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TORSION FREE EXTERIOR POWERS OF A MODULE AND THEIR RESOLUTIONS

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Abstract

We study the q-torsion freeness of (q a positive integer) of the exterior powers of a finitely generated module E over a commutative ring, of finite projective dimension. We obtain results by utilizing suitable associated complexes.

INTRODUCTION

Let E be a finitely generated module on a commutative noetherian ring R with unit element. Many results about the torsion freeness of the symmetric powers of E can be deduced from syzygietic properties of the module E (cf. [2], [3], [10], [11]). This paper contains results for the torsion freeness of the exterior powers of E, when the resolution of E is given. More precisely we give necessary and sufficient conditions for the q-torsion freeness of symmetric and exterior powers of a module E, modulo bounds on the grade of all th-determinantal ideals that are present in the resolution of E, when these powers have acyclic resolution given by the Weyman-Tchernev complexes [11]. For a module E of projective dimension is one, we particularize the results and we succeed to find necessary and sufficient conditions for the q-torsion freeness of the exterior powers of E, under weaker hypotheses. As a corollary, we obtain a global result for q-torsion freeness of the exterior algebra of a module E of rank r and such that its r-th exterior power is a torsion- free cyclic R-module.

Key Words: Approximation complex, Cohen-Macaulay ring, Symmetric Algebra, Exterior Algebra Approximation complex, Cohen-Macaulay ring, Symmetric Algebra, Exterior Algebra

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Let R be a commutative noetherian ring with unit element and let E be a finitely generated R-module. We shall denote by $\wedge E$ and $Sym_R(E)$ or simply $S_R(E)$ the exterior, respectively, the symmetric algebra of E on R. Their degree t component will be denoted by $\overset{t}{\wedge} E$, respectively $S_t(E)$.

Definition 1 *We shall say that E has rank r if, denoting by $Q(R)$ the total quotient ring of R, $E \otimes_R Q(R)$ is a free $Q(R)$-module of rank r.*

Definition 2 *Let R be a ring and E a finitely generated R-module. We say that E is q- torsion free if every R-regular sequence of length q is also E-regular.*

Proposition 1.1 *Let E be a finitely generated R-module, $q \geq 1$ an integer. Consider the following conditions:*

(a_q) *E is q-torsion free;*

(b_q) *for every prime ideal $\wp$ of R, $depthE_\wp \geq min\{q, depthE_\wp\}$;*

(c_q) *E is a q-th syzygy, i.e. there is an exact sequence of free R-modules of as the following*

$$E \to G_q \to G_{q-1} \to ... \to G_1 \to 0.$$

We have $(c_q) \Rightarrow (b_q) \Rightarrow (a_q)$. If $pd_R E < \infty$, then (a_q),(b_q),(c_q) are equivalent.

Proof : cf. [1], [2], [9].

In all this section, we suppose that the ring R contains a field of characteristic zero and the projective dimension of the finitely generated R-module E is finite.

Let $F. : 0 \to F_n \overset{f_n}{\to} F_{n-1} \overset{f_{n-1}}{\to} F_{n-2} \to ... \to F_1 \overset{f_1}{\to} F_0$ be a finite free resolution of E, where $E = Coker f_1$. In [10] and [11], $\forall i > 0$ the complexes $\mathcal{S}_i(F.)$ and $\mathcal{L}_i(F.)$ are considered. With some assumptions, they are finite resolutions of $S_i(E)$ and $\overset{i}{\wedge} E$ when R contains a field k of characteristic zero. More precisely, we denote by $I_t(f_j)$ the determinantal ideal of $t \times t$-minors of a matrix that represents f_j and $r_j = rank(F_j)$. The grade of an ideal I of R is the length of a maximal R-sequence contained in I. The $length\mathcal{S}_i(F)$ (respectively $length\mathcal{L}_i(F)$) is the length of the complex $\mathcal{S}_i(F)$ (respectively $\mathcal{L}_i(F)$).

