} f(z)}{\left(DR_\lambda^{m,n} f(z)\right)^{1+\delta}} \prec q(z), \quad z \in U, \tag{8}$$

and q *is the best dominant.*

Proof. Let the function p be defined by $p(z) := \frac{z^\delta DR_\lambda^{m,n+1} f(z)}{\left(DR_\lambda^{m,n} f(z)\right)^{1+\delta}}$, $z \in U$, $z \neq 0$, $0 < \delta \leq 1$, $f \in \mathcal{A}$. The function p is analytic in U and $p(0) = 1$.

Differentiating this function, with respect to z, we get

$$zp'(z) = \frac{z^\delta DR_\lambda^{m,n+1} f(z)}{\left(DR_\lambda^{m,n} f(z)\right)^{1+\delta}}\left[\delta + \frac{z\left(DR_\lambda^{m,n+1} f(z)\right)'}{DR_\lambda^{m,n+1} f(z)} - (1+\delta)\frac{z\left(DR_\lambda^{m,n} f(z)\right)'}{DR_\lambda^{m,n} f(z)}\right].$$

By using the identity (4), we obtain

$$\frac{zp'(z)}{p(z)} = \delta(n+1) - 1 + (n+2)\frac{DR_\lambda^{m,n+2} f(z)}{DR_\lambda^{m,n+1} f(z)} - (1+\delta)(n+1)\frac{DR_\lambda^{m,n+1} f(z)}{DR_\lambda^{m,n} f(z)}. \tag{9}$$

By setting $\theta(w) := \alpha w$ and $\phi(w) := \frac{\beta}{w}$, $\alpha, \beta \in \mathbb{C}$, $\beta \neq 0$ it can be easily verified that θ is analytic in $\mathbb{C}$, ϕ is analytic in $\mathbb{C}\backslash\{0\}$ and that $\phi(w) \neq 0$, $w \in \mathbb{C}\backslash\{0\}$.

Also, by letting $Q(z) = zq'(z)\phi(q(z)) = \frac{\beta z q'(z)}{q(z)}$, we find that $Q(z)$ is starlike univalent in U.

Let $h(z) = \theta(q(z)) + Q(z) = \alpha q(z) + \frac{\beta z q'(z)}{q(z)}$, $z \in U$.

If we derive the function Q, with respect to z, perform calculations, we have $Re\left(\frac{zh'(z)}{Q(z)}\right) = Re\left(1 + \frac{\alpha}{\beta}q(z) - \frac{zq'(z)}{q(z)} + \frac{zq''(z)}{q'(z)}\right) > 0$.

By using (9), we obtain $\alpha p(z) + \beta\frac{zp'(z)}{p(z)} = \alpha\frac{z^\delta DR_\lambda^{m,n+1}f(z)}{\left(DR_\lambda^{m,n}f(z)\right)^{1+\delta}} +$

$+\beta\left[\delta(n+1) - 1 + (n+2)\frac{DR_\lambda^{m,n+2}f(z)}{DR_\lambda^{m,n+1}f(z)} - (1+\delta)(n+1)\frac{DR_\lambda^{m,n+1}f(z)}{DR_\lambda^{m,n}f(z)}\right].$

By using (7), we have $\alpha p(z) + \frac{\beta z p'(z)}{p(z)} \prec \alpha q(z) + \frac{\beta z q'(z)}{q(z)}$.

Therefore, the conditions of Lemma 1 are met, so we have $p(z) \prec q(z)$, $z \in U$, i.e. $\frac{z^\delta DR_\lambda^{m,n+1}f(z)}{\left(DR_\lambda^{m,n}f(z)\right)^{1+\delta}} \prec q(z)$, $z \in U$, and q is the best dominant. $\square$

