An. Şt. Univ. Ovidius Constanţa Vol. 13(1), 2005, 27–54

FUZZY SETS AND NON COMPLETE 1-HYPERGROUPS*

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Abstract

In this paper it has been studied the sequence of membership functions and of join spaces (see [7]), determined by a class of 1–hypergroups which are not complete (see [2]).

Introduction

One knows since 1996, see [6], it is possible, given a fuzzy subset μ of a universe H, to associate with μ, a join space H_μ. In [7] one associated with every hypergroupoid $\langle H;*\rangle$ a fuzzy subset of H. So one can let to correspond to each hypergroupoid $\langle H;*\rangle$ a sequence of fuzzy subsets $\langle H,\mu_k\rangle_{k\in N}$ and a sequence of join spaces ${}^kH=\langle H;*_k\rangle$.

One already studied these sequences for several classes of hypergroups: in [6] one considered a class of canonical hypergroups not having canonical subhypergroups (see [2]), the hypergroups $C(n)$ (see [3]), and complete hypergroups (see for instance [5]). In [8] one considered i.p.s. hypergroups (see [5]), for the order less or equal to 6, in [9], i.p.s. hypergroups of order 7, in [13] one found the sequences determined by complete hypergroups of order less or equal to 6, which are also 1-hypergroups, in [14], the sequences determined by finite hypergroupoids, such that the associated membership functions satisfy an order condition.

In this paper we study the sequences of fuzzy subsets $\langle H,\mu_k\rangle_{k\in N}$ and of join spaces $\left\langle {}^kH\right\rangle_{k\in N}$ determined by a class of 1-hypergroups which are not complete, see [2].

Let us recall two correspondences, fundamental for what follows.

Key Words: Fuzzy sets, Join spaces.
*Work supported by MIUR.

Given a universe H and a fuzzy subset of H, $\langle H, \mu \rangle$, a join space H_μ is associated with μ, with hyperoperation defined by:

$$\forall\ (x,y) \in H^2, \quad x *_\mu y = \{z \mid \mu(x) \wedge \mu(y) \leq \mu(z) \leq \mu(x) \vee \mu(y)\}.$$

Given a hypergroupoid $H^0 = \langle H; * \rangle$, a fuzzy subset $\langle H, \widetilde{\mu} \rangle$ is associated with H^0, as follows: set

$$\forall\ u \in x * y, \quad A(u) = \sum_{u \in x*y} \frac{1}{|x * y|},$$

$$\forall\, u \in H, \qquad q(u) = |\{(x,y) \mid u \in x * y\}|,$$

$$\widetilde{\mu}(u) = A(u)/q(u).$$

We consider the following hypergroup $H = H_n = \{e\} \cup A \cup B$, where $|A| = \alpha$, $|B| = \beta$ with $\alpha, \beta \geq 2$ and $A \cap B = \emptyset$, $e \notin A \cup B$. We set $A = \{a_1, ..., a_\alpha\}$ and $B = \{b_1, ..., b_\beta\}$. The hyperoperation is defined in this way:

$\forall\, a \in A,\ a \circ a = b_1,$

$\forall\, (a_1, a_2) \in A^2$ such that $a_1 \neq a_2,\ a_1 \circ a_2 = B,$

$\forall\, (a, b) \in A \times B,\ a \circ b = b \circ a = e,$

$\forall\, (b, b') \in B^2,\ b \circ b' = A,$

$\forall\, a \in A,\ a \circ e = e \circ a = A,$

$\forall\, b \in B,\ b \circ e = e \circ b = B$ and $e \circ e = e.$

H_n is an 1-hypergroup which is not complete.

§1. Let us suppose $n = |H_6| = 6$, where $H = H_6 = \{e\} \cup A \cup B$, $\alpha = |A| = 2$, $\beta = |B| = 3$, $A \cap B = \emptyset$, $e \notin A \cup B$ with $A = \{a_1, a_2\}$, $B = \{b_1, b_2, b_3\}$. So the structure in H_6 is as follows:

H	e	a_1	a_2	b_1	b_2	b_3
e	e	A	A	B	B	B
a_1		b_1	B	e	e	e
a_2			b_1	e	e	e
b_1				A	A	A
b_2					A	A
b_3						A

therefore we have: $\widetilde{\mu}(e) = 1$; $\widetilde{\mu}(a_1) = \widetilde{\mu}(a_2) = 0,5$; $\widetilde{\mu}(b_1) = 0,467$; $\widetilde{\mu}(b_2) = \widetilde{\mu}(b_3) = 0,333$, whence the associated join space ${}^1H$ is

${}^1H$	e	a_1	a_2	b_1	b_2	b_3
e	e	$A \cup \{e\}$	$A \cup \{e\}$	$H \setminus \{b_1, b_2\}$	H	H
a_1		A	A	$A \cup \{b_1\}$	$H \setminus \{e\}$	$H \setminus \{e\}$
a_2			A	$A \cup \{b_1\}$	$H \setminus \{e\}$	$H \setminus \{e\}$
b_1				b_1	B	B
b_2					b_2, b_3	b_2, b_3
b_3						b_2, b_3

Now we obtain: ${}^1\widetilde{\mu}(e) = 0,32$; ${}^1\widetilde{\mu}(a_i) = 0,286$; ${}^1\widetilde{\mu}(b_1) = 0,280 \approx 0,2797$; ${}^1\widetilde{\mu}(b_2) = {}^1\widetilde{\mu}(b_3) = 0,280$ so $\widetilde{\mu}_1(b_1) < \widetilde{\mu}_1(b_j) < \widetilde{\mu}_1(a_i) < \widetilde{\mu}_1(e)$. Therefore the associated join space ${}^2H$ is the following

${}^2H$	e	a_1	a_2	b_1	b_2	b_3
e	e	$A \cup \{e\}$	$A \cup \{e\}$	H	$H \setminus \{b_1\}$	$H \setminus \{b_1\}$
a_1		A	A	$H \setminus \{e\}$	$H \setminus \{e, b_1\}$	$H \setminus \{e, b_1\}$
a_2			A	$H \setminus \{e\}$	$H \setminus \{e, b_1\}$	$H \setminus \{e, b_1\}$
b_1				b_1	B	B
b_2					b_2, b_3	b_2, b_3
b_3						b_2, b_3

${}^2\widetilde{\mu}(e) = {}^2\widetilde{\mu}(b_1) = 0,315$; ${}^2\widetilde{\mu}(a_1) = {}^2\widetilde{\mu}(a_2) = {}^2\widetilde{\mu}(b_2) = {}^2\widetilde{\mu}(b_3) = 0,279$. It follows that the associated join space ${}^3H$ is:

${}^3H$	e	a_1	a_2	b_1	b_2	b_3
e	e, b_1	H	H	e, b_1	H	H
a_1		$H \setminus \{e, b_1\}$	$H \setminus \{e, b_1\}$	H	$H \setminus \{e, b_1\}$	$H \setminus \{e, b_1\}$
a_2			$H \setminus \{e, b_1\}$	H	$H \setminus \{e, b_1\}$	$H \setminus \{e, b_1\}$
b_1				e, b_1	H	H
b_2					$H \setminus \{e, b_1\}$	$H \setminus \{e, b_1\}$
b_3						$H \setminus \{e, b_1\}$

Now we obtain: ${}^3\widetilde{\mu}(e) = {}^3\widetilde{\mu}(b_1) = 0,246$; ${}^3\widetilde{\mu}(a_i) = {}^3\widetilde{\mu}(b_2) = {}^3\widetilde{\mu}(b_3) = 0,208$, therefore $\forall\, r \geq 3$, we have ${}^rH = {}^3H$.

§2. More generally, set $H = H_n = \{e\} \cup A \cup B$, $|A| = \alpha$, $|B| = \beta$, $\alpha, \beta \geq 2$, $A \cap B = \emptyset$, $e \notin A \cup B$, where $A = \{a_1, ..., a_\alpha\}$, $B = \{b_1, ..., b_\beta\}$.

First we consider $\alpha = \beta \geq 2$ and therefore the hypergroup H is the following one:

H	e	a_1	a_2	$\cdots$	a_α	b_1	b_2	$\cdots$	b_α
e	e	A	A	$\cdots$	A	B	B	$\cdots$	B
a_1		b_1	B	$\cdots$	B	e	e	$\cdots$	e
a_2			b_1	$\cdots$	B	e	e	$\cdots$	e
$\vdots$				$\ddots$					
a_α					b_1	e	e	$\cdots$	e
b_1						A	A	$\cdots$	A
b_2							A	$\cdots$	A
$\vdots$								$\ddots$	
b_α									A

whence $\widetilde{\mu}(e) = 1$; $A(a_i) = \dfrac{2\alpha + \beta + 2(1 + 2 + \cdots + \beta - 1)}{\alpha} = \dfrac{2\alpha + \beta^2}{\alpha}$; $q(a_i) = 2\alpha + \beta^2$, so $\widetilde{\mu}(a_i) = \dfrac{1}{\alpha}\cdot$ In the same way we find for $i \neq 1$, $\widetilde{\mu}(b_i) = \dfrac{1}{\alpha}\cdot$ Moreover, we find $A(b_1) = 2\alpha + 1$, $q(b_1) = \alpha^2 + 2\alpha$, whence $\widetilde{\mu}(b_1) = \dfrac{2\alpha + 1}{\alpha^2 + 2\alpha}$, therefore we have $\forall\, i \in I(\alpha) = \{1, ..., \alpha\}$ and $\forall\, j \in I(\alpha) - \{1\}$,

$$\widetilde{\mu}(e) > \widetilde{\mu}(b_1) > \widetilde{\mu}(a_i) = \widetilde{\mu}(b_j).$$

Set $B_1 = B \setminus \{b_1\}$, $H^* = H \setminus \{e\}$.
The associated join space ${}^1H$ is

${}^1H$	e	b_1	a_1	a_2	$\cdots$	a_α	b_2	$\cdots$	b_α
e	e	e, b_1	H	H	$\cdots$	H	H	$\cdots$	H
b_1		b_1	H^*	H^*	$\cdots$	H^*	H^*	$\cdots$	H^*
a_1			$A \cup B_1$	$A \cup B_1$	$\cdots$	$A \cup B_1$	$A \cup B_1$	$\cdots$	$A \cup B_1$
a_2				$A \cup B_1$	$\cdots$	$A \cup B_1$	$A \cup B_1$	$\cdots$	$A \cup B_1$
$\vdots$					$\ddots$				
a_α						$A \cup B_1$	$A \cup B_1$	$\cdots$	$A \cup B_1$
b_2							$A \cup B_1$	$\cdots$	$A \cup B_1$
$\vdots$								$\ddots$	
b_α									$A \cup B_1$

Therefore we have: $A(e) = \dfrac{4n - 4}{n}$; $q(e) = 2n - 1$, whence ${}^1\widetilde{\mu}(e) = \dfrac{4(n-1)}{n(2n-1)}\cdot$
$A(b_1) = 2 + \dfrac{2\alpha + 2(\alpha - 1)}{n} + \dfrac{2\alpha + 2(\alpha - 1)}{n - 1} = 2\,\dfrac{3n^2 - 6n + 2}{n(n-1)}$; $q(b_1) = 4n - 5$,

so ${}^1\widetilde{\mu}(b_1) = \dfrac{2(3n^2-6n+2)}{n(4n^2-9n+5)}\cdot$ We have clearly:

$$ {}^1\widetilde{\mu}(e) > {}^1\widetilde{\mu}(b_1) \Longleftrightarrow \frac{2n-2}{2n-1} > \frac{3n^2-6n+2}{4n^2-9n+5} \Longleftrightarrow 8n^3-18n^2+10n-8n^2+18n-10 > $$
$$ > 6n^3-12n^2+4n-3n^2+6n-2 \Longleftrightarrow G = 2n^3-11n^2+18n-8>0. $$

