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Linear operators on Köthe spaces of vector fields

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Abstract

The study of Köthe spaces of vector fields was initiated by the present authors. In this paper linear operators on these spaces are studied. An integral representation theorem is given and special types of linear operators are introduced and studied.

1. INTRODUCTION

The theory of Köthe spaces is a generalization of the theory of the Lebesgue spaces L^p, being more general than the theory of Orlicz spaces which generalize the L^p spaces too. The theory of vector fields grew up from considerations inspired by differential geometry and mechanics, being by far more general that the generic theories.

In his seminal papers [4], [5], [6] and [7], N. Dinculeanu introduced and studied the Orlicz spaces of vector fields and the linear operations on them. A systematic exposure of this theory is contained in the monograph [8] by the same author.

Being inspired by the work of N. Dinculeanu (see also the fundamental monograph [9]), the present authors initiated in [2] a more general theory : the theory of Köthe spaces of vector fields.

Key Words: Function norm; Köthe space; Vector field; Continuous vector field; Measurable vector field; Linear and continuous operator; Integral representation; Dominated operator.

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The present paper is a continuation of [2], studying the linear (and continuous) operations on the Köthe spaces of vector fields.

We present the integral representation of such operations, in the spirit of a theory initiated by the first of the present authors in [1]. The paper continues with the study of some special types of linear operators on Köthe spaces of vector fields, the dominated operators being among them. In the final part we consider vector fields of linear operators, Köthe spaces of such vector fields and we construct a special type of linear operators generated by these Köthe spaces. A special example of such an operator is extensively studied at the end of the paper, including the exhibition of its representing measure.

2. PRELIMINARY FACTS.

I. Throughout the paper, K will be the scalar field (either $K = \mathbb{R}$ or $K = \mathbb{C}$) and the vector spaces will be considered over K. We shall write $\mathbb{R}_+ \overset{def}{=} [0, \infty)$ and $\overline{\mathbb{R}_+} \overset{def}{=} [0, \infty]$. As usual, $\mathbb{N} = \{0, 1, ...\}$ = the natural numbers and $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$.

If $(x_n)_n$ is a sequence such that $x_n \in X$ for any n, we shall write $(x_n)_n \subset X$. A topological space X is called separable (or of countable type) if there exists a sequence $(x_n)_n \subset X$ such that the set $\{x_n \mid n \in \mathbb{N}\}$ is dense in X. If X is a topological space and $a \in X$, we shall denote by $\mathcal{V}(a)$ the set of all neighborhoods of a.

Assuming that (X, d) is a semimetric space, $A \subset X$ is a dense set, (Y, ρ) is a complete metric space and $f : A \to Y$ is a uniformly continuous function, one knows that there exists an unique uniformly continuous $F : X \to Y$ such that $f = F|_A$ = the restriction of F to A.

II. Let (X, p) be a seminormed space. The null space of p is

$$Ker(p) = \{x \in X \mid p(x) = 0\}.$$

The quotient vector space

$$\widetilde{X} \overset{def}{=} X/_{Ker(p)}$$

becomes a normed space when equipped with the norm

$$||\widetilde{x}|| \overset{def}{=} p(x)$$

for any representative $x \in \widetilde{x}$. We call $(\widetilde{X}, || \cdot ||)$ the <u>*associate normed space of*</u> (X, p).

The space $(\widetilde{X}, ||\cdot||)$ is Banach if and only if (X,p) if a complete semimetric space (i.e. for any Cauchy sequence $(x_n)_n \subset X$ there exists at least one element $x \in X$ such that $x_n \to x$).

Now let us consider a seminormed space (X,p) and a normed space $(Y, |||\cdot|||)$.

As usual, we write

$$L(X,Y) = \{V : X \to Y \mid V \text{ is linear}\}$$

$$\mathcal{L}(X,Y) = \{V : X \to Y \mid V \text{ is linear and continuous}\}.$$

An element $V \in L(X,Y)$ is in $\mathcal{L}(X,Y)$ if and only if there exists $M \in \mathbb{R}_+$ such that

$$|||V(x)||| \leq Mp(x)$$

for any $x \in X$.