Corollary 1. *Let* $q(z) = \frac{1+Az}{1+Bz}$, $-1 \leq B < A \leq 1$, $m, n \in \mathbb{N}$, $\lambda \geq 0$, $z \in U$. *Assume that (5) holds. If* $f \in \mathcal{A}$ *and*

$$\psi_\lambda^{m,n}(\alpha, \beta, \delta; z) \prec \alpha\frac{1+Az}{1+Bz} + \beta\frac{(A-B)z}{(1+Az)(1+Bz)},$$

for $\alpha, \beta \in \mathbb{C}$, $\beta \neq 0$, $-1 \leq B < A \leq 1$, $0 < \delta \leq 1$ *where* $\psi_\lambda^{m,n}$ *is defined in (6), then*

$$\frac{z^\delta DR_\lambda^{m,n+1}f(z)}{\left(DR_\lambda^{m,n}f(z)\right)^{1+\delta}} \prec \frac{1+Az}{1+Bz}$$

and $\frac{1+Az}{1+Bz}$ *is the best dominant.*

Proof. For $q(z) = \frac{1+Az}{1+Bz}$, $-1 \leq B < A \leq 1$, in Theorem 1 we get the corollary. $\square$

Theorem 2. *Let* q *be convex and univalent in* U, *such that* $q(0) = 1$, $m, n \in \mathbb{N}$, $\lambda \geq 0$. *Assume that*

$$Re\left(\frac{\alpha}{\beta}q(z)\right) > 0, \text{ for } \alpha, \beta \in \mathbb{C}, \ \beta \neq 0, \ z \in U. \tag{10}$$

If $f \in \mathcal{A}, 0 < \delta \leq 1$, $\frac{z^\delta DR_\lambda^{m,n+1}f(z)}{\left(DR_\lambda^{m,n}f(z)\right)^{1+\delta}} \in \mathcal{H}\left[q(0), 1\right] \cap Q$ *and* $\psi_\lambda^{m,n}(\alpha, \beta, \delta; z)$ *is univalent in* U, *where* $\psi_\lambda^{m,n}(\alpha, \beta, \delta; z)$ *is as defined in (6), then*

$$\alpha q(z) + \frac{\beta z q'(z)}{q(z)} \prec \psi_\lambda^{m,n}(\alpha, \beta, \delta; z), \quad z \in U, \tag{11}$$

implies

$$q(z) \prec \frac{z^\delta DR_\lambda^{m,n+1}f(z)}{\left(DR_\lambda^{m,n}f(z)\right)^{1+\delta}}, \quad z \in U, \tag{12}$$

and q *is the best subordinant.*

Proof. Let the function p be defined by $p(z) := \frac{z^\delta DR_\lambda^{m,n+1} f(z)}{\left(DR_\lambda^{m,n} f(z)\right)^{1+\delta}}$, $z \in U$, $z \neq 0$,
$0 < \delta \leq 1$, $f \in \mathcal{A}$.

By setting $\nu(w) := \alpha w$ and $\phi(w) := \frac{\beta}{w}$ it can be easily verified that ν is
analytic in $\mathbb{C}$, ϕ is analytic in $\mathbb{C}\backslash\{0\}$ and that $\phi(w) \neq 0$, $w \in \mathbb{C}\backslash\{0\}$.

Since q is convex and univalent function, it follows that $Re\left(\frac{\nu'(q(z))}{\phi(q(z))}\right) =$
$Re\left(\frac{\alpha}{\beta} q(z)\right) > 0$, for $\alpha, \beta \in \mathbb{C}$, $\beta \neq 0$.

By using (11) we obtain

$$\alpha q(z) + \beta \frac{z q'(z)}{q(z)} \prec \alpha p(z) + \beta \frac{z p'(z)}{p(z)}.$$

Using Lemma 2, we have

$$q(z) \prec p(z) = \frac{z^\delta DR_\lambda^{m,n+1} f(z)}{\left(DR_\lambda^{m,n} f(z)\right)^{1+\delta}}, \quad z \in U,$$

and q is the best subordinant. $\qquad\qquad\square$

Corollary 2. *Let* $q(z) = \frac{1+Az}{1+Bz}$, $-1 \leq B < A \leq 1$, $m, n \in \mathbb{N}$, $\lambda \geq 0$. *Assume
that* (10) *holds. If* $f \in \mathcal{A}$, $\frac{z^\delta DR_\lambda^{m,n+1} f(z)}{\left(DR_\lambda^{m,n} f(z)\right)^{1+\delta}} \in \mathcal{H}[q(0), 1] \cap Q$ *and*