Theorem 1.2 *Let* $F. : 0 \to F_n \stackrel{f_n}{\to} F_{n-1} \stackrel{f_{n-1}}{\to} F_{n-2} \to ... \to F_1 \stackrel{f_1}{\to} F_0$ *be a finite free resolution of* E*,where* $E = Coker f_1$*. Then:*

1) $\mathcal{S}_i(F.)$ *is acyclic if and only if:*

- **i)** *for all* j *even,* $gradeI_{r_j}(f_j) \geq ji$;
- **ii)** *for all* j *odd,*
  $gradeI_{r_j-i+1}(f_j) \geq ji$, $gradeI_{r_j-i+2}(f_j) \geq ji-1$,
  $gradeI_{r_j}(f_j) \geq (j-1)i+1$.

2) $\mathcal{L}_i(F.)$ *is acyclic if and only if:*

- **i)** *for all* j *even,* $gradeI_{r_j}(f_j) \geq ji$;
- **ii)** *for all* j *odd,*
  $gradeI_{r_j-i+1}(f_j) \geq ji$, $gradeI_{r_j-i+2}(f_j) \geq ji-1$,
  $gradeI_{r_j}(f_j) \geq (j-1)i+1$.

If $\mathcal{S}_i(F.)$ *is exact, it is a finite free resolution of the symmetric power* $S_i(E)$*. If* $\mathcal{L}_i(F.)$ *is exact, it is a finite free resolution of the exterior power* $\overset{i}{\wedge} E$*.*

Proof : cf. [11].

Definition 3 *Let* $F. : 0 \to F_n \stackrel{f_n}{\to} F_{n-1} \stackrel{f_{n-1}}{\to} F_{n-2} \to ... \to F_1 \stackrel{f_1}{\to} F_0$ *be a finite free resolution of* E*, where* $E = Coker f_1$*. We say that* E *satisfies the property* (SW_i) *(respectively* (EW_i)*) if* $\mathcal{S}_i(F.)$ *(respectively* $\mathcal{L}_i(F.)$*) is a finite free resolution of* $S_i(E)$ *(respectively* $\overset{i}{\wedge} E$*).*

Theorem 1.3 *Let* $F. : 0 \to F_n \stackrel{f_n}{\to} F_{n-1} \stackrel{f_{n-1}}{\to} F_{n-2} \to ... \to F_1 \stackrel{f_1}{\to} F_0$ *be a finite free resolution of* E*,where* $E = Coker f_1$*. The following statement are equivalent:*
a)

1) $S_i(E)$ *is* q*-torsion free and* E *satisfies* (SW_i)*.*

2) *For all* j*,*

   if j *is odd,* $gradeI_{r_j}(f_j) \geq ji+q$;
   if j *is even,* $gradeI_{r_j-i+1}(f_j) \geq ji+q$, $gradeI_{r_j-i+2}(f_j) \geq ji-1+q$,
   $gradeI_{r_j}(f_j) \geq (j-1)i+q+1$.

b)

1) $\overset{i}{\wedge} E$ *is q-torsion free and E satisfies (EW_i).*

2) *for all j,*

if j is odd, $gradeI_{r_j}(f_j) \geq ji + q$;

if j is even, $gradeI_{r_j-i+1}(f_j) \geq ji + q$, $gradeI_{r_j-i+2}(f_j) \geq ji - 1 + q$, $gradeI_{r_j}(f_j) \geq (j-1)i + q + 1$.

Proof : We prove *a)*. The proof of *b)* is similar. 1) $\Rightarrow$ 2) Since $S_i(E)$ is q-torsion free, the exact sequence that gives the resolution of $S_i(E)$ can be prolonged to right by an exact sequence of q free modules. The result comes from the exactness criterion of Buchsbaum-Eisenbud for complexes [4] of free modules and from Peskine and Szpiro lemma [7].
2) $\Rightarrow$ 1) Let $\wp \in Spec(R)$ such that $depthR_\wp \geq length\mathcal{S}_i(F.) + q$.
We have $pd_R S_i(E) \leq length\mathcal{S}_i(F.)$. Then

$$depth\mathcal{S}_i(E)_\wp = depthR_\wp - pd_R S_i(E)_\wp \geq q = min\{q, depth(R_\wp)\}.$$