The space $\mathcal{L}(X,Y)$ is a normed space, when equipped with the usual operator norm

$$||V||_o = sup\{\ |||\ V(x)\ |||\ \mid x \in X, p(x) \leq 1\}.$$

With this norm, $\mathcal{L}(X,Y)$ becomes a Banach space whenever $(Y, |||\cdot|||)$ is Banach. Considering the above defined normed space $(\widetilde{X}, ||\cdot||)$, we can identify the spaces $\mathcal{L}(X,Y)$ and $\mathcal{L}(\widetilde{X},Y)$ as follows :

For any $V \in \mathcal{L}(X,Y)$, let us define $\widetilde{V} : \widetilde{X} \to Y$, via $\widetilde{V}(\widetilde{x}) \overset{def}{=} V(x)$ for any representative $x \in \widetilde{x}$. The definition is coherent. We got the linear and continuous operator $\widetilde{V} : \widetilde{X} \to Y$, acting as above.

It is seen that the map $\Omega : \mathcal{L}(X,Y) \to \mathcal{L}(\widetilde{X},Y)$, given via $\Omega(V) = \widetilde{V}$ is a linear and isometric ($||V||_o = ||\widetilde{V}||_o$) isomorphism. So, in almost all cases, instead of studying $\mathcal{L}(\widetilde{X},Y)$, one studies $\mathcal{L}(X,Y)$.

The reader can consult [10] for this part.

III. Assume (S, Σ, μ) is a measure space (i.e. S is a non empty set, Σ is a σ-algebra of subsets of S and $\mu : \Sigma \to \overline{\mathbb{R}_+}$ is a non null and complete measure.

Write

$M_+(\mu) = \{u : S \to \overline{\mathbb{R}_+} \mid u \text{ is } \mu \text{ - measurable}\}$.

A *function norm* on (S, Σ, μ) is a function $\rho : M_+(\mu) \to \overline{\mathbb{R}_+}$ having the following properties (here u, v are in $M_+(\mu)$ and $\alpha \in \mathbb{R}_+$) :

(i) $\rho(u) = 0$ if and only if $u(t) = 0$ μ - a.e. ;

(ii) $u \leq v \Rightarrow \rho(u) \leq \rho(v)$;

(iii) $\rho(u+v) \leq \rho(u) + \rho(v)$;

(iv) $\rho(\alpha\, u) = \alpha\, \rho(u)$,

with the convention $0 \cdot \infty = 0$.

One knows that $\rho(u) < \infty \Rightarrow u(t) < \infty$ μ - a.e. and $u(t) \leq v(t)$ μ - a.e. $\Rightarrow \rho(u) \leq \rho(v)$ (hence $u(t) = v(t)$ μ - a.e. $\Rightarrow \rho(u) = \rho(v)$).

We say that ρ has the *Riesz - Fischer property* (and write $\rho\ R-F$) in case

$$\rho(\sum_{n=0}^{\infty} u_n) \leq \sum_{n=0}^{\infty} \rho(u_n)$$

for any sequence $(u_n)_n \subset M_+(\mu)$.

We say that ρ has the *Fatou property* (and write $\rho\ F$) in case

$$\rho(\sup_n u_n) = \sup_n \rho(u_n)$$

for any increasing sequence $(u_n)_n \subset M_+(\mu)$.

It is known that $\rho\ F \Rightarrow \rho\ R-F$, the converse implication not being true.

For any $A \subset S$ the characteristic (indicator) function of A is φ_A. For any $A \in \Sigma$, we shall write

$$\rho(A) \stackrel{def}{=} \rho(\varphi_A).$$

Now write

$$M(\mu) = \{f : S \to K \mid f \text{ is } \mu- \text{ measurable}\}$$

$$\mathcal{L}_\rho = \{f \in M(\mu) \mid \rho|f| < \infty\}$$

(we write $\rho|f| \stackrel{def}{=} \rho(|f|)$).

Then $\mathcal{L}_\rho$ is a vector seminormed space, equipped with the seminorm

$$f \mapsto \rho|f|.$$

The null space of this seminorm is

$$\begin{aligned} N(\mu) = \{\, f \in \mathcal{L}_\rho |\, \rho|f| = 0\} &= \{f \in M(\mu) | f(t) = 0\ \mu\ -\ a.e.\} \\ &= \{f : S \to K | f(t) = 0\ \ \mu - a.e.\} \end{aligned}$$

The associate normed space is

$$L_\rho \stackrel{def}{=} \mathcal{L}_\rho/_{N(\mu)}$$

(the equivalence involved is given via $f \sim g \Leftrightarrow \rho|f-g| = 0 \Leftrightarrow f(t) = g(t)$ μ - a.e.) and L_ρ is normed with the norm

$$\widetilde{f} \mapsto ||\widetilde{f}|| \stackrel{def}{=} \rho|f|$$

for any representative $f \in \widetilde{f}$.

