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AN OPTIMAL METHOD OF GALERKIN TYPE FOR DIFFUSION-DISPERSION PROBLEMS

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Abstract

The aim of this paper is the study of steady state fluid flow and transport (diffusion and dispersion) of pollutants in porous media. The mathematical model of this phenomena yields to some elliptic equations with boundary conditions. In this paper we present a projection method of Galerkin type for solving such elliptic equations. The originality of the work consists in the choice of the system of functions used for the theoretical discretization, which is a complete system of eigenfunctions of the duality map between a Hilbert space and its dual. This choice is optimal because the discretization system is orthonormal, and therefore we don't need to use Gramm-matrix preconditioning. Using the properties of the duality map and the fact that the embedding of the space $H^1(\Omega)$ in $L^2(\Omega)$ is compact, we can also estimate the error obtained by the method in a easier way. We use this method for finding the weak solution of the diffusion-dispersion problems, where the duality map is one of the operators involved in the studied equations.

Keywords: Diffusion-Dispersion, Elliptic Problems, Eigenfunctions

1 Diffusion-Dispersion Problem Formulation

The problem formulation of steady state diffusion and dispersion of pollutants is based on two elliptic problems: the diffusion equation (the fluid flow) and the dispersion (transport) equation. First we will present these mathematical models.

Let Ω be a bounded domain in $\mathbf{R}^3$, with smooth enough boundary $\partial\Omega$ such that we can apply Green's formula and Sobolev-Kondrashov embedding

Key Words: Diffusion-Dispersion; Elliptic Problems; Eigenfunctions.

theorem. We denote by $v = (v_1, v_2, v_3)$ the velocity that generates the flow and by p the pressure of the fluid. Then p satisfies the following diffusion problem:

$$\begin{cases} \nabla \cdot v \equiv -\nabla \cdot D\nabla p = f \text{ in } \Omega \\ p = p_0 \text{ on } \Gamma \\ -D\nabla p \cdot n = p_1 \text{ on } \partial\Omega \setminus \Gamma, \end{cases} \tag{1}$$

where $\Gamma \subseteq \Omega$ with $meas(\Gamma) > 0$; $D = (d_{ij})_{1\le i,j\le 3}$ (the permeability of porous media) is a symmetric matrix defined in $\bar{\Omega}$ and $d_{ij} \in C^1(\bar{\Omega})$, $\forall\, 1 \le i, j \le 3$; n is the exterior normal unit vector to $\partial\Omega$; p_0 and p_1 are given functions and f is the given source term. We assume that D satisfies the ellipticity condition

$$\sum_{i,j=1}^{3} d_{ij}\xi_i\xi_j \ge \lambda|\xi|^2, \lambda > 0, \xi \in \mathbf{R}^3,$$

where by $|\cdot|$ we denote the euclidian norm on $\mathbf{R}^3$.

Denote now by c the concentration of a chemical dissolved in the fluid, which is distributed due to convection-diffusion-dispersion processes. Then, the steady state distribution of c (the transport equation) is described by the following problem:

$$\begin{cases} -\nabla \cdot A\nabla c + \nabla \cdot (bc) + \gamma c = g \text{ in } \Omega \\ c = c_0 \text{ on } \Gamma \\ (-A\nabla c + bc) \cdot n = c_1 \text{ on } \partial\Omega \setminus \Gamma, \end{cases} \tag{2}$$

where b is the advection vector field; $A = (a_{ij})_{1\le i,j\le 3}$ is the diffusion-dispersion tensor which we assume that is a symmetric matrix defined in $\bar{\Omega}$, $a_{ij} \in C^1(\bar{\Omega})$, $\forall\, 1 \le i, j \le 3$; $\gamma \in C(\bar{\Omega}), \gamma \ge 0$; c_0, c_1 and g are given functions. We also assume that A satisfies the ellipticity condition

$$\sum_{i,j=1}^{3} a_{ij}\xi_i\xi_j \ge \lambda|\xi|^2, \lambda > 0, \xi \in \mathbf{R}^3.$$

We can suppose, without losing the generality, that $c_0 = 0$, because making the translation $c - c_0$, we arrive to an equivalent problem with homogeneous Dirichlet conditions on Γ.

