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THE PERRON-FROBENIUS OPERATOR ON BV(I)

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Abstract

In this paper, we determine the upper bound γ_n of $varU^n f/varf$, when f varies in the collection of non-constant monotone functions on $I = [0, 1]$.

A very simple proof of a generalization of the Gauss-Kuzmin-Lévy theorem on continued fractions is given by considering the classical operator U defined by

$$Uf(x) = \sum_{n\geq 1} \frac{x+1}{(x+i)(x+i+1)} \cdot f\left(\frac{1}{x+i}\right), \quad x \in I = [0,1].$$

This as an operator on $BV(I)$, the collection of complex-valued functions of bounded variation defined on I under the supremum norm $|f| = \sup\{|f(x)|, x \in I\}$.

For any $n \in N^*$, we have

$$U^n f(x) = \sum_{i_1,\ldots,i_n \in N^*} p_{i_1 i_2 \ldots i_n}(x) f\left(u_{i_n \ldots i_1}(x)\right), \; x \in I, \tag{1}$$

where

$$u_{i_n \ldots i_1} = u_{i_n} \circ \ldots \circ u_{i_1},$$

$$p_{i_1 i_2 \ldots i_n}(x) = p_{i_1}(x) p_{i_2}(u_{i_1}(x)) \ldots p_{i_n}\left(u^i_{n-1 \ldots i_1}(x)\right), n \geq 2, \tag{2}$$

and the functions u_i and $p_i, i \in N^*$, are defined by

$$u_i(x) = \frac{1}{i+x}, \tag{3}$$

$$p_i(x) = \frac{x+1}{(x+i)(x+i+1)}, \; x \in I. \tag{4}$$

Key Words: functions of bounded variation; continued fractions.

Putting

$$\cfrac{1}{x_1+\ddots\,{}_{+\cfrac{1}{x_r}}}=\frac{p_r(x_1,x_2,...,x_r)}{q_r(x_1,x_2,...,x_r)},\ r\in N^*,$$

for arbitrary indeterminates $x_1,x_2,...,x_r$, we have

$$p_{i_1 i2 \ldots} i_n(x) = \tag{5}$$

$$=\frac{x+1}{(q_{n-1}(i_2,...,i_n)(x+i_1)+p_{n-1}(i_2,...,i_n))\,(q_n(i_2,...,i_{n,1})(x+i_1)+p_n(i_2,...,i_{n,1}))},$$

for all $n\geq 2,\ i_1,i_2,...,i_n\in N^*$, and $x\in I$.

Note that, in particular, we can write

$$p_{i_1 i_2 \ldots i_n}(0)=(-1)^n\left(\cfrac{1}{i_1+\cfrac{p_n(i_2,...,i_{n,1})}{q_n(i_2,...,i_{n,1})}}-\cfrac{1}{i_1+\cfrac{p_{n-1}(i_2,...,i_n)}{q_{n-1}(i_2,...,i_n)}}\right), \tag{5'}$$

for all $n\geq 2$ and $i_1,i_2,...,i_n\in N^*$.

To simplify the writing, put

$$p_{i_1,i_2\ldots i_n}(0)=\alpha_{i_1 i_2\ldots i_n},\ u_{i_1 i_2\ldots i_n}(0)=\beta_{i_1 i_2\ldots i_n}.$$

If n is odd, then, by Proposition 2 of [1] and equations (1), (2), (3) and (5), we have

$$varU^n f=U^n f(0)-U^n f(1)=$$

$$=\sum_{i_1,i_2,\ldots,i_n\in N^*}\left[p_{i_1 i_2\ldots i_n}(0)f(u_{i_n\ldots i_1}(0))-p_{i_1 i_2 .. i_n}(1)f\left(u_{i_n\ldots i_1}(1)\right)\right]= \tag{6}$$

$$=\sum_{i_2,\ldots,i_n\in N^*}\left[\alpha_{1 i_2\ldots i_n}f\left(\beta_{i_n\ldots i_2 1}\right)-\sum_{i_1\in N^*}\alpha_{(i_1+1)i_2\ldots i_n}f\left(\beta_{i_n\ldots i_2(i_1+1)}\right)\right].$$

Similarly, if n is even, then we have

$$varU^n f=U^n f(1)-U^n f(0)= \tag{7}$$

$$\sum_{i_2,\ldots,i_n\in N^*}\left[\sum_{i_1\in N^*}\alpha_{(i_1+1)i_2\ldots i_n}f\left(\beta_{i_n\ldots i_2(i_1+1)}\right)-\alpha_{1 i_2\ldots i_n}f\left(\beta_{i_n}\ldots i_{21}\right)\right].$$

The case $n=1$. In this case, writing i for i_1, equation (6) yields

$$varUf=\alpha_1 f(\beta_1)-\sum_{i\in N^*}\alpha_{i+1}f\left(\beta_{i+1}\right).$$

Since

$$\alpha_1 = \sum_{i\in N^*} \alpha_{i+1} = \frac{1}{2} \text{ and } 1 = \beta_1 > \beta_2 > ...,$$

we deduce that

$$varUf \leq \frac{1}{2}\left(f(1) - f(0) = \frac{1}{2}varf.\right) \tag{8}$$

The case $n = 2$. Write i for i_1 and j for i_2.