$$\alpha \frac{1+Az}{1+Bz} + \beta \frac{(A-B)z}{(1+Az)(1+Bz)} \prec \psi_\lambda^{m,n}(\alpha, \beta, \delta; z),$$

for $\alpha, \beta \in \mathbb{C}$, $\beta \neq 0$, $-1 \leq B < A \leq 1$, *where* $\psi_\lambda^{m,n}$ *is defined in* (6), *then*

$$\frac{1+Az}{1+Bz} \prec \frac{z^\delta DR_\lambda^{m,n+1} f(z)}{\left(DR_\lambda^{m,n} f(z)\right)^{1+\delta}}$$

and $\frac{1+Az}{1+Bz}$ *is the best subordinant.*

Proof. For $q(z) = \frac{1+Az}{1+Bz}$, $-1 \leq B < A \leq 1$ in Theorem 2 we get the corollary.
$\qquad\qquad\square$

Combining Theorem 1 and Theorem 2, we state the following sandwich
theorem.

Theorem 3. *Let* q_1 *and* q_2 *be analytic and univalent in* U *such that* $q_1(z) \neq 0$
and $q_2(z) \neq 0$, *for all* $z \in U$, *with* $z q_1'(z)$ *and* $z q_2'(z)$ *being starlike univalent.*
Suppose that q_1 *satisfies* (5) *and* q_2 *satisfies* (10). *If* $f \in \mathcal{A}$, $\frac{z^\delta DR_\lambda^{m,n+1} f(z)}{\left(DR_\lambda^{m,n} f(z)\right)^{1+\delta}} \in$

$\mathcal{H}[q(0),1] \cap Q$, $0 < \delta \leq 1$ and $\psi_\lambda^{m,n}(\alpha,\beta,\delta;z)$ is as defined in (6) univalent in U, then

$$\alpha q_1(z) + \frac{\beta z q_1'(z)}{q_1(z)} \prec \psi_\lambda^{m,n}(\alpha,\beta,\delta;z) \prec \alpha q_2(z) + \frac{\beta z q_2'(z)}{q_2(z)},$$

for $\alpha,\beta \in \mathbb{C}$, $\beta \neq 0$, implies

$$q_1(z) \prec \frac{z^\delta DR_\lambda^{m,n+1} f(z)}{\left(DR_\lambda^{m,n} f(z)\right)^{1+\delta}} \prec q_2(z), \quad \delta \in \mathbb{C}, \ \delta \neq 0,$$

and q_1 and q_2 are respectively the best subordinant and the best dominant.

For $q_1(z) = \frac{1+A_1 z}{1+B_1 z}$, $q_2(z) = \frac{1+A_2 z}{1+B_2 z}$, where $-1 \leq B_2 < B_1 < A_1 < A_2 \leq 1$, we have the following corollary.

Corollary 3. Let $m,n \in \mathbb{N}$, $\lambda \geq 0$. Assume that (5) and (10) hold for $q_1(z) = \frac{1+A_1 z}{1+B_1 z}$ and $q_2(z) = \frac{1+A_2 z}{1+B_2 z}$, respectively. If $f \in \mathcal{A}$, $0 < \delta \leq 1$, $\frac{z^\delta DR_\lambda^{m,n+1} f(z)}{\left(DR_\lambda^{m,n} f(z)\right)^{1+\delta}} \in \mathcal{H}[q(0),1] \cap Q$ and

$$\alpha \frac{1+A_1 z}{1+B_1 z} + \beta \frac{(A_1 - B_1) z}{(1+A_1 z)(1+B_1 z)} \prec \psi_\lambda^{m,n}(\alpha,\beta,\delta;z)$$

$$\prec \alpha \frac{1+A_2 z}{1+B_2 z} + \beta \frac{(A_2 - B_2) z}{(1+A_2 z)(1+B_2 z)},$$

for $\alpha,\beta \in \mathbb{C}$, $\beta \neq 0$, $-1 \leq B_2 \leq B_1 < A_1 \leq A_2 \leq 1$, where $\psi_\lambda^{m,n}$ is defined in (6), then

$$\frac{1+A_1 z}{1+B_1 z} \prec \frac{z^\delta DR_\lambda^{m,n+1} f(z)}{\left(DR_\lambda^{m,n} f(z)\right)^{1+\delta}} \prec \frac{1+A_2 z}{1+B_2 z},$$

hence $\frac{1+A_1 z}{1+B_1 z}$ and $\frac{1+A_2 z}{1+B_2 z}$ are the best subordinant and the best dominant, respectively.