We obtain that for $n \geq 5$, $G = n^2(2n-11)+2(9n-4) > 0$. On the other side $A(a_i) = \dfrac{n^3+n^2-8n+4}{n(n-1)}$; $q(a_i) = (n-2)(n+2)$, whence ${}^1\widetilde{\mu}(a_i) = \dfrac{n^2+3n-2}{n(n-1)(n+2)}\cdot$ We have also ${}^1\widetilde{\mu}(b_1) > {}^1\widetilde{\mu}(a_i) \Longleftrightarrow \dfrac{2(3n^2-6n+2)}{4n-5} > \dfrac{n^2+3n-2}{n+2} \Longleftrightarrow$ $F = 2n^3-7n^2+3n-2 > 0$, and since $F = n^2(2n-7) + (3n-2)$, we have finally, for every $n \geq 5$, $\forall\, j \in I(\alpha) \setminus \{1\}$ and $\forall\, i \in I(\alpha)$, ${}^1\widetilde{\mu}(b_1) > {}^1\widetilde{\mu}(a_i) = {}^1\widetilde{\mu}(b_j)$. Then

$$ \forall\, r \geq 1,\ {}^rH = {}^1H. $$

§3. Set now $\alpha > \beta \geq 2$, then we have $\widetilde{\mu}(e) = 1$, $\widetilde{\mu}(a_i) = \dfrac{1}{\alpha}$, $\widetilde{\mu}(b_j) = \dfrac{1}{\beta}$, for $j \in I(\beta) - \{1\}$, so $\widetilde{\mu}(a_i) < \widetilde{\mu}(b_j) < \widetilde{\mu}(e)$. On the other side $\widetilde{\mu}(b_1) = \dfrac{\alpha^2+\alpha\beta+2\beta-\alpha}{\beta(\alpha^2+2\beta)}$ from which $\widetilde{\mu}(b_1) > \widetilde{\mu}(b_j) \Longleftrightarrow \dfrac{\alpha^2+\alpha\beta+2\beta-\alpha}{\beta(\alpha^2+2\beta)} > \dfrac{1}{\beta}$, true. Hence $\widetilde{\mu}(a_i) < \widetilde{\mu}(b_j) < \widetilde{\mu}(b_1) < \widetilde{\mu}(e)$. By consequence, the associated join space is

${}^1H$	e	b_1	b_2	$\cdots$	b_β	a_1	$\cdots$	a_α
e	e	e, b_1	$H \setminus A$	$\cdots$	$H \setminus A$	H	$\cdots$	H
b_1		b_1	B	$\cdots$	B	H^*	$\cdots$	H^*
b_2			B_1	$\cdots$	B_1	$H \setminus \{e, b_1\}$	$\cdots$	$H \setminus \{e, b_1\}$
$\vdots$				$\ddots$				
b_β					B_1	$H \setminus \{e, b_1\}$	$\cdots$	$H \setminus \{e, b_1\}$
a_1						A	$\cdots$	A
$\vdots$							$\ddots$	
a_α								A

where $B_1 = B \setminus \{b_1\}$. Then $A(e) = 2 + \dfrac{2(\beta-1)}{n-\alpha} + \dfrac{2\alpha}{n} = \dfrac{6n\beta+2n-2\beta^2-4\beta-2}{n(\beta+1)}$, and $q(e) = 2n-1$, it follows $\widetilde{\mu}_1(e) = \frac{6n\beta + 2n - 2\beta^2 - 4\beta - 2}{n(2n-1)(\beta+1)}\cdot$ We have also

$$\begin{aligned} A(b_1) &= 2 + \frac{2(\beta-1)}{n-\alpha} + \frac{2\alpha}{n} + \frac{2(\beta-1)}{\beta} + \frac{2\alpha}{n-1} = \\ &= 2\,\frac{5n^2\beta^2 - 8n\beta^2 - 2n\beta^3 + \beta^3 + 2n^2\beta - 3n\beta + 2\beta^2 + \beta - n^2 + n}{n\beta(n-1)(\beta+1)}; \\ q(b_1) &= 1 + 2\beta + 4\alpha + 2\beta - 2 = 4n - 5. \end{aligned}$$

By consequence, $\widetilde{\mu}(e) > \widetilde{\mu}(b_1)$ if and only if $\frac{6n\beta + 2n - 2\beta^2 - 4\beta - 2}{n(2n-1)(\beta+1)} >$ $> 2\,\frac{5n^2\beta^2 - 8n\beta^2 - 2n\beta^3 + \beta^3 + 2n^2\beta - 3n\beta + 2\beta^2 + \beta - n^2 + n}{n\beta(n-1)(\beta+1)(4n-5)}$.

Set $P = \beta(n-1)(4n-5)(3n\beta + n - \beta^2 - 2\beta - 1)$, $Q = (2n-1)(5n^2\beta^2 - 8n\beta^2 - 2n\beta^3 + \beta^3 + 2n^2\beta - 3n\beta + 2\beta^2 + \beta - n^2 + n)$. We have

$$\begin{aligned} P > Q \Longleftrightarrow\ & P = n^3(12\beta^2 + 4\beta) + n^2(-4\beta^3 - 35\beta^2 - 13\beta) + \\ & + n(9\beta^3 + 33\beta^2 + 14\beta) - 5\beta^3 - 10\beta^2 - 5\beta > \\ & > Q = n^3(10\beta + 4\beta - 2) + n^2(-4\beta^3 - 21\beta^2 - 8\beta + 3) + \\ & + n(4\beta^2 + 12\beta^2 + 5\beta - 1) - \beta^3 - 2\beta^2 - \beta. \end{aligned}$$

This is true if and only if $P - Q = n^3(2\beta^2 + 2) + n^2(-14\beta^2 - 5\beta - 3) + n(5\beta^3 + 21\beta^2 + 9\beta + 1) - 4\beta^3 - 8\beta^2 - 4\beta > 0$. We have clearly $n > 2\beta + 1$.

Let be $P - Q = E_n$, so we have

$$\begin{aligned} E_n'' &= 6n(2\beta^2 + 2) - 28\beta^2 - 10\beta - 6 > 6(2\beta + 1)(2\beta^2 + 2) - \\ & - 28\beta^2 - 10\beta - 6 = 24\beta^3 - 16\beta^2 + 14\beta + 6 > 0, \end{aligned}$$

whence $\forall\, \beta \geq 2$. E_n' is a strictly increasing function, that is $\forall\, n : n > 2\beta + 1$

$$E_n' > E_{2\beta+1}'.$$

We obtain for $\forall\, \beta : \beta \geq 2$

$$\begin{aligned} E_n' > & 3(2\beta + 1)^2(2\beta^2 + 2) + 2(2\beta + 1)(-14\beta^2 - 5\beta - 3) + 5\beta^3 + \\ & + 21\beta^2 + 9\beta + 1 = 24\beta^4 - 27\beta^3 + 3\beta^2 + 11\beta + 1 > 0, \end{aligned}$$

therefore E_n is a strictly increasing function. So $\forall\, n : n > 2\beta + 1$ and we have

$$E_n > E_{2\beta+1}.$$

Finally, we find $\forall\,\beta \geq 2$

$$\begin{aligned} P-Q\ >&(8\beta^3{+}12\beta^3{+}6\beta+1)(2\beta^2{+}2)(\,4\beta^2+4\beta+1)(-14\beta^2-5\beta-3)+\\ &+(2\beta+1)(5\beta^3+21\beta^2+9\beta+1)-4\beta^3-8\beta^2-4\beta, \end{aligned}$$

whence $P-Q > 16\beta^5-22\beta^4-5\beta^3+11\beta^2+2\beta > 0$, then $\widetilde{\mu}_1(e) > \widetilde{\mu}_1(b_1)$. On the other side, we have

$$\begin{aligned} A(b_j)\ &= \frac{2(\beta-1)}{n-\alpha}+\frac{2\alpha}{n}+\frac{2(\beta-1)}{\beta}+\frac{2\alpha}{n-1}+\beta-1+\frac{2\alpha(\beta-1)}{n-2}=\\ &= \frac{\begin{array}{c}2n(n{-}1)(n{-}2)(\beta{-}1)(2\beta{+}1){+}2\beta(\beta{+}1)(n{-}2)(n{-}\beta{-}1)(2n{-}1){+}\\ {+}n(n{-}1)\beta(\beta{+}1)(\beta{-}1)(3n{-}2\beta-4)\end{array}}{n(n{-}1)(n{-}2)\beta(\beta{+}1)};\\ q(b_j)\ &= 4(\beta{-}1){+}4\alpha{+}(\beta{-}1)^2{+}2\alpha(\beta{-}1) = 2n{-}\beta^2{+}2n\beta{-}2\beta{-}5. \end{aligned}$$

Therefore we have $\widetilde{\mu}(b_1) > \widetilde{\mu}(b_j)$ if and only if

$$\frac{10n^2\beta^2{-}16n\beta^2{-}4n\beta^3{+}2\beta^3{+}4n^2\beta{-}6n\beta{+}4\beta^2{+}2\beta{-}2n^2{+}2n}{n\beta(n{-}1)(\beta{+}1)(4n{-}5)} >$$

$$> \frac{\begin{array}{c}(2n^3{-}6n^2{+}4n)(2\beta^2{-}\beta{-}1){+}(2n\beta^2{+}2n\beta{-}4\beta^2{-}4\beta)(2n^2{-}3n{-}2n\beta{+}\beta{+}1){+}\\ {+}(n^2\beta^2{+}n^2\beta{-}n\beta^2{-}n\beta)(3n\beta{-}2\beta^2{-}2\beta{-}3n{+}4)\end{array}}{n(n{-}1)(n{-}2)(\beta{+}1)(2n{-}\beta^2{+}2n\beta{-}2\beta{-}5)\beta}$$

this is true if and only if, setting
$P = (2n^2-9n-n\beta^2+2\beta^2+2n^2\beta-6n\beta+4\beta+10)(10n^2\beta^2-16n\beta^2-4n\beta^3+2\beta^3+4n^2\beta-6n\beta+4\beta^2+2\beta-2n^2+2n)$ and
$Q = (4n-5)(8n^3\beta^2-n^3\beta-2n^3-28n^2\beta^2+6n^2+30n\beta^2+6n\beta-4n-11n^2\beta^3+14n\beta^3-4\beta^3-8\beta^2-4\beta-n^2\beta+3n^3\beta^3-2n^2\beta^4+2n\beta^4)$,
we have $P > Q$.
One finds: $P = n^4(20\beta^3+28\beta^2+4\beta-4)+n^3(-18\beta^4-104\beta^3-156\beta^2-32\beta+22)+n^2(4\beta^5+64\beta^4+198\beta^3+302\beta^2+78\beta-38)+n(-10\beta^5-64\beta^4-160\beta^3-228\beta^2-70\beta+20)+4\beta^5+16\beta^4+48\beta^3+48\beta^2+20\beta$, and
$Q = n^4(12\beta^3+32\beta^2-4\beta-8)+n^3(-8\beta^4-59\beta^3-152\beta^2+\beta+34)+n^2(18\beta^4+111\beta^3+260\beta^2+29\beta-46)+n(-86\beta^3-10\beta^4-182\beta^2-46\beta+20)+20\beta^3+40\beta^2+20\beta$. Then one obtains $\mu(b_1) > \mu(b_j)$ if and only if

$$\begin{aligned} P-Q\ =\ &n^4(8\beta^3-4\beta^2+8\beta+4)+n^3(-10\beta^4-45\beta^3-4\beta^2-33\beta-12)+\\ &+n^2(4\beta^5+46\beta^4+87\beta^3+42\beta^2+49\beta+8)+n(-10\beta^5-54\beta^4-\\ &-74\beta^3-46\beta^2-24\beta)+4\beta^5+16\beta^4+20\beta^3+8\beta^2>0. \end{aligned}$$

Set now $E_n = P - Q$, $\forall\, \beta \geq 2$.