Then in this case $\alpha_{ij} = \dfrac{1}{(i_j+1)(i(j+1)+1)}$, $i, j \in N^*$, and equation (7) yields

$$varU^2 f = \sum_{j\in N^*}\left(\sum_{i\in N^*} \alpha_{(i+1)j} f\left(\beta_{j(i+1)}\right) - \alpha_{ij} f\left(\beta_{j1}\right)\right).$$

Clearly, $\beta_{(j+1)(i+1)} < \beta_{j1}$ for all $i, j \in N^*$. Hence

$$varU^2 f \leq f(1) \cdot \sum_{i\in N^*} \alpha_{(i+1)1} + \sum_{j\in N^*} f(\beta_{j1})\left(\sum_{i\in N^*} \alpha_{(i+1)(j+1)} - \alpha_{1j}\right). \tag{9}$$

But

$$\sum_{i\in N^*} \alpha_{(i+1)((j+1)} = \sum_{i\in N^*} \frac{1}{((i+1)(j+1))\,((i+1)(j+2)+1)} \leq \tag{10}$$

$$\leq \frac{1}{(j+1)(j+2)} \sum_{i\in N^*} \frac{1}{(i+1)^2} < \alpha_{1j},$$

for all $j \in N^*$.

Since $f(\beta_{j1}) \geq f(0)$, $j \in N^*$, and

$$\sum_{j\in N^*}\left(\sum_{i\in N^*} \alpha_{(i+1)(j+1)} - \alpha_{1j}\right) = -\sum_{i\in N^*} \alpha_{(i+1)1},$$

(9) and (10) imply that

$$varUf \leq \sum_{i\in N^*} \alpha_{(i+1)}\left(f_{(1)} - f_{(0)}\right) = \sum_{ij\in N^*} \alpha_{(i+1)1} varf. \tag{11}$$

Note that, for f defined by $f(x) = 0$, $0 \leq x \leq \dfrac{1}{2}$ and $f(x) = 1$, $1/2 < x \leq 1$, we have

$$varU^2 f = \sum_{i\in N^*} \alpha_{(i+1)1} varf, \tag{12}$$

that is the constant

$$\sum_{i\in N^*}\alpha_{(i+1)1}=\sum_{i\in N^*}\frac{1}{(i+2)(2i+3)}=2\sum_{i\in N^*}\left(\frac{1}{2i+3}-\frac{1}{2i+4}\right)=$$

$$=\log 4-\frac{7}{6}=0,21962...$$

occurring in (11) cannot be lowered.

Proposition 1.1. *For any* $n\geq 3$ *and* $i_2,...,i_n\in N^*$, *we have*

$$\sum_{i_1\in N^*}\alpha_{(i_1+1)(i_2+1)i_3...i_n}\leq\alpha_{1i_2i_3...i_n}.$$

Proof. As $\dfrac{p_{n-1}(i_2+1,i_3,...,i_n)}{q_{n-1}(i_2+1,i_3,...,i_n)}=$

$$\frac{1}{1+\dfrac{q_{n-1}(i_2,...,i_n)}{p_{n-1}(i_2,...,i_n)}}=\frac{p_{n-1}(i_2,..,i_n)}{p_{n-1}(i_2,...,i_n)+q_{n-1}(i_2,...,i_n)},$$

we have $p_{n-1}(i_2+1,i_3,...,i_n)=p_{n-1}(i_2,...,i_n)$ and

$$q_{n-1}(i_{2+1},i_3,...,i_n)=p_{n-1}(i_2,...,i_n)+q_{n-1}(i_2,...,i_n).$$

Consequently, putting for brevity

$$p^1_{n-1}=p_{n-1}(i_2,...,i_n),\ p^{11}_n=p_n(i_2,...,i_n,1)$$

$$q^1_{n-1}=q_{n-1}(i_2,...,i_n),\ \text{and}\ q^{11}_n=q_n(i_2,...,i_n,1),$$

by (5), we obtain

$$\sum_{i_1\in N^*}\alpha_{(i_1+1)(i_2+1)i_3...i_n}=$$

$$=\sum_{i_1\in N^*}\frac{1}{\left((i_1+1)(p^1_{n-1}+q^1_{n-1})+p^1_{n-1}\right)\left((i_1+1)\left(p^{11}_n+q^{11}_n\right)+p^{11}_n\right)}\leq$$

$$\leq\frac{1}{\left(p^1_{n-1}+q^1_{n-1}\right)\left(p^{11}_n+q^{11}_n\right)}\cdot\sum_{i_1\in N^*}\frac{1}{\left(i_1+1\right)^2}<\alpha_1 i_2 i_3...i_n.$$