Next, our purpose is to find sufficient conditions for certain normalized analytic functions f such that

$$q_1(z) \prec \left(\frac{aDR_\lambda^{m+1,n} f(z) + bDR_\lambda^{m,n} f(z)}{(a+b)z} \right)^\delta \prec q_2(z), \quad z \in U$$

where q_1 and q_2 are given univalent functions

Theorem 4. Let $\left(\frac{aDR_\lambda^{m+1,n} f(z) + bDR_\lambda^{m,n} f(z)}{(a+b)z} \right)^\delta \in \mathcal{H}(U), f \in \mathcal{A}, z \in U,$ $\delta, a, b \in \mathbb{C}, \delta \neq 0, a+b \neq 0, m,n \in \mathbb{N}, \lambda \geq 0$ and let the function $q(z)$ be convex and univalent in U such that $q(0) = 1$, $z \in U$. Assume that

$$Re \left(1 + \frac{\alpha}{\beta} q(z) - \frac{z q'(z)}{q(z)} + \frac{z q''(z)}{q'(z)} \right) > 0, \tag{13}$$

for $\alpha, \beta \in \mathbb{C}$, $\beta \neq 0$, $z \in U$, and

$$\psi_\lambda^{m,n}(a, b, \alpha, \beta, \delta; z) := \alpha \left(\frac{a DR_\lambda^{m+1,n} f(z) + b DR_\lambda^{m,n} f(z)}{(a+b) z} \right)^\delta +$$

$$\frac{\beta \delta \left[a DR_\lambda^{m+2,n} f(z) + (b-a) DR_\lambda^{m+1,n} f(z) - b DR_\lambda^{m,n} f(z) \right]}{\lambda \left(a DR_\lambda^{m+1,n} f(z) + b DR_\lambda^{m,n} f(z) \right)}. \tag{14}$$

If q satisfies the following subordination

$$\psi_\lambda^{m,n}(a, b, \alpha, \beta, \delta; z) \prec \alpha q(z) + \frac{\beta z q'(z)}{q(z)}, \tag{15}$$

for $\alpha, \beta \in \mathbb{C}$, $\beta \neq 0$, $z \in U$, then

$$\left(\frac{a DR_\lambda^{m+1,n} f(z) + b DR_\lambda^{m,n} f(z)}{(a+b) z} \right)^\delta \prec q(z), \quad z \in U, \ \delta \in \mathbb{C}, \ \delta \neq 0, \tag{16}$$

and q is the best dominant.

Proof. Let the function p be defined by $p(z) := \left(\frac{a DR_\lambda^{m+1,n} f(z) + b DR_\lambda^{m,n} f(z)}{(a+b) z} \right)^\delta$, $z \in U$, $z \neq 0$, $\delta, a, b \in \mathbb{C}, \delta \neq 0, a + b \neq 0, f \in \mathcal{A}$. The function p is analytic in U and $p(0) = 1$

Differentiating this function, with respect to z, we get

$$z p'(z) = \delta \left(\frac{a DR_\lambda^{m+1,n} f(z) + b DR_\lambda^{m,n} f(z)}{(a+b) z} \right)^{\delta-1}.$$

$$\cdot \frac{1}{a+b} \left[\frac{a \left(DR_\lambda^{m+1,n} f(z) \right)' + b \left(DR_\lambda^{m,n} f(z) \right)'}{z} - \frac{a DR_\lambda^{m+1,n} f(z) + b DR_\lambda^{m,n} f(z)}{z^2} \right].$$

We have

$$z p'(z) = \delta \left(\frac{a DR_\lambda^{m+1,n} f(z) + b DR_\lambda^{m,n} f(z)}{(a+b) z} \right)^\delta \frac{1}{a DR_\lambda^{m+1,n} f(z) + b DR_\lambda^{m,n} f(z)}.$$

$$\left[a z \left(DR_\lambda^{m+1,n} f(z) \right)' + b z \left(DR_\lambda^{m,n} f(z) \right)' - a DR_\lambda^{m+1,n} f(z) - b DR_\lambda^{m,n} f(z) \right] \tag{17}$$