2 Weak Solutions for Diffusion-Dispersion Problems

We will define the weak solutions for problems (1) and (2), and further we will describe a projection method to approximate the weak solutions. First, we remark that the two problems presented above are of the same type, so we will present the method only for problem (2) (problem (1) can be solved similarly).

Denote now by $H := L^2(\Omega)$ and $V := \{v \in H^1(\Omega) | v = 0 \text{ on } \Gamma\}$. Then, H and V are real Hilbert spaces with respect to the scalar products (see [5]):

$$(u, v) := \int_\Omega u(x)v(x)dx, \quad < u, v >:= \int_\Omega \nabla u(x) \cdot \nabla v(x)dx$$

and $V \hookrightarrow H$ with compact embedding by Sobolev-Kondrashov theorem.

Denote by u the solution of problem (2). Consider the operator $L : V \to V$

$$Lu(x) := -\nabla \cdot A(x)\nabla u(x) + \nabla \cdot (b(x)u(x)) + \gamma(x)u(x) =$$

$$= -\sum_{i,j=1}^{3} \frac{\partial}{\partial x_j} \left(a_{ij}(x) \cdot \frac{\partial u}{\partial x_i}(x) \right) + \sum_{i=1}^{3} \frac{\partial}{\partial x_i} (b(x)u(x)) + \gamma(x) \cdot u(x), \ x \in \Omega.$$

Then, the equation $-\nabla \cdot A\nabla u + \nabla \cdot (bu) + \gamma u = g$ in Ω, becomes

$$Lu(x) = g(x), x \in \Omega, \tag{3}$$

and L is an elliptic operator.

The weak solution of problem (2) is a function $u \in V$ such that

$$(Lu, \varphi) = (g, \varphi), \ \forall \varphi \in V, \tag{4}$$

or, equivalently (by Green's formula)

$$\int_\Omega \left[\sum_{i,j=1}^{3} a_{ij}(x) \cdot \frac{\partial u}{\partial x_i}(x) \cdot \frac{\partial \varphi}{\partial x_j}(x) + b(x)u(x)\mathrm{div}\varphi(x) + \gamma(x)\, u(x)\, \varphi(x) \right] dx =$$

$$= (g, \varphi) + \int_{\partial\Omega\setminus\Gamma} c_1(x)\varphi(x)ds, \ \forall \varphi \in V.$$

It is known that problem (4) has an unique weak solution (see [5], p. 50–51).

3 Numerical Projection Method

We will approximate the weak solution using a discretization of the problem. For this, we need the following result (see [5], p. 63):

Theorem 1. *Let V and H be two real Hilbert spaces (H identified with its dual H^*), V being compactly embedded in H. Then there exist the sequences $\{\varphi_n\} \subset V$ and $\{\lambda_n\} \subset (0, \infty)$ such that:*
(i) $\{\varphi_n\}$ is an orthogonal basis in V;
(ii) $\left\{\sqrt{\lambda_n}\varphi_n\right\}$ is an orthogonal basis in H;
(iii) $\{\lambda_n \varphi_n\}$ is an orthogonal basis in V^*;
(iv) $\{\lambda_n\}$ is a monotone increasing sequence that diverges to $+\infty$.

We have $V \hookrightarrow H \hookrightarrow V^*$. From the proof of this theorem (see [5]), we know that λ_n are the eigenvalues of the duality mapping $J : V \to V^*$, and φ_n are the corresponding eigenfunctions.

Remember the following well-known results:

Lemma 1. *If V_N is a finite dimensional subspace of V with the basis $\varphi_1, ..., \varphi_N$, then for any $u \in V$, there exists a unique $u_N \in V_N$ satisfying:*

$$< u - u_N, \varphi >_V = 0, \, \forall \varphi \in V_N. \tag{5}$$

(u_N is called the orthogonal projection of u on V_N).

Equivalently, we say that u_N is the best approximation of u in V_N in the norm of V, *i.e.*

$$\|u - u_N\|_V = \inf_{\varphi \in V_N} \|u - \varphi\|_V . \tag{6}$$

In our case ($H := L^2(\Omega)$ and $V := \{v \in H^1(\Omega) | v = 0 \text{ on } \Gamma\}$), we have that $J = -\Delta$, and consider the system $\{\varphi_n\}_{n \geq 1}$ obtained from the Theorem 1.