One knows that L_ρ is Banach if and only if $\rho\ R-F$. The spaces L_ρ generalize the Lebesgue spaces $L^p(\mu)=L^p, 1\le p\le\infty$. Namely, for these spaces the generating function norm is $\rho=||\cdot||_p, 1\le p\le\infty$ and one knows that $\rho\ F$. At the same time, the spaces L_ρ generalize the Orlicz spaces.

The spaces L_ρ are called *Köthe spaces*. For their theory, see [1] and [11].

IV. Let T be a separated locally compact topological space. We consider a family $\mathcal{E}=(E_t)_{t\in T}$ of Banach spaces. On each E_t, the norm will be denoted by $||z||$, $z\in E_t$ (no confusion will occur).

A *vector field* (with respect to $\mathcal{E}$) is a function $x:T\to\bigcup_{t\in T}E_t$ such that

$$x(t)\in E_t$$

for any $t\in T$. The set of all vector fields will be denoted by $\mathcal{C}(\mathcal{E})$.

Clearly, $\mathcal{C}(\mathcal{E})$ is a vector space with respect to the pointwise defined operations :

$$(x,y)\mapsto x+y \text{ where } (x+y)(t)\overset{def}{=}x(t)+y(t)$$

$$(\alpha,x)\mapsto \alpha x \text{ where } (\alpha x)(t)\overset{def}{=}\alpha x(t)$$

for any $x,y\in\mathcal{C}(\mathcal{E})$ and $\alpha\in K$.

For any $x\in\mathcal{C}(\mathcal{E})$ one can define the function $|x|:T\to\mathbb{R}_+$ given via

$$|x|(t)=||x(t)||.$$

A *fundamental family of continuous vector fields* is a vector subspace $\mathcal{A}$ of $\mathcal{C}(\mathcal{E})$, satisfying the following axioms :

(A_1) For any $x\in\mathcal{A}$, the function $|x|$ is continuous.

(A_2) For any $t\in T$, the set $\{x(t)\mid x\in\mathcal{A}\}$ is dense in E_t.

Particular Case : *The Unicity Case.*

Assume E is a fixed Banach space and $E_t=E$ for any $t\in T$. Then, we shall say that *we are in the unicity case* $\mathcal{C}(E)$.

In this case :

- A vector field $x\in\mathcal{C}(E)$ is a function $x:T\to E$.

- One can take as a fundamental family of continuous vector fields $\mathcal{A}=$ all the constant functions $x:T\to E$. We shall write $\mathcal{A}=E$, identifying each function $x\in\mathcal{A}$ with the constant value $x(t)\in E$, $t\in T$.

Now, let us return to the general situation and let $\mathcal{A}$ be a fundamental family of continuous vector fields. We shall say that $x\in\mathcal{C}(\mathcal{E})$ is *continuous at $t_0\in T$ with respect to $\mathcal{A}$* if, for any $\epsilon>0$, there exists $V\in\mathcal{V}(t_0)$ and $y\in\mathcal{A}$ such that

$$||x(t)-y(t)||<\epsilon$$

for any $t \in V$.

If $\varnothing \neq A \subset T$ we say that x is *continuous on A* if x is continuous at any $a \in A$. In case $A = T$, we say that A is *continuous*. It is seen that any $x \in \mathcal{A}$ is continuous (with respect to $\mathcal{A}$).

For $x \in \mathcal{C}(\mathcal{E})$ and $t_0 \in T$:

- If x is continuous at t_0 (with respect to $\mathcal{A}$), it follows that the function $|x| : T \to \mathbb{R}$ is continuous at t_0.

- The vector field x is continuous at t_0 (with respect to $\mathcal{A}$) if and only if the vector field $x + \alpha y$ is continuous at t_0 (with respect to $\mathcal{A}$) for any $y \in \mathcal{A}$ and any $\alpha \in K$.

In the *unicity case* $\mathcal{C}(E)$, a vector field $x \in \mathcal{C}(E)$ (i.e. a function $x : T \to E$) is continuous at t_0 with respect to E (see above) if and only if x is continuous at t_0 in the usual sense.