Next, to make a choice, assume n is odd. It is easy to see that

$$\beta_{i_n...(i_2+1)(i_1+1)}>\beta_{i_n...i_3i_21},\ \ \beta_{i_n...i_31(i_1+1)}>\beta_{i_n...i_31},\ \ \beta_{i_n...i_3i_21}<\beta_{i_n...i_3},$$

for all $i_1, i_2, ...i_n \in N^*$. Then by (6) we have

$$varV^n f \le \sum_{i_3,...,i_n \in N^*} \left[- \sum_{i_1 \in N^*} \alpha_{(i_1+1)1i_2...i_n} f\left(\beta_{i_n...i_3 1(i_1+1)}\right) + \right.$$

$$\left. + \sum_{i_2 \in N^*} \left(\alpha_{1i_2i_3...i_n} - \sum_{i_1 \in N^*} \alpha_{(i_1+1)(i_2+1)i_3...i_n} \right) f\left(\beta_{i_n...i_3 i_2 1}\right) \right] \le \tag{13}$$

$$\le \sum_{i_3,...,i_n \in N^*} \left[\sum_{i_2 \in N^*} \left(\alpha_{1i_2i_3...i_n} - \sum_{i_1 \in N^*} \alpha_{(i_1+1)i_2,...,i_n} \right) f\left(\beta_{i_n...i_3}\right) + \right.$$

$$\left. + \left(\sum_{i_1 \in N^*} \alpha_{(i_1+1)i_3...i_n} \right) \left(f\left(\beta_{i_n...i_3}\right) - f\left(\beta_{i_n...i_3 1}\right) \right) \right].$$

Put $\delta_{i_3...i_n} = (-1)^{n-1} \sum_{i_2 \in N^*} \left(\alpha_1 i_2 ... i_n - \sum_{i_1 \in N^*} \alpha_{(i_1+1)} i_2 ... i_n \right)$ for all $i_3, ..., i_n \in N^*$. Note that

$$\sum_{i_3,...,i_n \in N^*} \delta_{i_3...i_n} = (-1)^{n-1} \left(\alpha_1 - \sum_{i_1 \in N^*} \alpha_{i_1+1} \right) = 0. \tag{14}$$

Now, the problem is to find the best upper bound for

$$\delta^{(n)} f = \sum_{i_3,...,i_n \in N} \delta_{i_3...i_n} f\left(\beta_{i_n...i_3}\right).$$

First, note that, by (14), we have

$$\delta^n f \le \frac{1}{2} \sum_{i_3,...,i_n \in N^*} \left| \delta_{i_3...i_n} \right| \left(f_{(1)} - f_{(0)} \right). \tag{15}$$

Having in view that $\frac{1}{2} \sum_{i_3,...,i_n \in N^*} \left| \delta_{i_3...i_n} \right| = \sup \sum_{i_3,...,i_n \in A} \delta_{i_3...i_n}$, where the supremum is taken over all $A \subset (N^*)^{n-2}$, it follows that $\frac{1}{2} \sum_{i_3,...,i_n \in N^*} \left| \delta_{i_3...i_n} \right| \ge \frac{1}{2} \sum_{i \in N^*} \left| \delta_i \right|$.

Hence the right-hand side (15) does not have the limit 0 as $n \to \infty$. Thus (15) is useless for $n > 3$.

As a matter of fact, it is a general result which does not take into account that f is non-descreassing

$$\delta^{(n)} f \le \delta^n (f_n) (f_{(1)} - f_{(0)} \ n \ge 3, \tag{16}$$

where

$$f_{2m+1}(x) = \begin{cases} 1, & \text{if } c_{2m}/c_{2m+1} \le x \le 1 \\ 0, & \text{if } 0 \le x < c_{2m}/c_{2m+1} \end{cases}$$

and

$$f_{2m+2}(x) = \begin{cases} 1, & \text{if } c_{2m+1}/c_{2m+2} < x \le 1 \\ 0, & \text{if } 0 \le x < c_{2m+1}/c_{2m+2}, \end{cases}$$

for all $m \in N^*$. Here $c_n, n \in N$ are the Fibonacci numbers defined by

$$c_0 = c_1 = 1, \;\; c_n = c_{n-1} + c_{n-2}, \; n \ge 2.$$

We now can state:

Theorem 1.1. *If (16) holds, then, for any monotone function f, we have*

$$varU^n f \le \gamma_n varf \;\text{ for any } n \ge 3, \tag{17}$$

where $\gamma_n = \delta^{(n)}(f_n) + \sum\limits_{i_1 \in N^*} \alpha_{(i_1+1)1\ldots 1}$.

The constant γ_n cannot be lowered.

The proof is immediate using (13) on account of the fact that $\alpha_{(i_1+1)1i_3\ldots i_n} < \alpha_{(i_1+1)1\ldots 1}$ for all $(i_3, ..., i_n) \ne (1, 1, ..., 1)$, which follows from (5). Finally, using (6) and (7), it is easy to check that

$$varU^n f_n = \gamma_n varf_n.$$

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