Let $\wp \in Spec(R)$ such that $depth(R_\wp) < length\mathcal{S}_i(F) + q$. We proceed by induction on n. For $n = 1$, the assertion is in [2]. For $n > 1$, by induction on $rank(F_n)$. If n is even, we have $gradeI_1(f_n) \geq length\mathcal{S}_i(F.) + q = ni + q$, then $I_1(f_n) \subsetneqq \wp$. By changing the bases in F_n and F_{n-1}, at all the localizations $R_\wp$, with $depthR_\wp < ni + q$, we have:

$$0 \to F_n^{'} \oplus R \to F_{n-1}^{'} \oplus R \to ...$$

One concludes by dividing by R. For n odd, we have $gradeI_n(f_n) \geq length\mathcal{S}_i(F.) + q$, hence $gradeI_1(f_n) \geq ni + q$. Then we can divide for R at all the localizations $R_\wp$ with $depthR_\wp < length\mathcal{S}_i(F.) + q$. We conclude by induction on $rank(F_n)$.

Proposition 1.4 *Let E be a finitely generated R-module, $q \geq 1$ be an integer.*

1) *If $S_t(E)$ is q-torsion free and E satisfies (SW_t), then $S_i(E)$ is q-torsion free and E satisfies (SW_i) for every $i < t$*

2) *If $\overset{t}{\wedge} E$ is q-torsion free and E satisfies (EW_t), then $\overset{i}{\wedge} E$ is q-torsion free and E satisfies (EW_i) for every $i < t$*

Proof : It follows from *theorem 1.3*, *a)* and *b)*.

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In all this section we consider a finitely generated R-module E of projective dimension one.

Theorem 2.1 *Let E be a finitely generated R-module with resolution:*

$$0 \to R^m \xrightarrow{f_1} R^n \xrightarrow{f_0} E \to 0.$$

Let $i \geq 0$ and $q \geq 1$ be two integer such that $\forall \wp \in Spec(R)$, $depthR_\wp < i+q$, $gradeI_1(f_1)_\wp \geq i+q$. Then the following facts are equivalent:

1) *$gradeI_m(f_1) \geq i+q$.*

2) *$\overset{i}{\wedge} E$ is q-torsion free and $\forall \wp \in Spec(R), depthR_\wp \geq i+q$, $gradeI_m(f_1)_\wp \geq i+q$.*

Proof : 1) $\Rightarrow$ 2) Since $depthI_m(f_1) \geq i+q > i$, then

$$\mathcal{L}_i(F.) : 0 \to D_i(R^m) \xrightarrow{f_i} D_{i-1}(R^m) \otimes R^n \to ... \to \overset{i}{\wedge} R^n \to \overset{i}{\wedge} E \to 0$$

is an exact complex and a resolution of $\overset{i}{\wedge} E$ ([11], theorem 1 or theorem 1.2, 2)), where $D_i(R^m)$ is the i-th divided power of R^m (see:[11]). The length of $\mathcal{L}_i(F.)$ equals i and $\mathcal{L}_i(F.)$ is a minimal resolution of $\overset{i}{\wedge} E$, if the resolution of E is minimal.
We have to prove that $\overset{i}{\wedge} E$ is q-torsion free, that is, $\forall \wp \in Spec(R)$,

$$depth(\overset{i}{\wedge} E)_\wp \geq min\{q, depthR_\wp\}$$

Case 1: Let $\wp \in Spec(R)$ such that $depthR_\wp \geq pd_R \overset{i}{\wedge} E + q$.
Then $depth(\overset{i}{\wedge} E)_\wp = depthR_\wp - pd_R(\overset{i}{\wedge} E)_\wp \geq pd_R \overset{i}{\wedge} E + q - pd_R(\overset{i}{\wedge} E)_\wp$.
Since $pd_R(\overset{i}{\wedge} E)_\wp \leq pd_R \overset{i}{\wedge} E$

$$depth(\overset{i}{\wedge} E)_\wp \geq pd_R \overset{i}{\wedge} E + q - pd_R \overset{i}{\wedge} E = q = min\{q, depthR_\wp\}$$