V. In order to establish our working framework, we shall again consider that T is a separated locally compact space with Borel sets $\mathcal{B}$. Let $\mathcal{T}$ be a σ - algebra of subsets of T such that $\mathcal{B} \subset \mathcal{T}$ and let $\mu : \mathcal{T} \to \overline{\mathbb{R}_+}$ be a non null complete measure, which is regular and such that $\mu(A) < \infty$ for any compact $A \subset T$ (some people say that μ is a Radon measure). So, we have the measure space $(T, \mathcal{T}, \mu)$. We denote by $\mathcal{K}$ the class of all compact sets $H \subset T$.

We consider a family $\mathcal{E} = (E_t)_{t \in T}$ of Banach spaces, generating the space of vector fields $\mathcal{C}(\mathcal{E})$ and a fundamental family of continuous vector fields $\mathcal{A} \subset \mathcal{C}(\mathcal{E})$.

A vector field $x \in \mathcal{C}(\mathcal{E})$ is called *μ - measurable with respect to $\mathcal{A}$* (we say that x is $(\mathcal{A}, \mu)$ - measurable) if, for any $A \in \mathcal{K}$ and any $\epsilon > 0$, there exists $\mathcal{K} \ni A_\epsilon \subset A$ such that $\mu(A \setminus A_\epsilon) < \epsilon$ and x is continuous on A_ϵ with respect to $\mathcal{A}$.

Write :

$$\mathcal{M}(\mathcal{A}, \mu) = \{x \in \mathcal{C}(E) \mid x \text{ is } (\mathcal{A}, \mu) - \text{ measurable}\}$$

and notice that $\mathcal{M}(\mathcal{A}, \mu)$ is a vector subspace of $\mathcal{C}(\mathcal{E})$ having the following properties :

- If $\mathcal{C}_{\mathcal{A}}(\mathcal{E}) = \{x \in \mathcal{C}(\mathcal{E}) \mid x \text{ is continuous with respect to } A\}$, one has $\mathcal{C}_{\mathcal{A}}(\mathcal{E}) \subset \mathcal{M}(\mathcal{A}, \mu)$.

- If x, y are in $\mathcal{C}(\mathcal{E})$, such that $x \in \mathcal{M}(\mathcal{A}, \mu)$ and $y(t) = x(t)$ μ - a.e., then $y \in \mathcal{M}(\mathcal{A}, \mu)$.

- If $x \in \mathcal{M}(\mathcal{A}, \mathcal{E})$, it follows that the function $|x| : T \to \mathbb{R}$ is μ - measurable (i.e. $|x| \in M_+(\mu)$).

- If $(x_n)_n \subset \mathcal{M}(\mathcal{A}, \mathcal{E})$ and $x \in \mathcal{C}(\mathcal{E})$ is such that $x_n(t) \xrightarrow{n} x(t)$ μ - a.e., then $x \in \mathcal{M}(\mathcal{A}, \mathcal{E})$. Moreover (analogue of Egorov's theorem), it follows that, for any $A \in \mathcal{K}$ and any $\epsilon > 0$, there exists $\mathcal{K} \ni A_\epsilon \subset A$ such that $\mu(A \backslash A_\epsilon) < \epsilon$ and $(x_n)_n$ converges uniformly to x on A_ϵ.

In the unicity case, one can prove (analogue of Lusin's theorem) that a vector field $x \in \mathcal{C}(E)$, i.e. a function $x : T \to E$, is (E, μ) - measurable if and only if x is μ - measurable. In this case we write

$$M_E(\mu) \stackrel{def}{=} \mathcal{M}(E, \mu).$$

We are prepared to introduce the framework which will be used in the sequel.

Framework

Assume T is a separated locally compact space, with Borel sets $\mathcal{B}$, and $(T, \mathcal{T}, \mu)$ is a measure space, with $\mathcal{T} \supset \mathcal{B}$ and μ non null, complete, regular and such that $\mu(A) < \infty$ for any $A \in \mathcal{K}$. We shall also consider a function norm ρ on $(T, \mathcal{T}, \mu)$.

Let $\mathcal{E} = (E_t)_{t \in T}$ a family of Banach spaces, thus obtaining $\mathcal{C}(\mathcal{E})$ and let $\mathcal{A} \subset \mathcal{C}(\mathcal{E})$ be a fundamental family of continuous vector fields.