By using the identity (3) we obtain

$$\frac{z p'(z)}{p(z)} = \frac{\delta \left[a DR_\lambda^{m+2,n} f(z) + (b-a) DR_\lambda^{m+1,n} f(z) - b DR_\lambda^{m,n} f(z) \right]}{\lambda \left(a DR_\lambda^{m+1,n} f(z) + b DR_\lambda^{m,n} f(z) \right)} \tag{18}$$

By setting $\theta(w) := \alpha w$ and $\phi(w) := \frac{\beta}{w}$, $\alpha, \beta \in \mathbb{C}$, $\beta \neq 0$ it can be easily verified that θ is analytic in $\mathbb{C}$, ϕ is analytic in $\mathbb{C}\backslash\{0\}$ and that $\phi(w) \neq 0$, $w \in \mathbb{C}\backslash\{0\}$.

Also, by letting $Q(z) = zq'(z)\phi(q(z)) = \frac{\beta z q'(z)}{q(z)}$, we find that $Q(z)$ is starlike univalent in U.

Let $h(z) = \theta(q(z)) + Q(z) = \alpha q(z) + \frac{\beta z q'(z)}{q(z)}$, $z \in U$

If we derive the function Q, with respect to z, perform calculations, we have $Re\left(\frac{zh'(z)}{Q(z)}\right) = Re\left(1 + \frac{\alpha}{\beta}q(z) - \frac{zq'(z)}{q(z)} + \frac{zq''(z)}{q'(z)}\right) > 0.$

By using (9), we obtain

$$\alpha p(z) + \beta \frac{zp'(z)}{p(z)} = \alpha \left(\frac{aDR_\lambda^{m+1,n}f(z) + bDR_\lambda^{m,n}f(z)}{(a+b)z}\right)^\delta +$$

$$+\frac{\beta\delta\left[aDR_\lambda^{m+2,n}f(z) + (b-a)DR_\lambda^{m+1,n}f(z) - bDR_\lambda^{m,n}f(z)\right]}{\lambda\left(aDR_\lambda^{m+1,n}f(z) + bDR_\lambda^{m,n}f(z)\right)}.$$

By using (15), we have $\alpha p(z) + \beta\frac{zp'(z)}{p(z)} \prec \alpha q(z) + \beta\frac{zq'(z)}{p(z)}$.

From Lemma 1, we have $p(z) \prec q(z)$, $z \in U$, i.e.,

$$\left(\frac{aDR_\lambda^{m+1,n}f(z) + bDR_\lambda^{m,n}f(z)}{(a+b)z}\right)^\delta \prec q(z),$$

$z \in U$, $\delta \in \mathbb{C}$, $\delta \neq 0$, and q is the best dominant. $\qquad\square$

Corollary 4. *Let* $q(z) = \frac{1+Az}{1+Bz}$, $z \in U$, $-1 \leq B < A \leq 1$, $m, n \in \mathbb{N}$, $\lambda \geq 0$. *Assume that (13) holds. If* $f \in \mathcal{A}$ *and*

$$\psi_\lambda^{m,n}(a, b, \alpha, \beta, \delta; z) \prec \alpha\frac{1+Az}{1+Bz} + \beta\frac{(A-B)z}{(1+Az)(1+Bz)},$$

for $\alpha, \beta \in \mathbb{C}$, $\beta \neq 0$, $-1 \leq B < A \leq 1$, *where* $\psi_\lambda^{m,n}$ *is defined in (14), then*

$$\left(\frac{aDR_\lambda^{m+1,n}f(z) + bDR_\lambda^{m,n}f(z)}{(a+b)z}\right)^\delta \prec \frac{1+Az}{1+Bz}, \quad \delta \in \mathbb{C}, \delta \neq 0,$$

and $\frac{1+Az}{1+Bz}$ *is the best dominant.*

Proof. For $q(z) = \frac{1+Az}{1+Bz}$, $-1 \leq B < A \leq 1$, in Theorem 4 we get the corollary.
$\qquad\square$