$$\begin{aligned} E_n''' &= 24n(8\beta^3 - 4\beta^2 + 8\beta + 4) - 60\beta^4 - 270\beta^3 - 24\beta^2 - 198\beta - 72 > \\ &> 24(2\beta+1)(8\beta^3-4\beta^2+8\beta+4)-60\beta^4-270\beta^3-24\beta^2-198\beta-72 = \\ &= 324\beta^4 - 174\beta^3 + 264\beta^2 + 186\beta + 24 > 0, \end{aligned}$$

whence $E_n'' > E_{2\beta+1}'' = 272\beta^5 - 316\beta^4 + 144\beta^3 + 192\beta^2 + 44\beta - 8 > 0$, therefore E_n' is a strictly increasing function. It follows

$$E_n' > E_{2\beta+1}' = 152\beta^6 - 222\beta^5 + 24\beta^4 + 137\beta^3 + 50\beta^2 - 9\beta - 4 > 0,$$

so $E_n > E_{2\beta+1} = 64\beta^7 - 108\beta^6 - 18\beta^5 + 72\beta^4 + 40\beta^3 - 6\beta^2 - 8\beta > 0$, therefore

$$\widetilde{\mu}_1(b_1) > \widetilde{\mu}_1(b_j).$$

On the other side, we have

$$\begin{aligned} A(a_i) &= \frac{2\alpha}{n} + \frac{2\alpha}{n-1} + \frac{2\alpha(\beta-1)}{n-2} + \alpha = \\ &= \frac{(n-\beta-1)(n^3 - n^2 - 6n + 2n^2\beta - 2n\beta + 4)}{n(n-1)(n-2)} \end{aligned}$$

$$q(a_i) = 4\alpha + \alpha^2 + 2\alpha(\beta - 1) = (n - \beta - 1)(n + \beta + 1),$$

therefore

$$\widetilde{\mu}(b_j) > \widetilde{\mu}(a_i),$$

if and only if

$$\frac{\begin{array}{c} 8n^3\beta^2 - n^3\beta - 2n^3 - 28n^2\beta^2 + 6n^2 + 30n\beta^2 + 6n\beta - 4n - 11n^2\beta^3 + \\ +14n\beta^3 - 4\beta^3 - 8\beta^2 - 4\beta - n^2\beta + 3n^3\beta^3 - 2n^2\beta^4 + 2n\beta^4 \end{array}}{n(n-1)(n-2)\beta(\beta+1)(2n - \beta^2 + 2n\beta - 2\beta - 5)} >$$

$$> \frac{n^3 - n^2 - 6n + 2n^2\beta - 2n\beta + 4}{n(n-1)(n-2)(n+\beta+1)}.$$

This is true if and only if $P = (n + \beta + 1)(8n^3\beta^2 - n^3\beta - 2n^3 - 28n^2\beta^2 + 6n^2 + 30n\beta^2 + 6n\beta - 4n - 11n^2\beta^3 + 14n\beta^3 - 4\beta^3 - 8\beta^2 - 4\beta - n^2\beta + 3n^3\beta^3 - 2n^2\beta^4 + 2n\beta^4) > Q = (2n\beta^2 + 2n\beta - \beta^4 - \beta^3 + 2n\beta^3 + 2n\beta^2 - 2\beta^3 - 2\beta^2 - 5\beta^2 - 5\beta)(n^3 - n^2 - 6n + 2n^2\beta - 2n\beta + 4)$. One obtains
$P = n^4(3\beta^3 + 8\beta^2 - \beta - 2) + n^3(\beta^4 - 21\beta^2 - 4\beta + 4) + n^2(-2\beta^5 - 11\beta^4 - 25\beta^3 + \beta^2 + 11\beta + 2) + n(2\beta^5 + 16\beta^4 + 40\beta^3 + 28\beta^2 - 2\beta - 4) - 4\beta^4 - 12\beta^3 - 12\beta^2 - 4\beta$ and
$Q = n^4(2\beta^3 + 4\beta^2 + 2\beta) + n^3(3\beta^4 + 3\beta^3 - 7\beta^2 - 7\beta) + n^2(-2\beta^5 - 9\beta^4 - 31\beta^3 -$

$31\beta^2-7\beta)+n(2\beta^5+12\beta^4+40\beta^3+68\beta^2+38\beta)-4\beta^4-12\beta^3-28\beta^2-20\beta$. So $P-Q=n^4(\beta^3+4\beta^2-3\beta-2)+n^3(-2\beta^4-3\beta^3-14\beta^2+3\beta+4)+n^2(-2\beta^4+6\beta^3+32\beta^2+18\beta+2)+n(4\beta^4-40\beta^2-40\beta-4)+16\beta^2+16\beta$.

Set $E_n=P-Q$, then we have

$$\begin{aligned}E_n''' &= 24n(\beta^3+4\beta^2-3\beta-2)-12\beta^4-18\beta^3-84\beta^2+18\beta+24\geq\\ &\geq(48\beta+24)(\beta^3+4\beta^2-3\beta-2)-12\beta^4-18\beta^3-84\beta^2+18\beta+24=\\ &=48\beta^4+216\beta^3-48\beta^2-168\beta-48-12\beta^4-18\beta^3-84\beta^2+18\beta+24=\\ &=36\beta^4+198\beta^3-132\beta^2-150\beta-24>0,\end{aligned}$$

whence E_n'' is a strictly increasing function. It follows

$$\begin{aligned}&E_n''>E_{2\beta+1}''=12(4\beta^2+4\beta+1)(\beta^3+4\beta^2-3\beta-2)+6(2\beta+1)\cdot\\ &\cdot(-2\beta^4-3\beta^3-14\beta^2+3\beta+4)-4\beta^4+12\beta^3+64\beta^2+36\beta+4=\\ &=48\beta^5+192\beta^4-144\beta^3-96\beta^2+48\beta^4+192\beta^3-144\beta^2-96\beta+\\ &+12\beta^3+48\beta^2-36\beta-24-24\beta^5-36\beta^4-168\beta^3+36\beta^2+48\beta-\\ &-12\beta^4-18\beta^3-84\beta^2+18\beta+24-4\beta^4+12\beta^3+64\beta^2+36\beta+4=\\ &=24\beta^5+188\beta^4-114\beta^3-176\beta^2-30\beta+4>0,\end{aligned}$$

and, by consequence, E_n' is a strictly increasing function. So we have

$$\begin{aligned}&E_n'>E_{2\beta+1}'=4(8\beta^3+12\beta^2+6\beta+1)(\beta^3+4\beta^2-3\beta-2)+\\ &+3(4\beta^2+4\beta+1)(-2\beta^4-3\beta^3-14\beta^2+3\beta+4)+\\ &+2(2\beta+1)(-2\beta^4+6\beta^3+32\beta^2+18\beta+2)+4\beta^4-40\beta^2-40\beta-4=\\ &=8\beta^6+108\beta^5-66\beta^4-109\beta^3-14\beta^2+\beta+4>0,\end{aligned}$$

therefore E_n is a strictly increasing function and, by consequence,

$$\begin{aligned}&E_n>E_{2\beta+1}=(16\beta^4+32\beta^3+24\beta^2+8\beta+1)(\beta^3+4\beta^2-3\beta-2)+\\ &+(8\beta^3+12\beta^2+6\beta+1)(-2\beta^4-3\beta^3-14\beta^2+3\beta+4)+\\ &+(4\beta^2+4\beta+1)(-2\beta^4+6\beta^3+32\beta^2+18\beta+2)+\\ &+(2\beta+1)(4\beta^4-40\beta^2-40\beta-4)+16\beta^2+16\beta,\end{aligned}$$

whence $E_n=P-Q>E_{2\beta+1}=40\beta^6-32\beta^5-34\beta^4+4\beta^3-8\beta^2+2\beta>0$. Finally,

$$\widetilde{\mu}_1(b_j)>\widetilde{\mu}_1(a_i).$$

We conclude that

$$\widetilde{\mu}_1(a_i)<\widetilde{\mu}_1(b_j)<\widetilde{\mu}_1(b_1)<\widetilde{\mu}_1(e).$$

It follows $\forall r : r \geq 1$

$$^rH = {}^1H.$$

§4. Let $\beta > \alpha \geq 2 \Longrightarrow n > 2\alpha + 1$. We find in the same way as in §3:

$$\widetilde{\mu}(e) = 1, \quad \widetilde{\mu}(a_i) = \frac{1}{\alpha}, \quad \text{for } j \in \{2, 3, ..., \beta\},$$

$$\widetilde{\mu}(b_j) = \frac{1}{\beta}, \quad \widetilde{\mu}(b_1) = \frac{\alpha^2 + \alpha\beta + 2\beta - \alpha}{\beta(\alpha^2 + 2\beta)}$$

4.1. Let us consider now the case $\alpha = 2$, $\beta = n - 3$, whence we have:

$$\widetilde{\mu}(b_1) < \widetilde{\mu}(a_i) \text{ iff } \frac{4\beta + 2}{\beta(2\beta + 4)} < \frac{1}{2} \text{ iff } \beta^2 - 2\beta - 2 > 0 \text{ iff } \beta \geq 3.$$

Moreover, for $j \in I(\beta)$,

$$\widetilde{\mu}(b_j) < \widetilde{\mu}(b_1) \text{ iff } \frac{1}{\beta} < \frac{4\beta + 2}{\beta(2\beta + 4)} \text{ iff } 2\beta - 2 > 0, \text{ iff } \forall \beta \geq 3.$$

So, we have:

$$\widetilde{\mu}(b_j) < \widetilde{\mu}(b_1) < \widetilde{\mu}(a_i) < \widetilde{\mu}(e).$$

The associated join space is the following

^1H	b_2	$\cdots$	b_β	b_1	a_1	a_2	e
b_2	B_1	$\cdots$	B_1	B	H^*	H^*	H
$\vdots$		$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
b_β			B_1	B	H^*	H^*	H
b_1				b_1	$A \cup \{b_1\}$	$A \cup \{b_1\}$	$H \setminus B_1$
a_1					A	A	$A \cup \{e\}$
a_2						A	$A \cup \{e\}$
e							e

We obtain

$$A(e) = 1 + \frac{4}{3} + \frac{2}{4} + \frac{2(\beta-1)}{n} = \frac{29n-48}{6n}; \; q(e){=}2n{-}1; \; \widetilde{\mu}_1(e){=}\frac{29n-48}{6n(2n-1)}$$

$$A(a_i){=}2{+}\frac{8}{3}{+}\frac{2}{4}{+}\frac{2(\beta-1)}{n}{+}\frac{4(\beta-1)}{n-1}{=}\frac{67n^2-187n+48}{6n(n-1)}; \; q(a_i){=}6n{-}10,$$

whence

$$\widetilde{\mu}_1(a_i) = \frac{67n^2 - 187n + 48}{6n(n-1)(6n-10)}.$$

Moreover

$$A(b_1) = 1 + \frac{4}{3} + \frac{1}{2} + \frac{4(\beta-1)}{n-1} + \frac{2(\beta-1)}{n} + \frac{2(\beta-1)}{\beta} = \frac{65n^3-392n^2+615n-144}{6n(n-1)(n-3)}$$

$$q(b_1) = 8n - 25,$$

whence

$$\widetilde{\mu}_1(b_1) = \frac{65n^3 - 392n^2 + 615n - 144}{6n(n-1)(n-3)(8n-25)}.$$

Moreover, for $j \in I(\beta) - \{1\}$, we have

$$A(b_j) = n - 4 + \frac{2(n-4)}{n-3} + \frac{4(n-4)}{n-1} + \frac{2(n-4)}{n} = (n-4)\,\frac{n^3+4n^2-19n+6}{n(n-1)(n-3)},$$

$$q(b_j) = (n-4)(n+4),$$

whence

$$\widetilde{\mu}_1(b_j) = \frac{n^3 + 4n^2 - 19n + 6}{n(n-1)(n-3)(n+4)}.$$

For $\beta = 3$, so that $n = 6$, see §1.