Denote by $a : V \times V \to \mathbf{R}$,

$$a(u, v) = \int_\Omega [\sum_{i,j=1}^{3} a_{ij}(x) \cdot \frac{\partial u}{\partial x_i}(x) \cdot \frac{\partial v}{\partial x_j}(x) + b(x)u(x)\mathrm{div}v(x) + \\ + \gamma(x)\, u(x)\, v(x)]dx, \; u, v \in V. \tag{7}$$

In the case when there is no advection (so $b = 0$ on Ω), we easily see that $a(u, v)$ is a scalar product on V, and denote this product by $(\cdot, \cdot)_V$ and the induced norm by $||| \cdot |||_V$.

Let now $N \in \mathbf{N}^*$ and $S_N(\Omega)$ be the space generated by the eigenfunctions $\varphi_1, \varphi_2, ..., \varphi_N$.

Consider instead of V_N from the above theorem, the space $S_N(\Omega)$. In this case, the matrix $G = (G_{ij})$, $G_{ij} = (\varphi_i, \varphi_j)_V$ is the unity matrix, because $\{\varphi_i\}_{i=1,2,...,N}$ form an orthonormal system.

Denote by $T_N : V \to S_N(\Omega)$ the operator which satisfies:

$$(u - T_N u, \varphi)_V = 0, \forall \varphi \in S_N(\Omega) \tag{8}$$

or equivalently,

$$|||u - T_N u|||_V = \inf_{\varphi \in S_N(\Omega)} |||u - \varphi|||_V \ . \tag{9}$$

Now we state the approximate problem corresponding to the problem (4):

Find $u_N \in S_N(\Omega)$ such that:

$$(u_N, \varphi)_V = (g, \varphi) + \int\limits_{\partial\Omega\setminus\Gamma} c_1(x)\varphi(x)ds \text{ for any } \varphi \in S_N(\Omega) . \tag{10}$$

Because $u_N \in S_N(\Omega)$, we have that

$$u_N = \sum_{i=1}^{N} \alpha_i \varphi_i \tag{11}$$

and the relations (10) and (11) lead us to the algebraic system:

$$\sum_{i=1}^{N} \alpha_i (\varphi_i, \varphi_j)_V = (g, \varphi_j) + \int\limits_{\partial\Omega\setminus\Gamma} c_1(x)\varphi(x)ds,\ j = 1, 2, ..., N,$$

where α_i are not known and must be determined.

Further, we will prove the existence, the uniqueness and the error estimation for approximate problem (10).

Theorem 2. *In the above conditions, we have that*
(i) If $g \in L^2(\Omega)$, there exists an unique $u_N \in S_N(\Omega)$ satisfying (10);
(ii) If u is the solution of problem (4) and $u_N \in S_N(\Omega)$ satisfies (10), then $u - u_N$ satisfies the relation (8), i.e. $u_N = T_N u$ and we have:

$$|||u - u_N||| = \inf_{\varphi \in S_N(\Omega)} |||u - \varphi||| . \tag{12}$$

Proof. (i) As $S_N(\Omega) \subset V$, $(\cdot,\cdot)_V$ is also a scalar product on $S_N(\Omega)$. For a fixed g in $L^2(\Omega)$, $\hat{g}(\varphi) := (g, \varphi) + \int\limits_{\partial\Omega\setminus\Gamma} c_1(x)\varphi(x)ds$ is a linear and continuous functional on S_N, and by Riesz-Frechet theorem, it results that equation (10) has a unique solution in $S_N(\Omega)$, for any $g \in L^2(\Omega)$.

(ii) By (4) and (10), u_N satisfies:

$$(u - u_N, \varphi)_V = 0\,,\ \forall\varphi \in S_N(\Omega) \tag{13}$$

We have that

$$|||u - u_N|||^2 = (u - u_N, u - u_N)_V\ .$$

From (13), for any $\varphi \in S_N(\Omega)$, we have:

$$|||u - u_N|||^2 = (u - u_N, u - \varphi + \varphi - u_N)_V = (u - u_N, u - \varphi)_V + (u - u_N, \varphi - u_N)_V\,.$$

But $(u - u_N, \varphi - u_N)_V = 0$, because $\varphi - u_N \in S_N(\Omega)$ (see relation (13)). So,

$$|||u - u_N|||^2 = (u - u_N, u - \varphi)_V \leq |||u - u_N||| \cdot |||u - \varphi|||\,,$$

from Cauchy-Buniakowski-Schwartz inequality. From this it results that

$$|||u - u_N||| \leq |||u - \varphi|||\,, \forall\varphi \in S_N(\Omega)$$

i.e. (12), so $u_N = T_N u$.

