An. Şt. Univ. Ovidius Constanţa Vol. 12(1), 2004, 81–90

ON THE DIOPHANTINE EQUATION $x^4 - q^4 = py^3$

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Abstract

In this paper we study the Diophantine equation $x^4 - q^4 = py^3$, with the following conditions: p and q are prime distincts natural numbers, x is not divisible with p, $p \equiv 11$ (mod12), $q \equiv 1$ (mod3), $\overline{p}$ is a generator of the group $\left(\mathbf{Z}_q^*, \cdot\right)$, 2 is a cubic residue mod q.

1 Introduction

In some previous papers, [3], [4], [5], we have solved Diophantine equations of the form

$$x^4 - y^4 = pz^2,$$

where p is a prime natural number taken from the set $\{3, 5, 7, 11, 13, 19, 29, 37\}$. Here we try to solve an anlogous Diophantine equation, replacing the exponent 2 of z by 3 and considering y given, namely y being a prime number, q.
It is clear that it has been necessary to impose some additional conditions for p and q. In the proofs of the statements, one can see why those conditions are necessary.
But first we recall some results.

Proposition 1.1. ([7]). *If p is a prime natural number, $p \equiv 2$ $(mod\ 3)$and ϵ is a primitive root of unity of order p, then p is irreducible in the ring $Z[\epsilon]$.*

Proposition 1.2. ([7]). *If p is a prime natural number and $p \equiv 1$ $(mod$ $3)$, then its decomposition in irreducible factors in the ring $Z[\epsilon]$ is $p = \pi_1\pi_2$,*

Key Words: Diophantine equations, algebraic integers.

where π_1 *is not associated to* π_2.

Proposition 1.3. ([7]). *Let* $\pi \in Z[\epsilon]$ *be an irreducible element in* $Z[\epsilon]$ *with*$N(\pi) \neq 3$ *and* $\alpha \in Z[\epsilon]$, α *be not divisible with* π.*Then there exists a unique* $m \in \{0,1,2\}$, *such that* $\alpha^{\frac{N(\pi)-1}{3}} \equiv \epsilon^m \ (mod \pi)$.

Definition 1.4.([7]). *Let* $\pi \in Z[\epsilon]$ *be an irreducible element in* $Z[\epsilon]$ *with* $N(\pi) \neq 3$ *and* $\alpha \in Z[\epsilon]$. *We define the* **residual cubic symbol** $\left(\frac{\alpha}{\pi}\right)_3$ *in the following manner*:
i) $\left(\frac{\alpha}{\pi}\right)_3 = 0$ *if* π/α;
ii)$\left(\frac{\alpha}{\pi}\right)_3 = \epsilon^m$, *if* α *is not divisible with* π *where* $m \in \{0,1,2\}$ *and* $\alpha^{\frac{N(\pi)-1}{3}} \equiv \epsilon^m$ $(mod \pi)$.

Proposition 1.5.([7]). *Let* $\pi \in Z[\epsilon]$ *be an irreducible element in* $Z[\epsilon]$ *with* $N(\pi) \neq 3$ *and* $\alpha, \beta \in Z[\epsilon]$. *Then*:
i)$\alpha \equiv \beta \ (mod \ \pi)$ *implies* $\left(\frac{\alpha}{\pi}\right)_3 = \left(\frac{\beta}{\pi}\right)_3$;
ii)$\left(\frac{\alpha\beta}{\pi}\right)_3 = \left(\frac{\alpha}{\pi}\right)_3 \left(\frac{\beta}{\pi}\right)_3$;
iii)$\left(\frac{\alpha}{\pi}\right)_3 = 1$ *if and only if* α *is not divisible with* π *and the congruence* $x^3 \equiv \alpha$ $(mod \ \pi)$ *has at least one solution* $x \in Z[\epsilon]$.