For $\beta > 3$ we have

$$\widetilde{\mu}_1(e) > \widetilde{\mu}_1(a_i)$$

if and only if

$$\frac{29n - 48}{6n(2n-1)} > \frac{67n^2 - 187n + 48}{6n(n-1)(6n-10)}$$

which is equivalent with $40n^3 - 311n^2 + 775n - 432 > 0$, true for $n \geq 7$. So

$$\widetilde{\mu}_1(e) > \widetilde{\mu}_1(a_i).$$

Moreover

$$\widetilde{\mu}_1(b_1) > \widetilde{\mu}_1(b_j)$$

if and only if $\dfrac{65n^3 - 392n^2 + 615n - 144}{6n(n-1)(n-3)(8n-25)} > \dfrac{n^3 + 4n^2 - 19n + 6}{n(n-1)(n-3)(n+4)}$ which is equivalent with $17n^4 - 174n^3 + 559n^2 - 822n + 324 > 0$. Clearly, the inequivalence is true for any $n \geq 7$. So

$$\widetilde{\mu}_1(b_1) > \widetilde{\mu}_1(b_j).$$

Finally,

$$\widetilde{\mu}_1(a_i) > \widetilde{\mu}_1(b_1)$$

if and only if $\frac{67n^2-187n+48}{6n(n-1)(6n-10)} > \frac{65n^3-392n^2+615n-144}{6n(n-1)(n-3)(8n-25)} \Longleftrightarrow$ $146n^4-1777n^3+6962n^2-9363n+2160>0$.

In conclusion, for $\alpha=2$ and $\beta>3$ we have

$$\widetilde{\mu}_1(e)>\widetilde{\mu}_1(a_i)>\widetilde{\mu}_1(b_1)>\widetilde{\mu}_1(b_j)$$

so that

$$\forall\, r\geq 1,\quad {}^rH={}^1H.$$

For $\alpha=2$ and $\beta=3$ we have

$$\forall\, r\geq 1,\ {}^rH={}^3H.$$

Let us suppose now $\beta>\alpha\geq 3$. One can see that

$$1=\widetilde{\mu}(e)>\widetilde{\mu}(b_1)>\widetilde{\mu}(b_j)\ \text{ and }\ \widetilde{\mu}(a_i)>\widetilde{\mu}(b_j).$$

Moreover,

$$\widetilde{\mu}(b_1)>\widetilde{\mu}(a_i)\Longleftrightarrow\frac{\alpha^2+\alpha\beta+2\beta-\alpha}{\beta(\alpha^2+2\beta)}\geq\frac{1}{\alpha}\Longleftrightarrow\alpha^2(\alpha-1)\geq 2\beta(\beta-\alpha).$$

By consequence, we study two cases: when $\widetilde{\mu}(b_1)>\widetilde{\mu}(a_i)$ and when $\widetilde{\mu}(a_i)>\widetilde{\mu}(b_1)$.

4.2. In the case $\widetilde{\mu}(e)>\widetilde{\mu}(b_1)>\widetilde{\mu}(a_i)>\widetilde{\mu}(b_j)$ the associated join space ${}^1H$ is

${}^1H$	e	b_1	a_1	$\cdots$	a_α	b_2	$\cdots$	b_β
e	e	e,b_1	$H\setminus B_1$	$\cdots$	$H\setminus B_1$	H	$\cdots$	H
b_1		b_1	$H^*\setminus B_1$	$\cdots$	$H^*\setminus B_1$	H^*	$\cdots$	H^*
a_1			A	$\cdots$	A	$H^*\setminus\{b_1\}$	$\cdots$	$H^*\setminus\{b_1\}$
$\vdots$				$\ddots$	$\vdots$	$\vdots$		$\vdots$
a_α					A	$H^*\setminus\{b_1\}$	$\cdots$	$H^*\setminus\{b_1\}$
b_2						B_1	$\cdots$	B_1
$\vdots$							$\ddots$	$\vdots$
b_β								B_1

We have

$$A(e)=2+\frac{2\alpha}{\alpha+2}+\frac{2(n-\alpha-2)}{n}=\frac{6n\alpha+8n-2\alpha^2-8\alpha-8}{n(\alpha+2)};\quad q(e)=2n-1,$$

whence

$$\widetilde{\mu}_1(e)=\frac{6n\alpha+8n-2\alpha^2-8\alpha-8}{n(2n-1)(\alpha+2)}\,.$$

Moreover

$$A(b_1) = 2 + \frac{2\alpha}{\alpha+2} + \frac{2\alpha}{\alpha+1} + \frac{2(n-\alpha-2)}{n} + \frac{2(n-\alpha-2)}{n-1} =$$
$$= \frac{10n^2\alpha^2 + 24n^2\alpha + 12n^2 - 4n\alpha^3 - 28n\alpha^2 - 50n\alpha - 24n + 2\alpha^3 + 10\alpha^2 + 16\alpha + 8}{n(n-1)(\alpha+1)(\alpha+2)},$$

$q(b_1) = 4n - 5$, so

$$\widetilde{\mu}_1(b_1) = \frac{10n^2\alpha^2 + 24n^2\alpha + 12n^2 - 4n\alpha^3 - 28n\alpha^2 - 50n\alpha - 24n + 2\alpha^3 + 10\alpha^2 + 16\alpha + 8}{n(n-1)(\alpha+1)(\alpha+2)(4n-5)}.$$

We have also

$$A(a_i) = \frac{2\alpha(n-\alpha-2)}{n-2} + \frac{2(n-\alpha-2)}{n-1} + \frac{2(n-\alpha-2)}{n} + \frac{2\alpha}{\alpha+2} + \frac{2\alpha}{\alpha+1} + \alpha =$$
$$= \frac{\begin{array}{c}3n^3\alpha^3 + 17n^3\alpha^2 + 24n^3\alpha + 8n^3 - 2n^2\alpha^4 - 19n^2\alpha^3 - 73n^2\alpha^2 - 98n^2\alpha - 36n^2 + \\ +2n\alpha^4 + 22n\alpha^3 + 84n\alpha^2 + 116n\alpha + 48n - 4\alpha^3 - 20\alpha^2 - 32\alpha - 16\end{array}}{n(n-1)(n-2)(\alpha+1)(\alpha+2)},$$

$$q(a_i) = \alpha^2 + 4\alpha + 4(n-\alpha-2) + 2\alpha(n-\alpha-2) = 2n\alpha + 4n - \alpha^2 - 4\alpha - 8.$$

Then

$$\begin{aligned} A(b_j) &= \frac{2(\beta-1)}{n} + \frac{2(\beta-1)}{n-1} + \frac{2\alpha(\beta-1)}{n-2} + \beta - 1 = \\ &= (n-\alpha-2)\,\frac{n^3 + n^2 + 2n^2\alpha - 2n\alpha - 8n + 4}{n(n-1)(n-2)} \end{aligned}$$

and

$$q(b_j) = 4(\beta-1) + 2\alpha(\beta-1) + (\beta-1)^2 = (n-\alpha-2)(n+\alpha+2),$$

whence

$$\widetilde{\mu}_1(b_j) = \frac{n^3 + n^2(1+2\alpha) + n(-2\alpha-8) + 4}{n(n+\alpha+2)(n-1)(n-2)}.$$

Therefore we have $\widetilde{\mu}_1(b_1) > \widetilde{\mu}_1(a_i)$ if and only if

$$\frac{10n^2\alpha^2 + 24n^2\alpha + 12n^2 - 4n\alpha^3 - 28n\alpha^2 - 50n\alpha - 24n + 2\alpha^3 + 10\alpha^2 + 16\alpha + 8}{n(n-1)(\alpha+1)(\alpha+2)(4n-5)} >$$
$$> \frac{\begin{array}{c}3n^3\alpha^3 + 17n^3\alpha^2 + 24n^3\alpha + 8n^3 - 2n^2\alpha^4 - 19n^2\alpha^3 - 73n^2\alpha^2 - 98n^2\alpha - 36n^2 + \\ +2n\alpha^4 + 22n\alpha^3 + 84n\alpha^2 + 116n\alpha + 48n - 4\alpha^3 - 20\alpha^2 - 32\alpha - 16\end{array}}{n(n-1)(n-2)(\alpha+1)(\alpha+2)(2n\alpha + 4n - \alpha^2 - 4\alpha - 8)}$$

and this is true if and only if

$$P=(2n^2\alpha+4n^2-n\alpha^2-8n\alpha-16n+2\alpha^2+8\alpha+16)(10n^2\alpha^2+24n^2\alpha+$$
$$+12n^2-4n\alpha^3-28n\alpha^2-50n\alpha-24n+2\alpha^3+10\alpha^2+16\alpha+8)>$$
$$>Q=(4n-5)(3n^3\alpha^3+17n^3\alpha^2+24n^3\alpha+8n^3-2n^2\alpha^4-19n^2\alpha^3-$$
$$-73n^2\alpha^2-98n^2\alpha-36n^2+2n\alpha^4+22n\alpha^3+84n\alpha^2+116n\alpha+$$
$$+48n-4\alpha^3-20\alpha^2-32\alpha-16).$$

We have $P=n^4(20\alpha^3+88\alpha^2+120\alpha+48)+n^3(-18\alpha^4-176\alpha^3-576\alpha^2-728\alpha-288)+n^2(4\alpha^5+84\alpha^4+494\alpha^3+1320\alpha^2+1552\alpha+608)+n(-10\alpha^5-114\alpha^4-516\alpha^3-1192\alpha^2-1312\alpha-512)+4\alpha^5+36\alpha^4+144\alpha^3+304\alpha^2+320\alpha+128$ and $Q=n^4(12\alpha^3+68\alpha^2+96\alpha+32)+n^3(-8\alpha^4-91\alpha^3-377\alpha^2-512a-18)+n^2(18\alpha^4+183\alpha^3+701\alpha^2+954\alpha+372)+n(-10\alpha^4-126\alpha^3-500\alpha^2-708\alpha-304)+20\alpha^3+100\alpha^2+160\alpha+8$, whence

$$\begin{aligned}P-Q &= n^4(8\alpha^3+20\alpha^2+24\alpha+16)+\\ &+n^3(-10\alpha^4-85\alpha^3-199\alpha^2-216\alpha-104)+\\ &+n^2(4\alpha^5+66\alpha^4+311\alpha^3+619\alpha^2+598\alpha+236)+\\ &+n(-10\alpha^5-104\alpha^4-390\alpha^3-692\alpha^2-604\alpha-208)+\\ &+4\alpha^5+36\alpha^4+124\alpha^3+204\alpha^2+160\alpha+48.\end{aligned}$$

Set $P-Q=E_n$. Then we find $\forall\,\alpha,\ \alpha\geq 3$

$$E_n'''=324\alpha^4+642\alpha^3+438\alpha^2+48\alpha-240>0,$$

it follows that E_n'' is a strictly increasing function, so $E_n''{>}E_{2\alpha+1}''$ if and only if

$$\begin{aligned}E_n'' &> 12(4\alpha^2+4\alpha+1)(8\alpha^3+20\alpha^2+24\alpha+16)+\\ &+6(2\alpha+1)(-10\alpha^4-85\alpha^3-199\alpha^2-216\alpha-104)+\\ &+8\alpha^5+132\alpha^4+622\alpha^3+1238\alpha^2+1196\alpha+472=\\ &=272\alpha^5+396\alpha^4-68\alpha^3-388\alpha^2-292\alpha+40>0.\end{aligned}$$

It follows that E_n' is a strictly increasing function, so $\forall\, n>2\alpha+1$,

$$E_n'>E_{2\alpha+1}',$$

whence

$$\begin{aligned}E_n' &> 4(8\alpha^3+12\alpha^2+6\alpha+1)(8\alpha^3+20\alpha^2+24\alpha+16)+\\ &+3(4\alpha^2+4\alpha+1)(-10\alpha^4-85\alpha^3-199\alpha^2-216\alpha-104)+\\ &+(-10\alpha^5-104\alpha^4-390\alpha^3-692\alpha^2-604\alpha-208)+\\ &+2(2\alpha+1)(4\alpha^5+66\alpha^4+311\alpha^3+619\alpha^2+598\alpha+236)=\\ &=152\alpha^6+146\alpha^5-246\alpha^4-351\alpha^3-75\alpha^2+120\alpha+16>0.\end{aligned}$$

It follows that E_n is a strictly increasing function and therefore, $\forall\, n > 2\alpha + 1$, $E_n > E_{2\alpha+1}$.