The *seminormed Köthe space of vector fields* $\mathcal{L}_\rho(\mathcal{E}, \mathcal{A})$ is defined as follows :

$$\mathcal{L}_\rho(\mathcal{E}, \mathcal{A}) \stackrel{def}{=} \{x \in \mathcal{M}(\mathcal{A}, \mu) \mid \rho|x| < \infty\}.$$

Clearly, $\mathcal{L}_\rho(\mathcal{E}, \mathcal{A})$ is a vector subspace of $\mathcal{M}(\mathcal{A}, \mu)$, seminormed with the seminorm given via $x \mapsto \rho|x|$. The null space of this seminorm is

$$N_\rho(\mathcal{E}, \mathcal{A}) = \{x \in \mathcal{M}(\mathcal{A}, \mu) \mid \rho|x| = 0\} = \{x \in \mathcal{C}(\mathcal{E}) \mid x(t) = 0 \ \mu - a.e.\}.$$

The quotient space

$$L_\rho(\mathcal{E}, \mathcal{A}) \stackrel{def}{=} \mathcal{L}_\rho(\mathcal{E}, \mathcal{A})/_{N_\rho(\mathcal{E}, \mathcal{A})}$$

(the equivalence relation is given via $x \sim y \Leftrightarrow x(t) = y(t)\ \mu$ -a.e.) is normed with the norm $\widetilde{x} \mapsto ||\widetilde{x}|| = \rho|x|$ for any representative $x \in \widetilde{x}$.

We call the normed space $(L_\rho(\mathcal{E}, \mathcal{A}), || \cdot ||)$ *Köthe space of vector fields*.

One can prove that, in case $\rho\ R - F$, the space $L_\rho(\mathcal{E}, \mathcal{A})$ is Banach (Theorem 3 in [2]).

Two *particular cases* are of special interest :

1. Assume we are in the *unicity case* $\mathcal{C}(E)$. We shall write in this case

$$\mathcal{L}_\rho(E, \mu) \stackrel{def}{=} \mathcal{L}_\rho(\mathcal{E}, E).$$

It is seen that

$$\mathcal{L}_\rho(E, \mu) = \{x \in M_E(\mu) \mid \rho|x| < \infty\}$$

seminormed with the seminorm $x \mapsto \rho|x|$.

The associated normed space in this case will be denoted as follows :

$$L_\rho(E, \mu) \stackrel{def}{=} L_\rho(\mathcal{E}, E)$$

normed with $\widetilde{x} \mapsto ||\widetilde{x}|| = \rho|x|, x \in \widetilde{x}$, which is Banach in case ρ $R - F$.

Of course :

$$\mathcal{L}_\rho(K, \mu) = \mathcal{L}_\rho \text{ and } L_\rho(K, \mu) = L_\rho.$$

2. Assume we have, for any $t \in T$, a measure space (S_t, Σ_t, μ_t) and a function norm ρ_t on (S_t, Σ_t, μ_t). We shall take, $E_t \stackrel{def}{=} L_{\rho_t}$, for any $t \in T$. We shall also assume ρ_t $R - F$ for any $t \in T$ (consequently, all E_t are Banach, as stipulated in the definition). Taking $\mathcal{E} = (E_t)_{t \in T}$, we can construct $\mathcal{C}(\mathcal{E})$ and let us consider a fundamental family of continuous vector fields $\mathcal{A} \in \mathcal{C}(\mathcal{E})$.

In this case we shall write :

$$\begin{aligned} \mathcal{L}_\rho\left((\rho_t)_t, \mathcal{A}\right) &\stackrel{def}{=} \mathcal{L}_\rho(\mathcal{E}, \mathcal{A}) \\ L_\rho\left((\rho_t)_t, \mathcal{A}\right) &\stackrel{def}{=} L_\rho(\mathcal{E}, \mathcal{A}). \end{aligned}$$

The most "normal" situation is that one when all the measure spaces (S_t, Σ_t, μ_t) are equal to the same (S, Σ, μ). In this case, the variability is furnished by $\rho_t, t \in T$.

Finally, we finish the presentation of this framework by mentioning the fact that, from now on, F will be a fixed Banach space.

For vector fields, see [8], [2] and the seminal papers [4], [5], [6], [7].

3. INTEGRAL REPRESENTATIONS

Assume the Framework described at the end of paragraph 2. We want to describe $\mathcal{L}(L_\rho(\mathcal{E}, \mathcal{A}), F)$, or, which is the same, $\mathcal{L}(\mathcal{L}_\rho(\mathcal{E}, \mathcal{A}), F)$.