Proposition 1.6.([7]). *Let* $\pi \in Z[\epsilon]$ *be an irreducible element in* $Z[\epsilon]$ *with* $N(\pi) \neq 3$. *Then, for any* $\alpha \in Z[\epsilon]$, *we have*:

$$\overline{\left(\frac{\alpha}{\pi}\right)_3} = \left(\frac{\alpha^2}{\pi}\right)_3 = \left(\frac{\overline{\alpha}}{\pi}\right)_3.$$

Theorem 1.7.([7]). *Let* π_1 *and* π_2 *be two irreducible primary elements in* $Z[\epsilon]$ *such that* $N(\pi_1) \neq 3 \neq N(\pi_2)$ *and* $N(\pi_1) \neq N(\pi_2)$. *Then*:

$$\left(\frac{\pi_1}{\pi_2}\right)_3 = \left(\frac{\pi_2}{\pi_1}\right)_3.$$

Theorem 1.8.([2]). *Let* ξ *be a primitive root of l-order, of unity, where l is a prime natural number. Then a prime ideal P in the ring* $Z[\xi]$ *is in one of the cases*:
(i)*if* $\left\{\frac{\mu}{P}\right\} = 0$ *then* P *is in the ring of integers* A *in the Kummer field* $Q(M;\xi)$ (*where* $M = \sqrt[l]{\mu}$, $\mu \in \mathbf{Z}$) *equal with the l-power of a prime ideal, or*
(ii) *if* $\left\{\frac{\mu}{P}\right\} = 1$ *then* P *decomposes in l different prime ideals in the ring A*,

or
(iii) *if* $\left\{\frac{\mu}{P}\right\}$ *is equal with a root of order* l *of unity, different from* 1, *then* P *is a prime ideal in the ring* A.

Proposition 1.9.([6]).*Let* A *be the ring of integers of the Kummer field* $\mathbf{Q}\left(\sqrt[l]{p};\xi\right)$ *where* p *is a prime natural number and* ξ *is a primitive root of order* l *of unity. Let* G *be the Galois group of the Kummer field* $\mathbf{Q}\left(\sqrt[l]{p};\xi\right)$ *over* $\mathbf{Q}$. *Then for any* $\sigma\in G$ *and for any* $P\in Spec\,(A)$ *we have* $\sigma\,(P)\in Spec\,(A)$.

Proposition 1.10.([1]). *Let* p *be an odd prime natural number and* ξ *be a primitive root of order* p *of the unity. Then* $1-\xi^k = u_k\,(1-\xi)$, $k\notin p\mathbf{Z}$ *and* $u_k\in U\,(Z\,[\xi])$.

Proposition 1.11.([7]). *Let* p *be an odd prime natural number. Then*:
i) 2 *is a cubic residue mod* 3 (*in the case* $p=3$);
ii) *if* $p\equiv 2$ (*mod* 3), *then* 2 *is a cubic residue mod* p;
iii) *if* $p\equiv 1$ (*mod* 3), *then* 2 *is a cubic residue mod* p *if and only if there exist* $c,d\in\mathbf{Z}$ *such that* $p=c^2+27d^2$.

Proposition 1.12.([6]). *Let* ϵ *be a primitive root of* 3*-order of unity. Then the extension of fields* $\mathbf{Q}\subset\mathbf{Q}\left(\epsilon,\sqrt[3]{p}\right)$ *is a Galois extension and the Galois group* $G\cong(S_3,\circ)$. $G=\left\{1_{\mathbf{Q}\left(\epsilon,\sqrt[3]{p}\right)},v_1,v_1^2,v_2,v_1\circ v_2,v_1^2\circ v_2\right\}$, *where* $v_1\,(\epsilon)=\epsilon$, $v_1\left(\sqrt[3]{p}\right)=\epsilon\sqrt[3]{p}$, $v_2\,(\epsilon)=\epsilon^2$, $v_2\left(\sqrt[3]{p}\right)=\sqrt[3]{p}$.

2 Results

First, we state and prove two propositions that are necessary for solving the equation

$$x^4 - q^4 = py^3 \quad (1)$$

in the conditions (2):
(i) p *and* q *are different prime natural numbers;*
(ii) x *is not divisible with* p*;*
(iii)$\overline{p}$ *is a generator of the group* $(Z_q^*,\cdot)$*;*
(iv)$p\equiv 11$ *(* mod 12 *),* $q\equiv 1$ *(* mod 3 *);*
(v) 2 *is a cubic residue mod* q.

Lema 2.1. *Let* p *and* q *be prime integers satisfying the conditions* (2) *and* *take ϵ as a primitive root of order 3 of the unity. If $\mathbf{Q}\left(\epsilon;\sqrt[3]{p}\right)$ is the Kummer field with the ring of integers A, y_1 and y_2 are integer numbers such that $gcd(y_1,y_2)=1$, p does not divide y_2, then, taking $m,n\in\{0,1,2\}$, $m\neq n$,*

$$\left(y_2-\epsilon^m\sqrt[3]{p}y_1\right)A \text{ and } \left(y_2-\epsilon^n\sqrt[3]{p}y_1\right)A$$

are comaximal ideals of A.