One finds

$$\begin{aligned} E_n &= P - Q > (16\alpha^4 + 32\alpha^3 + 24\alpha^2 + 8\alpha + 1)(8\alpha^3 + 20\alpha^2 + 24\alpha + 16)+ \\ &+(8\alpha^3 + 12\alpha^2 + 6\alpha + 1)(-10\alpha^4 - 85\alpha^3 - 199\alpha^2 - 216\alpha - 104)+ \\ &+(4\alpha^2 + 4\alpha + 1)(4\alpha^5 + 66\alpha^4 + 311\alpha^3 + 619\alpha^2 + 598\alpha + 236)+ \\ &+(2\alpha + 1)(-10\alpha^5 - 104\alpha^4 - 390\alpha^3 - 692\alpha^2 - 604\alpha - 208)+ \\ &+4\alpha^5 + 36\alpha^4 + 124\alpha^3 + 204\alpha^2 + 160\alpha + 48 = \\ &= 64\alpha^7 + 36\alpha^6 - 158\alpha^5 - 130\alpha^4 + 82\alpha^3 + 112\alpha^2 - 6\alpha - 12 > 0 \end{aligned}$$

and by consequence

$$\widetilde{\mu}_1(b_1) > \widetilde{\mu}_1(a_i).$$

On the other side we have $\widetilde{\mu}_1(e) > \widetilde{\mu}_1(b_1)$ if and only if

$$\frac{6n\alpha + 8n - 2\alpha^2 - 8\alpha - 8}{n(2n-1)(\alpha+2)} >$$

$$> \frac{10n^2\alpha^2 + 24n^2\alpha + 12n^2 - 4n\alpha^3 - 28n\alpha^2 - 50n\alpha - 24n + 2\alpha^3 + 10\alpha^2 + 16\alpha + 8}{n(n-1)(\alpha+1)(\alpha+2)(4n-5)}.$$

This is true if and only if $P = (n\alpha + n - \alpha - 1)(4n-5)(6n\alpha + 8n - 2\alpha^2 - 8\alpha - 8) > Q = (2n-1)(10n^2\alpha^2 + 24n^2\alpha + 12n^2 - 4n\alpha^3 - 28n\alpha^2 - 50n\alpha - 24n + 2\alpha^3 + 10\alpha^2 + 16\alpha + 8)$. We obtain

$$\begin{aligned} P &= n^3(24\alpha^2 + 56\alpha + 32) + n^2(-8\alpha^3 - 94\alpha^2 - 190\alpha - 104)+ \\ &\quad + n(18\alpha^3 + 120\alpha^2 + 214\alpha + 112) - 10\alpha^3 - 50\alpha^2 - 80\alpha - 40 \end{aligned}$$

and

$$\begin{aligned} Q &= n^3(20\alpha^2 + 48\alpha + 24) + n^2(-8\alpha^3 - 66\alpha^2 - 124\alpha - 60)+ \\ &\quad + n(8\alpha^3 + 48\alpha^2 + 82\alpha + 40) - 2\alpha^3 - 10\alpha^2 - 16\alpha - 8, \end{aligned}$$

whence

$$\begin{aligned} P - Q &= n^3(4\alpha^2 + 8\alpha + 8) + n^2(-28\alpha^2 - 66\alpha - 44)+ \\ &\quad + n(10\alpha^3 + 72\alpha^2 + 132\alpha + 72) - 8\alpha^3 - 40\alpha^2 - 64\alpha - 32. \end{aligned}$$

Set $E_n = P - Q$, whence $\forall\, \alpha \geq 3$, we have that

$$\begin{aligned} E_n'' &= 6n(4\alpha^2 + 8\alpha + 8) - 56\alpha^2 - 132\alpha - 88 > \\ &> (12\alpha + 6)(4\alpha^2 + 8\alpha + 8) - 56\alpha^2 - 132\alpha - 88 = \\ &= 48\alpha^3 + 96\alpha^2 + 96\alpha + 24\alpha^2 + 48\alpha + 48 - 56\alpha^2 - 132\alpha - 88 > 0, \end{aligned}$$

therefore E_n' is a strictly increasing function; from this, one obtains $\forall\, \alpha \geq 3$

$$\begin{aligned} E_n' &> 3(4\alpha^2+4\alpha+1)(4\alpha^2+8\alpha+8)+2(2\alpha+1)(-28\alpha^2-66\alpha-44)+ \\ &+10\alpha^3+72\alpha^2+132\alpha+72 = 48\alpha^4+42\alpha^3-44\alpha^2-56\alpha+8>0, \end{aligned}$$

so E_n is a strictly increasing function and therefore

$$\begin{aligned} &E_n > E_{2\alpha+1} = (8\alpha^3+12\alpha^2+6\alpha+1)(4\alpha^2+8\alpha+8)+ \\ &+(4\alpha^2+4\alpha+1)(-28\alpha^2-66\alpha-44)+(2\alpha+1)(10\alpha^3+72\alpha^2+132\alpha+72)- \\ &-8\alpha^3-40\alpha^2-64\alpha-32 = 32\alpha^5+20\alpha^4-46\alpha^3-24\alpha^2+26\alpha+4>0. \end{aligned}$$

Then, $\forall\, i \in I(\alpha)$,

$$\widetilde{\mu}_1(e) > \widetilde{\mu}_1(b_1) > \widetilde{\mu}_1(a_i).$$

Moreover we have $\widetilde{\mu}_1(a_i) > \widetilde{\mu}_1(b_j)$ if and only if

$$\frac{\begin{array}{c}3n^3\alpha^3+17n^3\alpha^2+24n^3\alpha+8n^3-2n^2\alpha^4-19n^2\alpha^3-73n^2\alpha^2-98n^2\alpha-36n^2+ \\ +2n\alpha^4+22n\alpha^3+84n\alpha^2+116n\alpha+48n-4\alpha^3-20\alpha^2-32\alpha-16\end{array}}{n(n-1)(n-2)(\alpha+1)(\alpha+2)(2n\alpha+4n-\alpha^2-4\alpha-8)} >$$

$$> \frac{n^3+2n^2\alpha+n^2-2n\alpha-8n+4}{n(n-1)(n-2)(n+\alpha+2)}.$$

Set

$$\begin{aligned} P &= (n+\alpha+2)(3n^3\alpha^3+17n^3\alpha^2+24n^3\alpha+8n^3-2n^2\alpha^4- \\ &-19n^2\alpha^3-73n^2\alpha^2-98n^2\alpha-36n^2+2n\alpha^4+22n\alpha^3+ \\ &+84n\alpha^2+116n\alpha+48n-4\alpha^3-20\alpha^2-32\alpha-16) = \\ &= n^4(3\alpha^3+17\alpha^2+24\alpha+8)+n^3(\alpha^4+4\alpha^3-15\alpha^2-42\alpha-20)+ \\ &+n^2(-2\alpha^5-21\alpha^4-89\alpha^3-160\alpha^2-116\alpha-24)+n(2\alpha^5+26\alpha^4+ \\ &+124\alpha^3+264\alpha^2+248\alpha+80)-4\alpha^4-28\alpha^3-72\alpha^2-80\alpha-32 \end{aligned}$$

and

$$\begin{aligned} Q &= (n^3+2n^2\alpha+n^2-2n\alpha-8n+4)(\alpha^2+3\alpha+2)(2n\alpha+4n-\alpha^2-4\alpha-8) = \\ &= (n^3+2n^2\alpha+n^2-2n\alpha-8n+4)(2n\alpha^3+10n\alpha^2-\alpha^4-7\alpha^3-22\alpha^2+ \\ &+16n\alpha-32\alpha+8n-16) = n^4(2\alpha^3+10\alpha^2+16\alpha+8)+ \\ &+n^3(3\alpha^4+15\alpha^3+20\alpha^2-8)+n^2(-2\alpha^5-19\alpha^4-87\alpha^3-198\alpha^2-208\alpha-80)+ \\ &+n(2\alpha^5+22\alpha^4+108\alpha^3+280\alpha^2+352\alpha+160)-4\alpha^4-28\alpha^3-88\alpha^2-128\alpha-64 \end{aligned}$$

whence

$$\begin{aligned}P-Q &= n^4(\alpha^3+7\alpha^2+8\alpha)+n^3(-2\alpha^4-11\alpha^3-35\alpha^2-42\alpha-12)+\\&+n^2(-2\alpha^4-2\alpha^3+38\alpha^2+92\alpha+56)+\\&+n(4\alpha^4+16\alpha^3-16\alpha^2-104\alpha-80)+16\alpha^2+48\alpha+32.\end{aligned}$$

Set $E_n = P-Q$, so $\forall\,\alpha\geq 3$

$$\begin{aligned}E_n'' &= 24n(\alpha^3+7\alpha^2+8\alpha)-12\alpha^4-66\alpha^3-210\alpha^2-252\alpha-72>\\&>48\alpha^4+336\alpha^3+384\alpha^2+24\alpha^3+168\alpha^2+192\alpha-12\alpha^4-66\alpha^3-\\&-210\alpha^2-252\alpha-72=36\alpha^4+294\alpha^3+342\alpha^2-60\alpha-72>0,\end{aligned}$$

we obtain

$$\begin{aligned}E_n''>E_{2\alpha+1}'' &= 12(4\alpha^2+4\alpha+1)(\alpha^3+7\alpha^2+8\alpha)+\\&+6(2\alpha+1)(-2\alpha^4-11\alpha^3-35\alpha^2-42\alpha-12)-\\&-4\alpha^4-4\alpha^3+76\alpha^2+184\alpha+112=\\&=24\alpha^5+236\alpha^4+242\alpha^3-170\alpha^2-116\alpha+40>0.\end{aligned}$$

It follows

$$\begin{aligned}E_n'>E_{2\alpha+1}' &= 4(8\alpha^3+12\alpha^2+6\alpha+1)(\alpha^3+7\alpha^2+8\alpha)+\\&+3(4\alpha^2+4\alpha+1)(-2\alpha^4-11\alpha^3-35\alpha^2-42\alpha-12)+\\&+2(2\alpha+1)(-2\alpha^4-2\alpha^3-38\alpha^2+92\alpha+56)+\\&+4\alpha^4+16\alpha^3-16\alpha^2-104\alpha-80=\\&=8\alpha^6+108\alpha^5+50\alpha^4-237\alpha^3-105\alpha^2+66\alpha-4>0.\end{aligned}$$

Therefore E_n is a strictly increasing function. Then

$$\begin{aligned}P-Q &> (16\alpha^4+32\alpha^3+24\alpha^2+8\alpha+1)(\alpha^3+7\alpha^2+8\alpha)+\\&+(8\alpha^3+12\alpha^2+6\alpha+1)(-2\alpha^4-11\alpha^3-35\alpha^2-42\alpha-12)+\\&+(4\alpha^2+4\alpha+1)(-2\alpha^4-2\alpha^3+38\alpha^2+92\alpha+56)+\\&+(2\alpha+1)(4\alpha^4+16\alpha^3-16\alpha^2-104\alpha-80)+16\alpha^2+48\alpha+32=\\&=24\alpha^6-56\alpha^5-214\alpha^4-70\alpha^3+62\alpha^2-6\alpha-4>0,\end{aligned}$$

so we have for every $\beta>\alpha\geq 3$, for every $i\in I(\alpha)$ and $j\in I(\beta)\setminus\{1\}$

$$\widetilde{\mu}_1(e)>\widetilde{\mu}_1(b_1)>\widetilde{\mu}_1(a_i)>\widetilde{\mu}_1(b_j)$$

and therefore, $\forall\, r\geq 1$,

$$^{r}H={}^{1}H.$$

4.3. Set $\beta > \alpha \geq 3$ and

$$\widetilde{\mu}(b_j) < \widetilde{\mu}(b_1) < \widetilde{\mu}(a_i) < \widetilde{\mu}(e).$$