Theorem 5. *Let q be convex and univalent in U such that $q(0) = 1$. Assume that*

$$Re\left(\frac{\alpha}{\beta}q(z)\right) > 0, \text{ for } \alpha, \beta \in \mathbb{C}, \ \beta \neq 0. \tag{19}$$

*If $f \in \mathcal{A}$, $\delta, a, b \in \mathbb{C}, \delta \neq 0, a+b \neq 0, \left(\frac{aDR_\lambda^{m+1,n}f(z)+bDR_\lambda^{m,n}f(z)}{(a+b)z}\right)^\delta \in$
$\mathcal{H}[q(0), 1] \cap Q$ and
$\psi_\lambda^{m,n}(a, b, \alpha, \beta, \delta; z)$ is univalent in U, where $\psi_\lambda^{m,n}(a, b, \alpha, \beta, \delta; z)$ is as defined
in (14), then*

$$\alpha q(z) + \frac{\beta z q'(z)}{q(z)} \prec \psi_\lambda^{m,n}(a, b, \alpha, \beta, \delta; z) \tag{20}$$

implies

$$q(z) \prec \left(\frac{aDR_\lambda^{m+1,n}f(z) + bDR_\lambda^{m,n}f(z)}{(a+b)z}\right)^\delta, \quad \delta \in \mathbb{C}, \ \delta \neq 0, \ z \in U, \tag{21}$$

and q is the best subordinant.

Proof. Let the function p be defined by $p(z) := \left(\frac{aDR_\lambda^{m+1,n}f(z)+bDR_\lambda^{m,n}f(z)}{(a+b)z}\right)^\delta$,
$z \in U$, $z \neq 0$, $a, b \in \mathbb{C}, a+b \neq 0, \delta \in \mathbb{C}, \delta \neq 0, \ f \in \mathcal{A}$. The function p is
analytic in U and $p(0) = 1$.

By setting $\nu(w) := \alpha w$ and $\phi(w) := \frac{\beta}{w}$ it can be easily verified that ν is
analytic in $\mathbb{C}$, ϕ is analytic in $\mathbb{C}\backslash\{0\}$ and that $\phi(w) \neq 0$, $w \in \mathbb{C}\backslash\{0\}$.

Since q is convex and univalent function, it follows that $Re\left(\frac{\nu'(q(z))}{\phi(q(z))}\right) =$
$Re\left(\frac{\alpha}{\beta}q(z)\right) > 0$, for $\alpha, \beta \in \mathbb{C}, \beta \neq 0$.

By using (11) we obtain

$$\alpha q(z) + \beta\frac{zq'(z)}{q(z)} \prec \alpha p(z) + \beta\frac{zp'(z)}{p(z)}.$$

From Lemma 2, we have

$$q(z) \prec p(z) = \left(\frac{aDR_\lambda^{m+1,n}f(z) + bDR_\lambda^{m,n}f(z)}{(a+b)z}\right)^\delta, \quad z \in U, \ \delta \in \mathbb{C}, \ \delta \neq 0,$$

and q is the best subordinant. $\qquad\square$

Corollary 5. *Let $q(z) = \frac{1+Az}{1+Bz}$, $-1 \leq B < A \leq 1$, $z \in U$, $m, n \in \mathbb{N}$,
$\lambda \geq 0$. Assume that (19) holds. If $f \in \mathcal{A}$, $\left(\frac{aDR_\lambda^{m+1,n}f(z)+bDR_\lambda^{m,n}f(z)}{(a+b)z}\right)^\delta \in$*

$\mathcal{H}[q(0),1] \cap Q$, $\delta \in \mathbb{C}$, $\delta \neq 0$ and

$$\alpha \frac{1+Az}{1+Bz} + \beta \frac{(A-B)z}{(1+Az)(1+Bz)} \prec \psi_\lambda^{m,n}(a,b,\alpha,\beta,\delta;z),$$

for $\alpha, \beta \in \mathbb{C}$, $\beta \neq 0$, $-1 \leq B < A \leq 1$, where $\psi_\lambda^{m,n}$ is defined in (14), then

$$\frac{1+Az}{1+Bz} \prec \left(\frac{aDR_\lambda^{m+1,n}f(z) + bDR_\lambda^{m,n}f(z)}{(a+b)z} \right)^\delta, \delta \in \mathbb{C}, \ \delta \neq 0, \ a,b \in \mathbb{C}, a+b \neq 0$$

and $\frac{1+Az}{1+Bz}$ is the best subordinant.