Now we can estimate the error as follows:

Theorem 3. *For any $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbf{N}^*$ such that for any $N \geq N_\varepsilon$, then*

$$\|u - T_N u\|_H^2 \leq \varepsilon \cdot \|u\|_V^2\ .$$

Proof. From the Theorem 1, we have

$$J\varphi_n = \lambda_n\varphi_n,\ J : V \to V^*. \tag{14}$$

We have that $\left\{\sqrt{\lambda_n}\varphi_n\right\}$ is an orthonormal basis in $H := L^2(\Omega)$, so in $L^2(\Omega)$ we can write:

$$u = \sum_{n=1}^{\infty} c_n\sqrt{\lambda_n}\varphi_n,$$

where $c_n = (u, \sqrt{\lambda_n}\varphi_n) = \sqrt{\lambda_n}(u, \varphi_n)$.

We have, using (14) and Green's formula for functions in the space $V = \{v \in H^1(\Omega) | v = 0 \text{ on } \Gamma\}$ (see [5], p.28), that

$$(u, \varphi_n) = \int_\Omega u(x)\, \varphi_n(x)\, dx = \frac{1}{\lambda_n} \int_\Omega u(x)\, \lambda_n \varphi_n(x)\, dx =$$

$$= \frac{1}{\lambda_n} \int_\Omega u(x)\, J\varphi_n(x)\, dx = \frac{1}{\lambda_n} \int_\Omega Ju(x) \cdot \varphi_n(x)\, dx$$

so

$$u(x) = \sum_{n=1}^{\infty} \frac{1}{\lambda_n} \left(\int_\Omega Ju(x)\, \sqrt{\lambda_n} \varphi_n(x)\, dx \right) \sqrt{\lambda_n} \varphi_n = \sum_{n=1}^{\infty} \frac{1}{\lambda_n} (Ju, \sqrt{\lambda_n} \varphi_n) \cdot \sqrt{\lambda_n} \varphi_n \,.$$

Because $T_N u = \sum_{n=1}^{N} c_n \sqrt{\lambda_n} \varphi_n$, we obtain:

$$u - T_N u = \sum_{n=N+1}^{\infty} \frac{1}{\lambda_n} \cdot (Ju, \sqrt{\lambda_n} \varphi_n) \cdot \sqrt{\lambda_n} \varphi_n.$$

It results from this that:

$$\|u - T_N u\|_H^2 = \sum_{n=N+1}^{\infty} \frac{1}{\lambda_n^2} \cdot (Ju, \sqrt{\lambda_n} \varphi_n)^2.$$

Let now be $\varepsilon > 0$ arbitrary fixed. Because $\lambda_n \nearrow \infty$, we have that there exists $N_\varepsilon \in \mathbf{N}$ such that

$$\frac{1}{\lambda_n} < \sqrt{\varepsilon}, \, \forall n \geq N_\varepsilon.$$

If $N \geq N_\varepsilon$, then:

$$\|u - T_N u\|_H^2 \leq \varepsilon \sum_{n=N+1}^{\infty} (Ju, \sqrt{\lambda_n} \varphi_n)^2 \leq$$

$$\leq \varepsilon \sum_{n=1}^{\infty} (Ju, \sqrt{\lambda_n} \varphi_n)^2 = \varepsilon \, \|Ju\|_H^2 \,,$$

so

$$\|u - T_N u\|_H^2 \leq \varepsilon \, \|Ju\|_H^2 = \varepsilon \, \|u\|_V^2 \,.$$

4 Conclusions

Using the theoretical discretization presented above, we can find a very good approximation of the weak solution for the diffusion-dispersion problem we have considered. The method is optimal with respect to other discretization methods because the discretization system $(\varphi_n)_{n\geq 1}$, that is formed with eigenfunctions of Laplacian (involved in the equations), is orthonormal, and therefore we don't need to use the Gramm-matrix preconditioning. The method can be also used to approximate the weak solution of other problems from continuous mechanics in which are involved elliptic operators, such as problems for the elasticity theory or steady state Stokes equation.

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