Proof. Let J be the ideal of A generated by $y_2-\epsilon^m\sqrt[3]{p}y_1$ and $y_2-\epsilon^n\sqrt[3]{p}y_1$. It is sufficient to prove that $J=A$. We may suppose $m<n$. Using Proposition 1.10., we obtain:
$\left(y_2-\epsilon^m\sqrt[3]{p}y_1\right)-\left(y_2-\epsilon^n\sqrt[3]{p}y_1\right)=\epsilon^m\sqrt[3]{p}y_1\left(\epsilon^{n-m}-1\right)=\epsilon^m\sqrt[3]{p}y_1u_{n-m}\left(\epsilon-1\right)$,
where u_{n-m} and ϵ^m are units in $\mathbf{Z}\left[\epsilon\right]$ and in A, since $U\left(\mathbf{Z}\left[\epsilon\right]\right)\subset U\left(A\right)$.
Therefore $\sqrt[3]{p}y_1\left(\epsilon-1\right)\in J$. But $\sqrt[3]{p^2}\in A$, hence it results that

$$py_1\left(\epsilon-1\right)\in \mathrm{J}. \quad (3)$$

$\left(y_2-\epsilon^m\sqrt[3]{p}y_1\right)\in J$ and $\epsilon^{n-m}\in A$ implies $\left(y_2\epsilon^{n-m}-\epsilon^n\sqrt[3]{p}y_1\right)\in J$.

But $\left(y_2-\epsilon^n\sqrt[3]{p}y_1\right)\in J$, therefore $y_2\left(\epsilon^{n-m}-1\right)\in J$ and, by using the Proposition1.10.,
we gt

$$y_2\left(\epsilon-1\right)\in J. \quad (4)$$

Since $(y_1,y_2)=1$ and y_2 is not divisible with p, we get $(py_1,y_2)=1$, therefore there exist $k_1,k_2\in\mathbf{Z}$ such that $py_1k_1+y_2k_2=1$. Multiplying the last equality with $\epsilon-1$ and using the relations (3) and (4), we obtain that $\epsilon-1\in$J.
But $3=\left(\epsilon-1\right)^2\left(-\epsilon^2\right)$ and $-\epsilon^2\in U\left(\mathbf{Z}\left[\epsilon\right]\right)\subset U\left(A\right)$, therefore

$$3\in \mathrm{J}. \quad (5)$$

$\left(y_2-\epsilon^m\sqrt[3]{p}y_1\right)\left(y_2-\epsilon^n\sqrt[3]{p}y_1\right)\in$J. Let $k\in\{0,1,2\}-\{m,n\}$. Knowing that $\left(y_2-\epsilon^k\sqrt[3]{p}y_1\right)\in$A and that J is an ideal in A, we get:

$$\left(y_2-\epsilon^m\sqrt[3]{p}y_1\right)\left(y_2-\epsilon^n\sqrt[3]{p}y_1\right)\left(y_2-\epsilon^k\sqrt[3]{p}y_1\right)\in J.$$

This relation is equivalent with

$$y_2^3-py_1^3\in \mathrm{J}. \quad (6)$$

But $y_2^3 - py_1^3 \equiv 1 \pmod 3$, therefore $\left(3, y_2^3 - py_1^3\right) = 1$. We obtain that there exists $h_1, h_2 \in \mathbf{Z}$ such that $3h_1 + \left(y_2^3 - py_1^3\right) h_2 = 1$. Using the relations (5) and (6) we get that $J = A$.
In the same way we may prove the next lemma.

Lema 2.2. *Let us consider p and q as in the above conditions* (2) *and take ϵ as a primitive root of order* 3 *of the unity. If $\mathbf{Q}\left(\epsilon; \sqrt[3]{2p}\right)$ is the Kummer field with the ring of integers A, y_1 and y_2 are integers numbers, $gcd(y_1, y_2) = 1$, p does not divide y_2, then, taking, $m,n \in \{0,1,2\}$, $m \neq n$,*

$$\left(y_2 - \epsilon^m \sqrt[3]{2p}y_1\right) A \text{ and } \left(y_2 - \epsilon^n \sqrt[3]{2p}y_1\right) A$$

are comaximal ideals of A.