The associated join space ${}^1H$ is

${}^1H$	b_2	$\cdots$	b_β	b_1	a_1	$\cdots$	a_α	e
b_2	B_1	$\cdots$	B_1	B	H^*	$\cdots$	H^*	H
$\vdots$		$\ddots$						
b_β			B_1	B	H^*	$\cdots$	H^*	H
b_1				b_1	$A \cup \{b_1\}$	$\cdots$	$A \cup \{b_1\}$	$H \setminus B_1$
a_1					A	$\cdots$	A	$A \cup \{e\}$
$\vdots$						$\ddots$		
a_α							A	$A \cup \{e\}$
e								e

where $B_1 = B \setminus \{b_1\},\ H^* = H \setminus \{e\}$, $\alpha + \beta + 1 = n$. We have

$$\begin{aligned} A(e) &= 1 + \frac{2\alpha}{\alpha+1} + \frac{2}{n-\beta+1} + \frac{2(\beta-1)}{n} = \\ &= \frac{5\alpha^2 n - 2\alpha^3 - 10\alpha^2 + 15n\alpha - 16\alpha + 8n - 8}{n(\alpha+1)(\alpha+2)} \end{aligned}$$

$q(e) = 2n - 1$, therefore

$$\widetilde{\mu}_1(e) = \frac{5\alpha^2 n - 2\alpha^3 - 10\alpha^2 + 15n\alpha - 16\alpha + 8n - 8}{n(2n-1)(\alpha+1)(\alpha+2)}.$$

Moreover

$$A(a_i) = \frac{2\alpha(\beta-1)}{n-1} + \frac{2(\beta-1)}{n} + \frac{4\alpha}{\alpha+1} + \frac{2}{\alpha+2} + \alpha$$

One obtains

$$A(a_i) = \frac{\begin{array}{c} 3n^2\alpha^3 + 15n^2\alpha^2 + 22n^2\alpha + 6n^2 - 2n\alpha^4 - 13n\alpha^3 - \\ -35n\alpha^2 - 42n\alpha - 14n + 2\alpha^3 + 10\alpha^2 + 16\alpha + 8 \end{array}}{n(n-1)(\alpha+1)(\alpha+2)}.$$

We have also $q(a_i) = 2\alpha(\beta - 1) + 2(\beta - 1) + 4\alpha + \alpha^2 + 2$, where $q(a_i) = 2n\alpha + 2n - \alpha^2 - 2\alpha - 2$, therefore

$$\widetilde{\mu}_1(a_i) = \frac{A(a_i)}{2n\alpha + 2n - \alpha^2 - 2\alpha - 2}.$$

We have $\widetilde{\mu}_1(e) > \widetilde{\mu}_1(a_i)$ if and only if

$$\frac{5n\alpha^2+15n\alpha+8n-2\alpha^3-10\alpha^2-16\alpha-8}{n(2n-1)(\alpha+1)(\alpha+2)} >$$
$$> \frac{\begin{array}{c}3n^2\alpha^3+15n^2\alpha^2+22n^2\alpha+6n^2-2n\alpha^4-13n\alpha^3-35n\alpha^2-\\ -42n\alpha-14n+2\alpha^3+10\alpha^2+16\alpha+8\end{array}}{n(n-1)(\alpha+1)(\alpha+2)(2n\alpha+2n-\alpha^2-2\alpha-2)}.$$

This is true if and only if

$$\begin{aligned}P = &(5n\alpha^2+15n\alpha+8n-2\alpha^3-10\alpha^2-16\alpha-8)\cdot\\ &\quad\cdot(2n^2\alpha-2n\alpha+2n^2-2n-n\alpha^2+\alpha^2-2n\alpha+2\alpha-2n+2) >\\ &> Q = (2n-1)(3n^2\alpha^3+15n^2\alpha^2+22n^2\alpha+6n^2-2n\alpha^4-13n\alpha^3-\\ &-35n\alpha^2-42n\alpha-14n+2\alpha^3+10\alpha^2+16\alpha+8).\end{aligned}$$

We obtain $P = n^3(10\alpha^3+40\alpha^2+46\alpha+16)+n^2(-9\alpha^4-59\alpha^3-140\alpha^2-140\alpha-48)+n(2\alpha^5+23\alpha^4+89\alpha^3+160\alpha^2+142\alpha+48)-2\alpha^5-14\alpha^4-40\alpha^3-60\alpha^2-48\alpha-16$ and $Q = n^3(6\alpha^3+30\alpha^2+44\alpha+12)+n^2(-4\alpha^4-29\alpha^3-85\alpha^2-106\alpha-34)+n(2\alpha^4+17\alpha^3+55\alpha^2+74\alpha+30)-2\alpha^3-10\alpha^2-16\alpha-8$. Then

$$\begin{aligned}P-Q = &n^3(4\alpha^3+10\alpha^2+2\alpha+4)+n^2(-5\alpha^4-30\alpha^3-55\alpha^2-34\alpha-14)+\\ &+n(2\alpha^5+21\alpha^4+72\alpha^3+105\alpha^2+68\alpha+18)-2\alpha^5-14\alpha^4-38\alpha^3-50\alpha^2-32\alpha-8.\end{aligned}$$

Let E_n be $P-Q$. We will show that E_n it is a strictly increasing function. We know that $\beta > \alpha \geq 2 \Longrightarrow n > 2\alpha+1$ $(n \geq 6)$.

$$\begin{aligned}E_n'' &= 6n(4\alpha^3+10\alpha^2+2\alpha+4)-10\alpha^4-60\alpha^3-110\alpha^2-68\alpha-28 >\\ &> 6(2\alpha+1)(4\alpha^3+10\alpha^2+2\alpha+4)-10\alpha^4-60\alpha^3-110\alpha^2-68\alpha-28 =\\ &= 38\alpha^4+84\alpha^3-26\alpha^2-8\alpha-4 > 0.\end{aligned}$$

So, $\forall n \geq 6$, $E_n'' > 0$, whence E_n' is a strictly increasing function. Since $n > 2\alpha+1$ we have $E_n' > E_{2\alpha+1}'$, therefore

$$\begin{aligned}E_n' &> 3(4\alpha^2+4\alpha+1)(4\alpha^3+10\alpha^2+2\alpha+4)-\\ &-2(2\alpha+1)(5\alpha^4+30\alpha^3+55\alpha^2+34\alpha+14)+\\ &+2\alpha^5+21\alpha^4+72\alpha^3+105\alpha^2+68\alpha+18 =\\ &= 30\alpha^5+59\alpha^4-52\alpha^3-39\alpha^2-2\alpha+2 > 0.\end{aligned}$$

So E_n is a strictly increasing function. By consequence, $E_n > E_{2\alpha+1}$, that is

$$\begin{aligned} P-Q &> (8\alpha^3+12\alpha^2+6\alpha+1)(4\alpha^3+10\alpha^2+2\alpha+4)+ \\ &+(4\alpha^2+4\alpha+1)(-5\alpha^4-30\alpha^3-55\alpha^2-34\alpha-14)+ \\ &+(2\alpha+1)(2\alpha^5+21\alpha^4+72\alpha^3+105\alpha^2+68\alpha+18)- \\ &-2\alpha^5-14\alpha^4-38\alpha^3-50\alpha^2-32\alpha-8= \\ &=16\alpha^6+30\alpha^5-34\alpha^4-22\alpha^3+14\alpha^2+8\alpha>0. \end{aligned}$$

Then

$$\widetilde{\mu}_1(e) > \widetilde{\mu}_1(a_i)$$

Moreover we have

$$\begin{aligned} A(b_1) &= \frac{2(\beta-1)}{\beta}+\frac{2\alpha(\beta-1)}{n-1}+\frac{2(\beta-1)}{n}+1+\frac{2\alpha}{\alpha+1}+\frac{2}{\alpha+2}= \\ &= \frac{\begin{array}{c}(2n\alpha^2+6n\alpha+4n-2\alpha^3-10\alpha^2-16\alpha-8)\cdot \\ \cdot(2n^2+n^2\alpha-\alpha^2n-2n\alpha-3n+\alpha+1)+ \\ +(3\alpha^2+9\alpha+4)(n^3-2n^2-n^2\alpha+n\alpha+n)\end{array}}{(\alpha+1)(\alpha+2)n(n-1)(n-\alpha-1)} \\ q(b_1) &= 2(\beta-1)+2\alpha(\beta-1)+2(\beta-1)+1+2\alpha+2= \\ &= 2n\alpha+4n-2\alpha^2-6\alpha-5. \end{aligned}$$

We have also for $j>1$

$$A(b_j)=\beta-1+\frac{2(\beta-1)}{\beta}+\frac{2\alpha(\beta-1)}{n-1}+\frac{2(\beta-1)}{n}=(\beta-1)\left(\frac{2}{n}+\frac{2\alpha}{n-1}+\frac{2}{\beta}+1\right)$$

$$q(b_j)=(\beta-1)^2+2(\beta-1)+2\alpha(\beta-1)+2(\beta-1)=(\beta-1)(n+\alpha+2),$$

so

$$\begin{aligned} \widetilde{\mu}_1(b_j) &= \frac{1}{n+\alpha+2}\left(\frac{2}{n}+\frac{2\alpha}{n-1}+\frac{2}{n-\alpha-1}+1\right)= \\ &= \frac{n^3+n^2\alpha+2n^2-2n\alpha^2-3n\alpha-5n+2\alpha+2}{n(n-1)(n-\alpha-1)(n+\alpha+2)}. \end{aligned}$$

Therefore $\widetilde{\mu}_1(a_i) > \widetilde{\mu}_1(b_j)$ if and only if

$$\frac{\begin{array}{c}3n^2\alpha^3+15n^2\alpha^2+22n^2\alpha+6n^2-2n\alpha^4-13n\alpha^3-35n\alpha^2-42n\alpha \\ -14n+2\alpha^3+10\alpha^2+16\alpha+8\end{array}}{n(n-1)(\alpha+1)(\alpha+2)(2n\alpha+2n-\alpha^2-2\alpha-2)} >$$

$$> \frac{n^3+n^2\alpha+2n^2-2n\alpha^2-3n\alpha-5n+2\alpha+2}{n(n-1)(n-\alpha-1)(n+\alpha+2)}$$

if and only if

$$P = (n^2 + n - \alpha^2 - 3\alpha - 2)(3n^2\alpha^3 + 15n^2\alpha^2 + 22n^2\alpha + 6n^2 - 2n\alpha^4 -$$
$$-13n\alpha^3 - 35n\alpha^2 - 42n\alpha - 14n + 2\alpha^3 + 10\alpha^2 + 16\alpha + 8) >$$
$$> Q = (2n\alpha^3 + 2n\alpha^2 - \alpha^4 - 2\alpha^3 - 2\alpha^2 + 6n\alpha^2 + 6n\alpha - 3\alpha^3 - 6\alpha^2 - 6\alpha +$$
$$+4n\alpha + 4n - 2\alpha^2 - 4\alpha - 4)(n^3 + n^2\alpha + 2n^2 - 2n\alpha^2 - 3n\alpha - 5n + 2\alpha + 2).$$