Case 2: Let $\wp \in Spec(R)$ such that $depthR_\wp < pd_R(\overset{i}{\wedge} E) + q$.
Since $gradeI_m(f_1) \geq i+q$, $I_m(f_1) \nsubseteq \wp$ and this implies $I_1(f_1) \nsubseteq \wp$. In fact, if $I_1(f_1) \subset \wp$,$I_m(f_1) \subset I_1(f_1)$ and then $I_m(f_1) \subset \wp$, contradiction. If $I_1(f_1) \nsubseteq \wp$, there exists an entry a_{ij}, $1 \leq i \leq m$ and $1 \leq j \leq n$, of the matrix that represents f_1, that is invertible in R. We can suppose that this entry is a_{11} after a

change of rows and columns of the matrix.

Then we can change the bases in R^m and R^n in such a way that $f_1 = f^{'} \oplus 1 : R^{m-1} \oplus R \to R^{n-1} \oplus R$ and $I_m(f_1) = I_{m-1}(f^{'})$. We proceed by induction on n. If $n = 0$ the assertion follows, because E is a free module. After a change of the bases, we have for E the presentation:

$$0 \to R^{m-1} \to R^{n-1} \to E \to 0$$

and by $I_m(f_1) = I_{m-1}(f^{'})$, we have: $gradeI_m(f_1) = grade_{m-1}I(f^{'}) \geq i + q$, hence the assertion.
2) $\Rightarrow$ 1) By induction on $rank(R^m)$. For $m = 1$ the assertion is true by hypothesis. Suppose $m > 1$. We prove that $gradeI_m(f_1)_\wp \geq i + q, \forall \wp \in Spec(R)$. By assumption, we have the assertion for all $\wp \in Spec(R)$, $depthR_\wp \geq i + q$. Then we have to prove that $\forall \wp \in Spec(R), depthR_\wp < i+q, gradeI_m(f_1)_\wp \geq i+q$. By our assumptions, $gradeI_1(f_1)_\wp \geq i + q$, so that $I_1(f_1) \nsubseteq \wp$ and so $I_1(f_1) \nsubseteq \wp$. As in the preceding proof, we obtain a presentation of $E_\wp$ of the form

$$0 \to R_\wp^{m-1} \stackrel{f_1^{'}}{\to} R_\wp^{n-1} \to E_\wp \to 0$$

and $gradeI_{m-1}(f_1^{'})_\wp = gradeI_m(f_1)_\wp$. We have moreover $depth(\stackrel{t}{\wedge} E)_\wp \geq q = min\{q, depthR_\wp\}$, then the conclusion follows by induction on m.

Remark 2.2 Let $(R, \mathfrak{m})$ *be a local ring containing a field* k. *Let* $i > 0$ *an integer for which one of the conditions* a) *and* b) *is true. Then we must have that* $i \leq rank(E) - 1 - q$.

Proof : Suppose $i > rank(E) - 1 - q$, then $i + q > rank(E) - 1$, $gradeI_m(f_1) > rank(E) - 1$ and $depthR > rank(E) - 1$, $depthR \geq rank(E)$. But this implies E is a free R-module, by syzygy theorem [1].

Proposition 2.3 *Let* E *be an* R*-module of rank* r *with resolution*

$$F. : 0 \to R^m \stackrel{f_1}{\to} R^n \stackrel{f_0}{\to} E \to 0$$

and let $i \geq 1$ *be an integer. The following statement are equivalent:*

1) $\forall \wp \in Spec(R)$, $depthR_\wp < i$, $I_1(f_1) \nsubseteq \wp$ *and* $I_1(f_1^{'}) \nsubseteq \wp$ *for any application* $f_1^{'}$ *given by a sub-matrix of the matrix that represents* f_1.

2) $\mathcal{L}_i(F.)$ *is exact and* $gradeI_m(f_1) \geq i$.