Proof. For $q(z) = \frac{1+Az}{1+Bz}$, $-1 \leq B < A \leq 1$, in Theorem 5 we get the corollary. $\qquad \square$

Combining Theorem 4 and Theorem 5, we state the following sandwich theorem.

Theorem 6. *Let q_1 and q_2 be convex and univalent in U such that $q_1(z) \neq 0$ and $q_2(z) \neq 0$, for all $z \in U$. Suppose that q_1 satisfies (13) and q_2 satisfies (19).*

If $f \in \mathcal{A}$, $\left(\frac{aDR_\lambda^{m+1,n}f(z) + bDR_\lambda^{m,n}f(z)}{(a+b)z} \right)^\delta \in \mathcal{H}[q(0),1] \cap Q$, $\delta \in \mathbb{C}, \delta \neq 0$, $a,b \in \mathbb{C}, a+b \neq 0$ and $\psi_\lambda^{m,n}(a,b,\alpha,\beta,\delta;z)$ is as defined in (14) univalent in U, then

$$\alpha q_1(z) + \frac{\beta z q_1'(z)}{q_1(z)} \prec \psi_\lambda^{m,n}(a,b,\alpha,\beta,\delta;z) \prec \alpha q_2(z) + \frac{\beta z q_2'(z)}{q_2(z)},$$

for $\alpha, \beta \in \mathbb{C}$, $\beta \neq 0$, implies

$$q_1(z) \prec \left(\frac{aDR_\lambda^{m+1,n}f(z) + bDR_\lambda^{m,n}f(z)}{(a+b)z} \right)^\delta \prec q_2(z), \quad z \in U, \delta \in \mathbb{C}, \ \delta \neq 0,$$

and q_1 and q_2 are respectively the best subordinant and the best dominant.

For $q_1(z) = \frac{1+A_1 z}{1+B_1 z}$, $q_2(z) = \frac{1+A_2 z}{1+B_2 z}$, where $-1 \leq B_2 < B_1 < A_1 < A_2 \leq 1$, we have the following corollary.

Corollary 6. *Let $m,n \in \mathbb{N}$, $\lambda \geq 0$. Assume that (13) and (19) hold for $q_1(z) = \frac{1+A_1 z}{1+B_1 z}$ and $q_2(z) = \frac{1+A_2 z}{1+B_2 z}$, respectively. If $f \in \mathcal{A}$, $\left(\frac{DR_\lambda^{m+1,n}f(z)}{DR_\lambda^{m,n}f(z)} \right)^\delta \in \mathcal{H}[q(0),1] \cap Q$ and*

$$\alpha \frac{1+A_1 z}{1+B_1 z} + \beta \frac{(A_1 - B_1)z}{(1+A_1 z)(1+B_1 z)} \prec \psi_\lambda^{m,n}(a,b,\alpha,\beta,\delta;z)$$

$$\prec \alpha \frac{1 + A_2 z}{1 + B_2 z} + \beta \frac{(A_2 - B_2) z}{(1 + A_2 z)(1 + B_2 z)},$$

for $\alpha, \beta \in \mathbb{C}$, $\beta \neq 0$, $-1 \leq B_2 \leq B_1 < A_1 \leq A_2 \leq 1$, where $\psi_\lambda^{m,n}$ is defined in (14), then

$$\frac{1 + A_1 z}{1 + B_1 z} \prec \left(\frac{a DR_\lambda^{m+1,n} f(z) + b DR_\lambda^{m,n} f(z)}{(a + b) z} \right)^\delta \prec \frac{1 + A_2 z}{1 + B_2 z},$$

$$z \in U, \ \delta \in \mathbb{C}, \ \delta \neq 0, a, b \in \mathbb{C}, a + b \neq 0$$

hence $\frac{1+A_1 z}{1+B_1 z}$ and $\frac{1+A_2 z}{1+B_2 z}$ are the best subordinant and the best dominant, respectively.

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