We obtain

$$P = n^4(3\alpha^3 + 15\alpha^2 + 22\alpha + 6) + n^3(-2\alpha^4 - 10\alpha^3 - 20\alpha^2 - 20\alpha - 8)+$$
$$+n^2(-3\alpha^5 - 26\alpha^4 - 84\alpha^3 - 127\alpha^2 - 88\alpha - 18) + n(2\alpha^6 + 19\alpha^5 + 79\alpha^4 +$$
$$+175\alpha^3 + 220\alpha^2 + 142\alpha + 36) - 2\alpha^5 - 16\alpha^4 - 50\alpha^3 - 76\alpha^2 - 56\alpha - 16$$

and

$$Q = (2n\alpha^3 + 8n\alpha^2 + 10n\alpha + 4n - \alpha^4 - 5\alpha^3 - 10\alpha^2 - 10\alpha - 4)(n^3 + n^2\alpha + 2n^2 -$$
$$-2n\alpha^2 - 3n\alpha - 5n + 2\alpha + 2) = n^4(2\alpha^3 + 8\alpha^2 + 10\alpha + 4) + n^3(\alpha^4 + 7\alpha^3 + 16\alpha^2 +$$
$$+14\alpha + 4) + n^2(-5\alpha^5 - 29\alpha^4 - 74\alpha^3 - 108\alpha^2 - 86\alpha - 28) + n(2\alpha^6 + 13\alpha^5 + 44\alpha^4 +$$
$$+95\alpha^3 + 124\alpha^2 + 90\alpha + 28) - 2\alpha^5 - 12\alpha^4 - 30\alpha^3 - 40\alpha^2 - 28\alpha - 8$$

whence

$$P - Q = n^4(\alpha^3 + 7\alpha^2 + 12\alpha + 2) + n^3(-3\alpha^4 - 17\alpha^3 - 36\alpha^2 - 34\alpha - 12)+$$
$$+n^2(2\alpha^5 + 3\alpha^4 - 10\alpha^3 - 19\alpha^2 - 2\alpha + 10) + n(6\alpha^5 + 34\alpha^4 + 80\alpha^3 + 96\alpha^2 +$$
$$+52\alpha + 18) - 4\alpha^4 - 20\alpha^3 - 36\alpha^2 - 28\alpha - 8.$$

Set $P - Q = E_n$. We have

$$\begin{aligned} E_n''' &= 24n(\alpha^3 + 7\alpha^2 + 12\alpha + 2) - 18\alpha^4 - 102\alpha^3 - 216\alpha^2 - 204\alpha - 72 > \\ &> (48\alpha+24)(\alpha^3+7\alpha^2+12\alpha+2) - 18\alpha^4 - 102\alpha^3 - 216\alpha^2 - 204\alpha - 72 = \\ &= 30\alpha^4 + 258\alpha^3 + 528\alpha^2 + 180\alpha - 24 > 0, \end{aligned}$$

whence E_n'' is a strictly increasing function, so $\forall\, n > 2\alpha + 1$ we have $E_n'' > E_{2\alpha+1}''$. Since

$$\begin{aligned} E_n'' &> 12(4\alpha^2 + 4\alpha + 1)(\alpha^3 + 7\alpha^2 + 12\alpha + 2)+ \\ &+ 6(2\alpha + 1)(-3\alpha^4 - 17\alpha^3 - 36\alpha^2 - 34\alpha - 12)+ \\ &+ 4\alpha^5 + 6\alpha^4 - 20\alpha^3 - 38\alpha^2 - 4\alpha + 20 = \\ &= 16\alpha^5 + 168\alpha^4 + 370\alpha^3 + 94\alpha^2 - 112\alpha - 28 > 0, \end{aligned}$$

it follows that E'_n is a strictly increasing function, so $E'_n > E'_{2\alpha+1}$. We have

$$\begin{aligned} E'_n &> 4(8\alpha^3+12\alpha^2+6\alpha+1)(\alpha^3+7\alpha^2+12\alpha+2)+\\ &+3(4\alpha^2+4\alpha+1)(-3\alpha^4-17\alpha^3-36\alpha^2-34\alpha-12)+\\ &+(4\alpha+2)(2\alpha^5+3\alpha^4-10\alpha^3-19\alpha^2-2\alpha+10)+\\ &+6\alpha^5+34\alpha^4+80\alpha^3+96\alpha^2+52\alpha+8=\\ &= 4\alpha^6+54\alpha^5+99\alpha^4-95\alpha^3-198\alpha^2-62\alpha>0, \end{aligned}$$

therefore E_n is a strictly increasing function, and so $\forall\, n \geq 2\alpha+2$ we have $E_n \geq E_{2\alpha+2}$. Then

$$\begin{aligned} E_n &\geq (16\alpha^4+64\alpha^3+96\alpha^2+64\alpha+16)(\alpha^3+7\alpha^2+12\alpha+2)+\\ &+(8\alpha^3+24\alpha^2+24\alpha+8)(-3\alpha^4-17\alpha^3-36\alpha^2-34\alpha-12)+\\ &+(4\alpha^2+8\alpha+4)(2\alpha^5+3\alpha^4-10\alpha^3-19\alpha^2-2\alpha+10)+\\ &+(2\alpha+2)(6\alpha^5+34\alpha^4+80\alpha^3+96\alpha^2+52\alpha+8)-\\ &-4\alpha^4-20\alpha^3-36\alpha^2-28\alpha-8=\\ &= 8\alpha^6+40\alpha^5+48\alpha^4-36\alpha^3-112\alpha^2-76\alpha-32>0. \end{aligned}$$

By consequence

$$\widetilde{\mu}_1(a_i) > \widetilde{\mu}_1(b_j).$$

On the other side, $\widetilde{\mu}_1(a_i) > \widetilde{\mu}_1(b_1)$ if and only if

$$\frac{\begin{array}{c}3n^2\alpha^3+15n^2\alpha^2+22n^2\alpha+6n^2-2n\alpha^4-13n\alpha^3-35n\alpha^2-4n\alpha-\\ -14n+2\alpha^3+10\alpha^2+16\alpha+8\end{array}}{n(n-1)(\alpha+1)(\alpha+2)(2n\alpha+2n-\alpha^2-2\alpha-2)} >$$

$$> \frac{\begin{array}{c}(2n\alpha^2+6n\alpha+4n-2\alpha^3-10\alpha^2-16\alpha-8)(2n^2+n^2\alpha-n\alpha^2-2n\alpha-\\ -3n+\alpha+1)+(3\alpha^2+9\alpha+4)(n^3-2n^2-n^2\alpha+n\alpha+n)\end{array}}{n(n-1)(\alpha+1)(\alpha+2)(n-\alpha-1)(2n\alpha+4n-2\alpha^2-6\alpha-5)}$$

Set $P=(2n^2\alpha+4n^2-2n\alpha^2-6n\alpha-5n-2n\alpha^2-4n\alpha+2\alpha^3+6\alpha^2+5\alpha-2n\alpha-4n+2\alpha^2+6\alpha+5)(3n^2\alpha^3+15n^2\alpha^2+22n^2\alpha+6n^2-2n\alpha^4-13n\alpha^3-$

$35n\alpha^2 - 42n\alpha - 14n + 2\alpha^3 + 10\alpha^2 + 16\alpha + 8)$.We obtain

$$\begin{aligned} P &= n^4(6\alpha^4 + 42\alpha^3 + 104\alpha^2 + 100\alpha + 24) + n^3(-16\alpha^5 - 130\alpha^4 - 417\alpha^3 - \\ &- 647\alpha^2 - 466\alpha - 110) + n^2(14\alpha^6 + 130\alpha^5 + 515\alpha^4 + 1101\alpha^3 + 1312\alpha^2 + \\ &+ 802\alpha + 188) + n(-4\alpha^7 - 42\alpha^6 - 204\alpha^5 - 518\alpha^4 - 1016\alpha^3 - 1063\alpha^2 - \\ &- 604\alpha - 142) + 4\alpha^6 + 36\alpha^5 + 134\alpha^4 + 264\alpha^3 + 290\alpha^2 + 168\alpha + 40 \\ Q &= (2n\alpha + 2n - \alpha^2 - 2\alpha - 2)(2n^3\alpha^3 + 13n^3\alpha^2 + 25n^3\alpha + 12n^3 - \\ &- 4n^2\alpha^4 - 27n^2\alpha^3 - 73n^2\alpha^2 - 88n^2\alpha - 36n^2 + 2n\alpha^5 + 14n\alpha^4 + \\ &+ 47n\alpha^3 + 90n\alpha^2 + 87n\alpha + 32n - 2\alpha^4 - 12\alpha^3 - 26\alpha^2 - 24\alpha - 8), \end{aligned}$$

whence

$$\begin{aligned} Q &= n^4(4\alpha^4 + 30\alpha^3 + 76\alpha^2 + 74\alpha + 24) + \\ &+ n^3(-10\alpha^5 - 79\alpha^4 - 255\alpha^3 - 430\alpha^2 - 342\alpha - 96) + \\ &+ n^2(8\alpha^6 + 67\alpha^5 + 257\alpha^4 + 562\alpha^3 + 732\alpha^2 + 506\alpha + 136) + \\ &+ n(-2\alpha^7 - 18\alpha^6 - 83\alpha^5 - 240\alpha^4 - 437\alpha^3 - 486\alpha^2 - 302\alpha - 72) + \\ &+ 2\alpha^6 + 16\alpha^5 + 54\alpha^4 + 124\alpha^3 + 108\alpha^2 + 64\alpha + 16. \end{aligned}$$

So

$$\begin{aligned} P - Q &= n^4(2\alpha^4 + 12\alpha^3 + 28\alpha^2 + 26\alpha) + \\ &+ n^3(-6\alpha^5 - 51\alpha^4 - 162\alpha^3 - 217\alpha^2 - 124\alpha - 14) + \\ &+ n^2(6\alpha^6 + 63\alpha^5 + 258\alpha^4 + 539\alpha^3 + 580\alpha^2 + 296\alpha + 52) + \\ &+ n(-2\alpha^7 - 24\alpha^6 - 121\alpha^5 - 341\alpha^4 - 579\alpha^3 - 577\alpha^2 - 302\alpha - 70) + \\ &+ 2\alpha^6 + 20\alpha^5 + 80\alpha^4 + 140\alpha^3 + 182\alpha^2 + 104\alpha + 34. \end{aligned}$$

For $n > 2\alpha + 1$ and $\alpha \geq 2$ we have

$$\begin{aligned} &P - Q > (16\alpha^4 + 32\alpha^3 + 24\alpha^2 + 8\alpha + 1)(2\alpha^4 + 12\alpha^3 + 28\alpha^2 + 26\alpha) + \\ &+ (8\alpha^3 + 12\alpha^2 + 6\alpha + 1)(-6\alpha^5 - 51\alpha^4 - 162\alpha^3 - 217\alpha^2 - 124\alpha - 14) + \\ &+ (4\alpha^2 + 4\alpha + 1)(6\alpha^6 + 63\alpha^5 + 258\alpha^4 + 539\alpha^3 + 580\alpha^2 + 296\alpha + 52) + \\ &+ (2\alpha + 1)(-2\alpha^7 - 24\alpha^6 - 121\alpha^5 - 341\alpha^4 - 579\alpha^3 - 577\alpha^2 - 302\alpha - 70) + \\ &+ 2\alpha^6 + 20\alpha^5 + 80\alpha^4 + 140\alpha^3 + 182\alpha^2 + 104\alpha + 34. \end{aligned}$$

Therefore we find

$$P - Q > 14\alpha^8 + 2\alpha^7 - 38\alpha^6 + 92\alpha^5 + 238\alpha^4 + 246\alpha^3 + 80\alpha^2 - 16\alpha + 2 > 0.$$

It follows

$$\widetilde{\mu}_1(a_i) > \widetilde{\mu}_1(b_1)$$

Therefore we have obtained

$$\widetilde{\mu}_1(e) > \widetilde{\mu}_1(a_i) > \widetilde{\mu}_1(b_1), \quad \widetilde{\mu}_1(a_i) > \widetilde{\mu}_1(b_j)$$

For $j > 1$ we have $\widetilde{\mu}_1(b_j) < \widetilde{\mu}_1(b_1)$ if and only if

$$\frac{n^3 + n^2\alpha + 2n^2 - 2n\alpha^2 - 3n\alpha - 5n + 2\alpha + 2}{n(n-1)(n-\alpha-1)(n+\alpha+2)} <$$

$$< \frac{\begin{array}{c}(2n\alpha^2 + 6n\alpha + 4n - 2\alpha^3 - 10\alpha^2 - 16\alpha - 8)\cdot \\ \cdot(2n^2 + n^2\alpha - \alpha^2 n - 2n\alpha - 3n + \alpha + 1)+ \\ +(3\alpha^2 + 9\alpha + 4)(n^3 - 2n^2 - n^2\alpha + n\alpha + n)\end{array}}{n(n-1)(\alpha+1)(\alpha+2)(n-\alpha-1)(2n\alpha + 4n - 2\alpha^2 - 6\alpha - 5)}$$