Now we try to solve the equation $x^4 - q^4 = py^3$.

Theorem 2.3. *The equation $x^4 - q^4 = py^3$ does not have nontrivial integer solutions in the conditions (2).*

Proof. We suppose that the equation (1)has nontrivial integer solutions $(x, y) \in \mathbf{Z}^2$ satisfying the conditions (2). We consider two cases, wether x is odd or even.
Case I: x **is an odd number**
Knowing that q is a prime natural number, $q \geq 3$, we get $x^2, q^2 \equiv 1 \pmod 4$ and therefore $x^2 - q^2 \equiv 0 \pmod 4$, $x^2 + q^2 \equiv 2 \pmod 4$.
We denote $d = gcd\left(x^2 - q^2, x^2 + q^2\right)$. Then $d/2x^2$ and $d/2q^2$. But $gcd\,(x, y) = 1$ implies x is not divisible with q. Therefore $d = 2$. We get either that

$$x^2 - q^2 = 4py_1^3,\ x^2 + q^2 = 2y_2^3,$$

where $y_1, y_2 \in \mathbf{Z}$, $2y_1y_2 = y$, y_2 is an odd number, $gcd\,(y_1, y_2) = 1$, or that

$$x^2 - q^2 = 4y_1^3,\ x^2 + q^2 = 2py_2^3,$$

where $y_1, y_2 \in \mathbf{Z}$, $2y_1y_2 = y$, y_2 is an odd number, $g.c.d.\,(y_1, y_2) = 1$.
In the last case, we obtain that $p/\left(x^2 + q^2\right)$, in contradiction with the fact that $p \equiv 3$ (mod4). It remains to study the case

$$x^2 - q^2 = 4py_1^3,\ x^2 + q^2 = 2y_2^3.$$

By substracting the two equations, we obtain $q^2 = y_2^3 - 2py_1^3$.
Let A be the ring of integers of the Kummer field $\mathbf{Q}\left(\epsilon; \sqrt[3]{2p}\right)$, where ϵ is a primitive root of order 3 of unity. In A, the last equality becomes:

$$q^2 = \left(y_2 - y_1 \sqrt[3]{2p}\right)\left(y_2 - y_1\epsilon \sqrt[3]{2p}\right)\left(y_2 - y_1\epsilon^2 \sqrt[3]{2p}\right). \quad (7)$$

But $q \equiv 1 \pmod 3$ implies (by using Proposition 1.2.) $q = \pi_1\pi_2$, where π_1, π_2 are prime elements in the ring $\mathbf{Z}[\epsilon]$, π_1 is not associate in divisibility with π_2.
We get: $q^2 = N(q) = N(\pi_1)N(\pi_2)$, π_1, $\pi_2 \notin U(\mathbf{Z}[\epsilon])$, therefore $N(\pi_1) = N(\pi_2) = q$. As p is a prime natural number, $p \equiv 2 \pmod 3$ implies (from Proposition 1.1.) that p remains prime in the ring $\mathbf{Z}[\epsilon]$ and $N(p) = p^2 \neq 3$.
We obtain $N(p) \neq N(\pi_i)$, $i = 1, 2$. Using Theorem 1.7., we have that

$$\left(\frac{p}{\pi_i}\right)_3 = \left(\frac{\pi_i}{p}\right)_3, \; i = 1, 2. \quad (8)$$

But 2 is a cubic residue mod q and this implies that there exists $x \in \mathbf{Z}$ such that $x^3 \equiv 2 \pmod q$, therefore there exists $x \in \mathbf{Z}$ such that $x^3 \equiv 2 \pmod{\pi_i}$, for any $i = 1, 2$. Hence $\left(\frac{2}{\pi_i}\right)_3 = \left(\frac{\pi_i}{2}\right)_3 = 1$, for any $i = 1, 2$. Using Proposition 1.5., we obtain: $\left(\frac{2p}{\pi_i}\right)_3 = \left(\frac{2}{\pi_i}\right)_3 \left(\frac{p}{\pi_i}\right)_3 = \left(\frac{p}{\pi_i}\right)_3$.
From the proof of the Theorem 1.7. and from the fact that p, q are prime natural numbers, $p \equiv 2 \pmod 3$, $q \equiv 1 \pmod 3$, we have that $\left(\frac{p}{q}\right)_3 = 1$. This is equivalent to