In order to obtain an integral representation of the operators in $\mathcal{L}(\mathcal{L}_\rho(\mathcal{E}, \mathcal{A}), F)$, we shall use the "simple fields" (see infra) under supplementary assumptions.

Write first

$$\mathcal{C}(\rho) = \{A \in \mathcal{T} \mid \rho(A) < \infty\}$$

and notice that $\mathcal{C}(\rho)$ is a T - ring.

Definition 1. The <u>*simple fields*</u> are vector fields $x \in \mathcal{C}(\mathcal{E})$ of the form $x = \sum_{i=1}^{n} \varphi_{Ai} x_i$ hence

$$x(t) = \sum_{i=1}^{n} \varphi_{A_i}(t) x_i(t) \in E_t$$

for $t \in T$, where $A_i \in \mathcal{C}(\rho)$ and $x_i \in \mathcal{A}$. One can assume that the sets A_i are mutually disjoint, hence $x(t) = x_i(t)$ for any $t \in A_i$ and $x(t) = 0$ for any $t \notin \bigcup_{i=1}^{n} A_i$.

The set of all simple fields will be denoted by $\mathcal{E}_\rho(\mathcal{A})$. Obviously, $\mathcal{E}_\rho(\mathcal{A})$ is a vector subspace of $\mathcal{C}(\mathcal{E})$. In the *unicity case* $\mathcal{C}(E)$ with $\mathcal{A} \equiv E$, the elements in $\mathcal{E}_\rho(\mathcal{A}) \stackrel{def}{=} \mathcal{E}_\rho(E)$ have the form $x = \sum_{i=1}^{n} \varphi_{A_i} x_i$ with $A_i \in \mathcal{C}(\rho)$ and $x_i \in E$. In case $E = K$, we write $\mathcal{E}_\rho(K) \stackrel{def}{=} \mathcal{E}_\rho$.

Because $\mathcal{A} \in \mathcal{M}(\mathcal{A}, \mu)$, one has $\mathcal{E}_\rho(\mathcal{A}) \subset \mathcal{M}(\mathcal{A}, \mu)$. Practically, all the time we shall assume more, namely

$$(P_1) \quad \mathcal{E}_\rho(\mathcal{A}) \subset \mathcal{L}_\rho(\mathcal{E}, \mathcal{A}).$$

Assumption (P_1) is automatically verified in the *unicity case* $\mathcal{C}(E)$ or in case $\mathcal{A} \subset \mathcal{L}_\rho(\mathcal{E}, \mathcal{A})$.

Caution : *Throughout this paragraph we shall assume* (P_1).

In order to obtain a satisfactory integral representation, we shall use an assumption which is stronger that (P_1), namely

$$(P_2) \quad \mathcal{E}_\rho(\mathcal{A}) \text{ is dense in } \mathcal{L}_\rho(\mathcal{E}, \mathcal{A}).$$

Assumption (P_2) is automatically verified in the *unicity case* $\mathcal{C}(K)$, if the *function norm ρ is of absolutely continuous type*, i.e. : for any decreasing sequence $(u_n)_n$ in $M_+(\mu)$, such that $\rho(u_1) < \infty$ and $u_n \downarrow 0$ (which means $\lim_n u_n = 0$ pointwise), one has $\lim_n \rho(u_n) = \inf_n \rho(u_n) = 0$ (see [1]). For instance, if $1 \le p < \infty$, the function norms $||\cdot||_p$ are of absolutely continuous type, whereas $||\cdot||_\infty$ does not have this property in most cases (see [1]).

Write

$$L(\mathcal{A}, F) = \{H : \mathcal{A} \to F \mid H \text{ is linear}\}$$

and consider an additive measure $m : \mathcal{C}(\rho) \to L(\mathcal{A}, F)$ (i.e. $m(A \cup B) = m(A) + m(B)$, for any $A, B \in \mathcal{C}(\rho)$ such that $A \cap B = \varnothing$).