Proof: 1) $\Rightarrow$ 2) By induction on m. For $m = 1$, we have the resolution

$$0 \to R \xrightarrow{f_1} R^n \xrightarrow{f_0} E \to 0.$$

Since $I_1(f_1) \subsetneqq \wp$, $E_\wp \cong R_\wp^{n-1}$, hence $(\mathcal{L}_i(F.))_\wp$ is exact and by Peskine-Szpiro [7], $\mathcal{L}_i(F.)$ is exact. This forces $gradeI_1(f_1) \geq i$. Since $I_1(f_1) \subsetneqq \wp$, localizing $F.$ at the prime ideal $\wp$, we have:

$$0 \to R_\wp^{m-1} \oplus R_\wp \xrightarrow{f_1^{'} \oplus id} R_\wp^{n-1} \oplus R_\wp \to E_\wp \to 0$$

and $I_{m-1}(f_1^{'}) = I_m(f_1)_\wp$. We conclude by the inductive hypothesis.
2) $\Rightarrow$ 1) Let $gradeI_m(f_1) \geq i$, hence $gradeI_1(f_1) \geq i$ and $I_1(f_1) \subsetneqq \wp$, $\forall \wp \in Spec(R)$, $depthR_\wp < i$. By localization at $\wp$, we have that $\mathcal{L}_i(F.^{'})$ is acyclic where

$$F.^{'} : 0 \to R_\wp^{m-1} \xrightarrow{f_1^{'}} R_\wp^{n-1} \to E_\wp \to 0$$

and $gradeI_m(f_1^{'}) \geq i$. Hence $gradeI_1(f_1^{'}) \geq i$, and $I_1(f_1^{'}) \subsetneqq \wp$. This process can be continued and we have the assertion.

Remark 2.4 If $\overset{l}{\wedge} E$ *non zero for* $l > rank(E) = r$, *it is useful to out down the highest exteriors powers of* E, *more precisely the powers* $\overset{l}{\wedge} E$, *for* $l > rank(E)$. *This may be done in several ways, for example by requiring that* $(\overset{r}{\wedge} E)$ *is a cyclic* R*-module.*

Proposition 2.5 *Let* E *be an* R*-module of rank* r. *Then*

1) *if* $\overset{r}{\wedge} E$ *is a free* R*-module, then* E *is a free* R*-module;*

2) *if* $\overset{r}{\wedge} E$ *is a cyclic* R*-module, then* $\overset{l}{\wedge} E = 0$, *for* $l > r$.

Proof: See [5].

Corollary 2.6 *($q = 1$) Let* E *be a finitely generated* R*-module of rank* r *and with resolution*

$$0 \to R^m \xrightarrow{f_1} R^n \xrightarrow{f_0} E \to 0.$$

Suppose that $\forall \wp \in Spec(R), depthR_\wp < r, gradeI_1(f_1)_\wp \geq r$ *and* $\overset{r}{\wedge} E$ *is a torsion free cyclic* R*-module. Then the following facts are equivalent:*

1) $gradeI_m(f_1) \geq r$.

2) *The exterior algebra* $\wedge E = \oplus_{i=0}^{r} \overset{i}{\wedge} E$ *is torsion free and* $\forall \wp \in Spec(R)$, $depthR_\wp \geq r$, *we have* $gradeI_m(f_1)_\wp \geq r$.

Proof : 1) $\Rightarrow$ 2) We have to consider only the exterior powers $\overset{i}{\wedge} E$ with $i < r$. The hypothesis $gradeI_m(f_1) \geq (r-1)+1$ implies that $\mathcal{L}_i(F.)$ acyclic (*Theorem 1.2*, 2)) and $\overset{i}{\wedge} E$ 1-torsion free (*Theorem 2.1*), for all $i \leq r-1$. Finally, $\wedge E$ is torsion free.

2) $\Rightarrow$ 1) From *Theorem 2.1*, $gradeI_m(f_1) \geq i+1$, for all $i \geq r-1$, hence 1).

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