$$\left(\frac{\pi_1}{p}\right)_3 \left(\frac{\pi_2}{p}\right)_3 = 1. \quad (9)$$

From the relations (4) and (5), we have that $\left(\frac{p}{\pi_1}\right)_3 = \left(\frac{p}{\pi_2}\right)_3 = 1$ or $\left(\frac{p}{\pi_1}\right)_3 = \epsilon$, $\left(\frac{p}{\pi_2}\right)_3 = \epsilon^2$ or $\left(\frac{p}{\pi_1}\right)_3 = \epsilon^2$, $\left(\frac{p}{\pi_2}\right)_3 = \epsilon$.
If $\left(\frac{p}{\pi_1}\right)_3 = \left(\frac{p}{\pi_2}\right)_3 = 1$, then $p^{\frac{N(\pi_i)-1}{3}} \equiv 1 \pmod{\pi_i}$, $i = 1, 2$.
Since $N(\pi_i) = q$, $i = 1, 2$ and π_1,π_2 are irreducible elements in the rings $\mathbf{Z}[\epsilon]$, π_1 is not associate in divisibility with π_2, we get that $p^{\frac{q-1}{3}} \equiv 1 \pmod q$, in contradiction with the fact that $\overline{p}$ is a generator of the group $\left(\mathbf{Z}_q^*, \cdot\right)$.
Therefore $\left(\frac{2p}{\pi_1}\right)_3 = \left(\frac{p}{\pi_1}\right)_3 = \epsilon^i$ and $\left(\frac{2p}{\pi_2}\right)_3 = \left(\frac{p}{\pi_2}\right)_3 = \epsilon^j$, with $i, j \in \{1, 2\}$, $i \neq j$.
According to Theorem 1.8., we get that $\pi_1 A$ and $\pi_2 A$ are prime ideals in the ring A.

Passing to ideals in the relation (7), we obtain:

$$(\pi_1 A)^2 (\pi_2 A)^2 = \left(y_2 - y_1 \sqrt[3]{2p}\right) A \left(y_2 - y_1 \epsilon \sqrt[3]{2p}\right) A \left(y_2 - y_1 \epsilon^2 \sqrt[3]{2p}\right) A. \quad (10)$$

According to Lema 2.2., the ideals $\left(y_2 - y_1 \sqrt[3]{2p}\right) A$, $\left(y_2 - y_1 \epsilon \sqrt[3]{2p}\right) A$, $\left(y_2 - y_1 \epsilon^2 \sqrt[3]{2p}\right) A$ are comaximal in pair, therefore the equality (10) is impossible. We get that the equation (1) does not have nontrivial integer solutions, in the case when x is an odd number.

Case II: x **is an even number**.

In this case, $x^2 - q^2$ and $x^2 + q^2$ are odd numbers.

We prove that $gcd\left(x^2 - q^2, x^2 + q^2\right) = 1$. Suppose that there exists an odd prime natural number d such that $d/\left(x^2 - q^2\right)$ and $d/\left(x^2 + q^2\right)$. Hence d/x and d/q. Using the hypothesis we obtain that d/y, in contradiction with the fact $(x, y) = 1$. Therefore $gcd\left(x^2 - q^2, x^2 + q^2\right) = 1$. Then (1) becomes either the system:

$x^2 - q^2 = py_1^3$, $x^2 + q^2 = y_2^3$, with $y_1, y_2 \in \mathbf{Z}$, $y_1 y_2 = y$, $gcd\,(y_1, y_2) = 1$

or the system:

$x^2 - q^2 = y_1^3$, $x^2 + q^2 = py_2^3$, with $y_1, y_2 \in \mathbf{Z}$, $y_1 y_2 = y$, $gcd\,(y_1, y_2) = 1$.

In the last case, we obtain that $p/\left(x^2 + q^2\right)$, in contradiction with the fact that $p \equiv 3$ (mod4). It remains to study the case

$$x^2 - q^2 = py_1^3,\ x^2 + q^2 = y_2^3.$$

Substracting the two equations, we get $2q^2 = y_2^3 - py_1^3$.