Definition 2. *The integral of* $f = \sum_{i=1}^{n} \varphi_{A_i} x_i \in \mathcal{E}_\rho(\mathcal{A})$ *with respect to* m is

$$\int f dm \stackrel{def}{=} \sum_{i=1}^{m} m(A_i)(x_i).$$

The definition does not depend on the representation of f. The integral furnishes the linear application

$$\int : \mathcal{E}_\rho(\mathcal{A}) \to F$$

given via

$$\int (f) \stackrel{def}{=} \int f dm.$$

We shall consider those additive measures $m : \mathcal{C}(\rho) \to L(\mathcal{A}, F)$ which make the above defined application $\int$ continuous, i.e. those m having the property

$$|||m||| = sup\{|| \int f dm|| \mid f \in \mathcal{E}_\rho(\mathcal{A}), \rho|f| \leq 1\} < \infty.$$

Namely, $\mathcal{E}_\rho(\mathcal{A})$ is equipped with the topology induced by the topology of $\mathcal{L}_\rho(\mathcal{E}, \mathcal{A})$. So, we shall consider the vector space

$$M_F(\mathcal{A}, \rho) = \{m : \mathcal{C}(\rho) \to L(\mathcal{A}, F) \mid m \text{ is additive and } |||m||| < \infty\}$$

normed with $m \mapsto |||m|||$. It follows that, for any $f \in \mathcal{E}_\rho(\mathcal{A})$ and any $m \in M_F(\mathcal{A}, \rho)$, one has

$$|| \int f dm|| \leq |||m|||\rho|f|.$$

Now, we are prepared for

Theorem 3. **(Integral Representation Theorem)** *Assume* (P_2). *Then, the Banach spaces :* $\mathcal{L}(\mathcal{L}_\rho(\mathcal{E}, \mathcal{A}), F)$ *with operator norm* $||V||_0$ *and* $M_F(\mathcal{A}, \rho)$ *with norm* $|||m|||$ *are linearly and isometrically isomorphic, via the isomorphisms :*

$$\begin{cases} a : \mathcal{L}\left(\mathcal{L}_\rho(\mathcal{E}, \mathcal{A}), F\right) \to M_F(\mathcal{A}, \rho) \\ a(V) = m, \text{ where } m(A)(x) = V\left(\varphi_A x\right), A \in \mathcal{C}(\rho), x \in \mathcal{A} \end{cases}$$

$$\begin{cases} b : M_F(\mathcal{A}, \rho) \to \mathcal{L}(\mathcal{L}_\rho(\mathcal{E}, \mathcal{A}), F) \\ b(m) = V, \text{ where } \ V(f) = \lim\limits_n \int f_n dm, f \in \mathcal{L}_\rho(\mathcal{E}, \mathcal{A}). \end{cases}$$

Here, $(f_n)_n \subset \mathcal{E}_\rho(\mathcal{A})$ *is a sequence having the property that* $f_n \xrightarrow{n} f$ *in* $\mathcal{L}_\rho(\mathcal{E}, \mathcal{A})$. *(the definition does not depend on the approximant sequence* $(f_n)_n$ *).*

The isomorphisms a and b are mutually inverse. We shall say that m represents V (or that m is a representing measure for V).

Proof : 1) We show that a is well defined. Let $V \in \mathcal{L}(\mathcal{L}_\rho(\mathcal{E},\mathcal{A}),F)$.

a) Let $A \in \mathcal{C}(\rho)$. We can define $m(A) : \mathcal{A} \to F$, via

$$m(A)(x) \overset{def}{=} V(\varphi_A x)$$

and $m(A) \in L(\mathcal{A},F)$. One sees immediately that m is additive, because $\varphi_{A\cup B}x = \varphi_A x + \varphi_B x$ whenever $A \cap B = \varnothing$, $x \in \mathcal{A}$.

b) For any $f = \sum_{i=1}^{n} \varphi_{A_i} x_i \in \mathcal{E}_\rho(\mathcal{A})$, one has

$$V(f) = \sum_{i=1}^{n} V(\varphi_{A_i} x_i) = \sum_{i=1}^{n} m(A_i)(x_i) = \int f dm$$

hence

$$|| \int f dm || = ||V(f)|| \leq ||V||_0 \rho|f|$$

which implies $a(V) \in M_F(\mathcal{A},\rho)$.

2) We show that b is well defined. Let $m \in M_F(\mathcal{A},\rho)$.

a) One can define $U : \mathcal{E}_\rho(\mathcal{A}) \to F$, via

$$U(f) = \int f dm$$

and U is linear and continuous, because

$$||U(f)|| = || \int f dm|| \leq ||m||\rho|f|.$$

b) Because F is a complete metric space and $\mathcal{E}_\rho(\mathcal{A})$ is dense in $\mathcal{L}_\rho(\mathcal{E},\mathcal{A})$, one can extend uniquely the linear and continuous operator U, obtaining the linear and continuous operator $b(m) = V : \mathcal{L}_\rho(\mathcal{E},\mathcal{A}) \to F$, according to the enunciation (use the extension theory for uniformly continuous maps).