This is true if and only if,by setting

$$P = (2n\alpha + 4n - 2\alpha^2 - 6\alpha - 5)(n^3\alpha^2 + 3n^3\alpha + 2n^3 + n^2\alpha^3 + 5n^2\alpha^2 + 8n^2\alpha +$$
$$+ 4n^2 - 2n\alpha^4 - 9n\alpha^3 - 18n\alpha^2 - 21n\alpha - 10n + 2\alpha^3 + 8\alpha^2 + 10\alpha + 4)$$

whence $P = n^4(2\alpha^3 + 10\alpha^2 + 16\alpha + 8) + n^3(2\alpha^3 + 9\alpha^2 + 13\alpha + 6) + n^2(-6\alpha^5 - 42\alpha^4 - 123\alpha^3 - 195\alpha^2 - 168\alpha - 60) + n(4\alpha^6 + 30\alpha^5 + 104\alpha^4 + 219\alpha^3 + 288\alpha^2 + 213\alpha + 66) - 4\alpha^5 - 28\alpha^4 - 78\alpha^3 - 108\alpha^2 - 74\alpha - 20$ and
$Q = (n + \alpha + 2)(2n^3\alpha^3 + 13n^3\alpha^2 + 25n^3\alpha + 12n^3 - 4n^2\alpha^4 - 27n^2\alpha^3 - 73n^2\alpha^2 - 88n^2\alpha - 36n^2 + 2n\alpha^5 + 14n\alpha^4 + 47n\alpha^3 + 90n\alpha^2 + 87n\alpha + 32n - 2\alpha^4 - 12\alpha^3 - 26\alpha^2 - 24\alpha - 8)$, one obtains

$$Q = n^4(2\alpha^3 + 13\alpha^2 + 25\alpha + 12) + n^3(-2\alpha^4 - 10\alpha^3 - 22\alpha^2 - 26\alpha - 12)+$$
$$+ n^2(-2\alpha^5 - 21\alpha^4 - 80\alpha^3 - 144\alpha^2 - 125\alpha - 40) + n(2\alpha^6 + 18\alpha^5 + 73\alpha^4 +$$
$$+ 172\alpha^3 + 241\alpha^2 + 182\alpha + 56) - -2\alpha^5 - 16\alpha^4 - 50\alpha^3 - 76\alpha^2 - 56\alpha - 16.$$

Hence

$$P - Q = n^4(-3\alpha^2 - 9\alpha - 4) + n^3(2\alpha^4 + 12\alpha^3 + 31\alpha^2 + 39\alpha + 18)+$$
$$+ n^2(-4\alpha^5 - 21\alpha^4 - 43\alpha^3 - 51\alpha^2 - 43\alpha - 20) + n(2\alpha^6 + 12\alpha^5 + 31\alpha^4 +$$
$$+ 47\alpha^3 + 47\alpha^2 + 31\alpha + 10) - 2\alpha^5 - 12\alpha^4 - 28\alpha^3 - 32\alpha^2 - 18\alpha - 4.$$

We have $\forall\, j \geq 2$ $\widetilde{\mu}_1(b_j) < \widetilde{\mu}_1(b_1)$ if and only if $P - Q < 0$.
Set $\alpha = 3$. We have

$$\text{for } n = 8\ P - Q > 0 \text{ whence } \widetilde{\mu}_1(b_j) > \widetilde{\mu}_1(b_1)$$
$$\text{for } n \geq 9,\ P - Q < 0 \text{ whence } \widetilde{\mu}_1(b_j) < \widetilde{\mu}(b_1)$$

Set $\alpha = 4$. We have $P - Q < 0$ if and only if $n \geq 13$, so

$$\text{for } n < 13,\ \widetilde{\mu}_1(b_j) > \widetilde{\mu}_1(b_1) \quad \text{and} \quad \text{for } n \geq 13,\ \widetilde{\mu}(b_j) < \widetilde{\mu}_1(b_1)$$

Set $\alpha = 5$. We have $P - Q < 0$ if and only if $n \geq 25$. So

$$\text{for } n < 25,\ \widetilde{\mu}_1(b_j) > \widetilde{\mu}_1(b_1) \quad \text{and} \quad \text{for } n \geq 25,\ \widetilde{\mu}_1(b_j) < \widetilde{\mu}_1(b_1)$$

The following problem remains open:
To find a function $n(\alpha)$ such that $P - Q < 0$ if and only if $n \geq n(\alpha)$.

Let us consider now the case $\alpha = 3$, $\beta = 4$ whence $n = 8$.
Set $H = A \cup B \cup \{e\}$ where $A = \{a_1, a_2, a_3\}$, $B = \{b_1, b_2, b_3, b_4\}$ and let us suppose $\forall\, i$, $\forall\, j \in B' = B - \{b_1\}$

$$\widetilde{\mu}(e) > \widetilde{\mu}(a_i) > \widetilde{\mu}(b_j) > \widetilde{\mu}(b_1)$$

so we have ${}^1H$ as follows, where $H' = H \setminus \{b_1\}$.

${}^1H$	b_1	b_2	b_3	b_4	a_1	a_2	a_3	e
b_1	b_1	B	B	B	A,B	A,B	A,B	H
b_2	B	B'	B'	B'	A,B'	A,B'	A,B'	H'
b_3	B	B'	B'	B'	A,B'	A,B'	A,B'	H'
b_4	B	B'	B'	B'	A,B'	A,B'	A,B'	H'
a_1	A,B	A,B'	A,B'	A,B'	A	A	A	A,e
a_2	A,B	A,B'	A,B'	A,B'	A	A	A	A,e
a_3	A,B	A,B'	A,B'	A,B'	A	A	A	A,e
e	H	H'	H'	H'	A,e	A,e	A,e	e

Therefore $A(b_1) = 1 + \dfrac{6}{4} + \dfrac{6}{7} + \dfrac{2}{8} = \dfrac{101}{28} \cdot q(b_1) = 15$, whence ${}^2\widetilde{\mu}(b_1) = 0,240$. We have also $\forall\, j > 1$, $A(b_j) = \dfrac{6}{4} + \dfrac{6}{7} + \dfrac{2}{8} + \dfrac{9}{3} + \dfrac{18}{6} + \dfrac{6}{7} = \dfrac{265}{28}$, $q(b_j) = 47$ whence ${}^2\widetilde{\mu}(b_j) = 0,20136$. Moreover, $\forall\, i$, $A(a_i) = \dfrac{6}{7} + \dfrac{2}{8} + \dfrac{18}{6} + \dfrac{6}{7} + \dfrac{9}{3} + \dfrac{6}{4} = \dfrac{265}{28}$, $q(a_i) = 47$, whence ${}^2\widetilde{\mu}(a_i) = {}^2\widetilde{\mu}(b_j)$ and, finally,

$$A(e) = \frac{2}{8} + \frac{6}{7} + \frac{6}{4} + 1 = \frac{101}{28}, \quad q(e) = 15,$$

whence

$${}^2\widetilde{\mu}(e) = {}^2\widetilde{\mu}(b_1).$$

One obtains ${}^2H$, where $B' \cup A = C$.

${}^2H$	e	b_1	b_2	b_3	b_4	a_1	a_2	a_3
e	e, b_1	e, b_1	H	H	H	H	H	H
b_1	e, b_1	e, b_1	H	H	H	H	H	H
b_2	H	H	C	C	C	C	C	C
b_3	H	H	C	C	C	C	C	C
b_4	H	H	C	C	C	C	C	C
a_1	H	H	C	C	C	C	C	C
a_2	H	H	C	C	C	C	C	C
a_3	H	H	C	C	C	C	C	C

We have $A(e) = A(b_1) = \dfrac{4}{2} + \dfrac{24}{8} = 5$ and $q(e) = q(b_1) = 28$, whence

$$ {}^3\widetilde{\mu}(b_1) = {}^3\widetilde{\mu}(e) = 0,178. $$

Moreover, $\forall\, j > 1,\ \forall\, i$,

$$ A(b_j) = A(a_i) = \frac{24}{8} + \frac{36}{6} = 9, \quad q(b_j) = q(a_i) = 60, $$

whence

$$ {}^3\widetilde{\mu}(b_j) = {}^3\widetilde{\mu}({}_i) = 0,15. $$

It follows

$$ {}^3H = {}^2H, $$

hence

$$ \forall\, r > 2,\ {}^rH = {}^2H. $$

References

[1] Ameri, R. and Zahedi, M.M., *Hypergroup and join space induced by a fuzzy subset*, PU.M.A., vol. 8 (1997).

[2] Bonansinga, P. and Corsini, P., *Sugli omomorfismi di semi–ipergruppi e di ipergruppi*, Bolletino U.M.I., 1-B (1982), 717–727.

[3] Corsini, P., *Hypergroupes et Groupes Ordonnés*, Rend. Sem. Mat. Univ. Padova, vol. 48 (1973).

[4] Corsini, P., *Feebly canonical and* 1*-ipergruppi*, Acta Universitatis Carolinae, Mathematica et Physica, vol. 24, no. 2 (1983).

[5] Corsini, P., *Prolegomena of Hypergroups*, Aviani Editore (1993).

[6] Corsini, P., *Join Spaces, Power Sets, Fuzzy Sets*, Proc. Fifth International Congress on A.H.A., 1993, Iaşi, Romania, Hadronic Press (1994).

[7] Corsini, P., *A new connection between Hypergroups and Fuzzy Sets,* Southeast Asian Bulletin of Mathematics, 27 (2003), pp. 221–229.

[8] Corsini, P. and Cristea, I., *Fuzzy grade of i.p.s. hypergroups of order less or equal to* 6, PU.M.A., vol.14, n.4 (2003), pp. 275–288.

[9] Corsini, P. and Cristea, I., *Fuzzy grade of i.p.s. hypergroups of order* 7, Iranian Journal of Fuzzy Systems, vol.1, n.2 (2004), pp. 15–32.

[10] Corsini, P. and Leoreanu, V., *Applications of Hyperstructure Theory*, Kluwer Academic Publishers, Advances in Mathematics (2003).

[11] Corsini, P., Leoreanu, V., *Join Spaces associated with Fuzzy Sets,* J. of Combinatorics, Information and System Sciences, vol. 20, no. 1 (1995).

[12] Corsini, P., Tofan, I., *On fuzzy hypergroups,* P.U.M.A., vol. 8, n.1 , Budapest (1997).

[13] Cristea, I., *Complete Hypergroups,* 1*-Hypergroups and Fuzzy Sets*, An. Şt. Univ. Ovidius Constanţa, vol.10 (2) (2002), pp.25–38.

[14] Cristea, I., *A property of the connection between fuzzy sets and hypergroupoids*, accepted by Italian Journal of Pure and Applied Mathematics (2005).

[15] Leoreanu, V., *About hyperstructures associated with fuzzy sets of type 2*, Italian Journal of Pure and Applied Mathematics, no. 17 (2005), 137-150.

[16] Prenowitz, W., Jantosciak, J., *Geometries and Join Spaces,* J. reine und angewandte Math., 257, (1972).

[17] Serafimidis, K, Kehagias, Ath., Konstantinidou, M., *The L-Fuzzy Corsini Join Hyperoperation*, Italian Journal of Pure and Applied Mathematics, no. 12 (2003).

[18] Ştefănescu, M., *On constructions of hypergroups*, Contributions to general algebra 9, Proceedings of the Conference, Linz, Austria, June 1994, Wien: Hölder–Pichler–Tempsky, 297–308 (1995).

[19] Tofan, I. and Volf, C., *On some connections between hyperstructures and fuzzy sets*, Italian Journal of Pure and Applied Mathematics, no. 7, (2000).

[20] Zadeh, L.A., *Fuzzy Sets,* Information and Control, 8, (1965).

[21] Zahedi, M.M., Bolurian, M., Hasankhani, A., *On polygroups and fuzzy–subpolygroups,* J. of Fuzzy Math., vol. 3, no. 1, (1995).

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