Let A be the ring of integers of the Kummer field $\mathbf{Q}\left(\epsilon; \sqrt[3]{p}\right)$, where ϵ is a primitive root of order 3 of the unity. In A, the last equality becomes:

$$\left(y_2 - y_1 \sqrt[3]{p}\right)\left(y_2 - y_1 \epsilon \sqrt[3]{p}\right)\left(y_2 - y_1 \epsilon^2 \sqrt[3]{p}\right) = 2q^2. \quad (11)$$

Similarly with the case when x is an odd number, we obtain $qA = \pi_1 A \cdot \pi_2 A$, where π_1,π_2 are ireduccible elements in the rings $\mathbf{Z}[\epsilon]$, π_1 is not associate in divisibility with π_2.

Since $p \equiv 2$ (mod 3) and using Proposition 1.5. and Proposition 1.11., we get that $\left(\frac{2}{p}\right)_3 = 1$.

Using Theorem 1.8., we get that, in the ring A, $2A = P_1 P_2 P_3$, where P_k, $k = 1, 2, 3$ are prime ideals in the ring A.

Considering the corresponding ideals in the relation (11), we obtain:

$$\left(y_2 - y_1 \sqrt[3]{p}\right) A \left(y_2 - y_1 \epsilon \sqrt[3]{p}\right) A \left(y_2 - y_1 \epsilon^2 \sqrt[3]{p}\right) A = P_1 P_2 P_3 (\pi_1 A)^2 (\pi_2 A)^2. \quad (12)$$

According to Proposition 1.12., $\mathbf{Q} \subset \mathbf{Q}\left(\epsilon, \sqrt[3]{p}\right)$ and the Galois group $G \cong (S_3, \circ)$. Hence $G = \left\{1_{\mathbf{Q}\left(\epsilon, \sqrt[3]{p}\right)}, v_1, v_1^2, v_2, v_1 \circ v_2, v_1^2 \circ v_2\right\}$, where $v_1(\epsilon) = \epsilon$,

$v_1\left(\sqrt[3]{p}\right)=\epsilon\sqrt[3]{p}$, $v_1^2\left(\epsilon\right)=\epsilon$, $v_1^2\left(\sqrt[3]{p}\right)=\epsilon^2\sqrt[3]{p}$.
Case (i): if there exists $k\in\{1,2,3\}$ such that $\left(y_2-y_1\sqrt[3]{p}\right)A=P_k\in$Spec(A), we use Proposition 1.9., and we obtain that

$$v_2\left(\left(y_2-y_1\sqrt[3]{p}\right)A\right)=\left(y_2-y_1\epsilon\sqrt[3]{p}\right)A\in Spec(A)$$

and

$$v_2^2\left(\left(y_2-y_1\sqrt[3]{p}\right)A\right)=\left(y_2-y_1\epsilon^2\sqrt[3]{p}\right)A\in Spec(A),$$

therefore the equality (12) is impossible.
Case (ii): if there exist k and h in $\{1,2,3\}$, $k\neq h$ such that $\left(y_2-y_1\sqrt[3]{p}\right)A=P_kP_h$, where P_k,$P_h\in$Spec(A), we use Proposition 1.9. obtaining that
$\left(y_2-y_1\epsilon\sqrt[3]{p}\right)A=v_2\left(\left(y_2-y_1\sqrt[3]{p}\right)A\right)=\left(\pi_1A\right)P_3$ and
$\left(y_2-y_1\epsilon^2\sqrt[3]{p}\right)A=v_2^2\left(\left(y_2-y_1\sqrt[3]{p}\right)A\right)=\left(\pi_1A\right)\left(\pi_2A\right)$
or similar equalities. This fact implies that the ideals $\left(y_2-y_1\sqrt[3]{p}\right)A$, $\left(y_2-y_1\epsilon\sqrt[3]{p}\right)A$, $\left(y_2-y_1\epsilon^2\sqrt[3]{p}\right)A$ are comaximal to each other.
Case (iii): if $\left(y_2-y_1\sqrt[3]{p}\right)A=\left(\pi_1A\right)^2$, then $\left(y_2-y_1\epsilon\sqrt[3]{p}\right)A=\left(\pi_2A\right)^2$, $\left(y_2-y_1\epsilon^2\sqrt[3]{p}\right)A=P^2$, $P\in$Spec(A), in contradiction with (12).
From the cases (i), (ii), (iii), it results that the equality (12) is impossible. We get that the equation (1) does not have nontrivial integer solutions satisfying the conditions (2).

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