3) We show that the linear applications a and b are mutually inverse.

a) Let us start with $V \in \mathcal{L}(\mathcal{L}_\rho(\mathcal{E},\mathcal{A}),F)$ and get $a(V) = m$. We must show that $W = V$, where $W = b(m)$. (and thus we show that $(b \circ a)(V) = V$).

For any $f = \sum_{i=1}^{n} \varphi_{A_i} x_i \in \mathcal{E}_\rho(\mathcal{A})$, we can take the approximant sequence $(f_n)_n$ from the enunciation to be constant : $f_n = f$ for any n. According to the definition :

$$W(f) = \int f dm = \sum_{i=1}^{n} m(A_i)(x_i) = \sum_{i=1}^{n} V(\varphi_{A_i} x_i) = V(f).$$

The linear and continuous maps W and V coincide on $\mathcal{E}_\rho(\mathcal{A})$, which is dense in $\mathcal{L}_\rho(\mathcal{E},\mathcal{A})$, hence $W = V$.

b) Now, let us start with $m \in M_F(\mathcal{A},\rho)$ and get $b(m) = V$. We must show that $u = m$, where $u = a(V)$. (and thus we show that $(a \circ b)(m) = m$).

According to the definition, for any $A \in \mathcal{C}(\rho)$ and any $x \in \mathcal{A}$, one has

$$u(A)(x) = V(\varphi_A x) = \int \varphi_A x dm = m(A)(x)$$

(approximating $\varphi_A x$ with the constant sequence $f_n = \varphi_A x$). Due to the fact that x is arbitrary, we get $m(A) = u(A)$, hence $u = m$. (A is arbitrary too.)

4) It remains to prove that a and b are isometries. Because they are mutually inverse, it suffices to prove that a is an isometry.

Let $V \in \mathcal{L}(\mathcal{L}_\rho(\mathcal{E},\mathcal{A}), F)$ and $m = a(V)$. Then

$$\begin{gathered} ||V||_0 = \sup\{||V(f)|| \,|f \in \mathcal{L}_\rho(\mathcal{E},\mathcal{A}) \;, \rho|f| \le 1\} \overset{!}{=} \\ \overset{!}{=} \sup\{||V(f)|| \,|f \in \mathcal{E}_\rho(\mathcal{A}) \;, \rho|f| \le 1\} = \\ = \sup\left\{|| \int f dm|| \,|f \in \mathcal{E}_\rho(\mathcal{A}) \;, \rho|f| \le 1\right\} = |||m|||. \end{gathered}$$

The critical equality $\overset{!}{=}$ is explained as follows. Let us write

$$B = \{f \in \mathcal{L}_\rho(\mathcal{E},\mathcal{A}) \mid \rho|f| \le 1\}$$

and use a sequential justification to see that (closure in $\mathcal{L}_\rho(\mathcal{E},\mathcal{A})$)

$$\overline{B \cap \mathcal{E}_\rho(\mathcal{A})} = B \cap \overline{\mathcal{E}_\rho(\mathcal{A})}.$$

(The reader can check that for any $f \in \mathcal{L}_\rho(\mathcal{E},\mathcal{A})$, with $\rho|f| = 1$, it is possible to find a sequence $(f_n)_n \subset \mathcal{E}_\rho(\mathcal{A})$ such that $f_n \xrightarrow{n} f$ and $\rho|f_n| = 1$).

The last set equals $B \cap \mathcal{L}_\rho(\mathcal{E},\mathcal{A}) = B$.

We have the real continuous function $f \mapsto ||V(f)||$ whose supremum on $B \cap \mathcal{E}_\rho(\mathcal{A})$ coincides with the supremum on $\overline{B \cap \mathcal{E}_\rho(\mathcal{A})}$ a.s.o.. ■

Remark. There is an important situation when assumption (P_2) holds, namely when the following two conditions are fulfilled :

(i) For any $A \in \mathcal{K}$, one has $\rho(A) < \infty$.

(ii) The function norm $\rho = ||\cdot||_p, 1 \leq p \leq \infty$.

Indeed, from (i) if follows that $\mathcal{B}_0 \subset \mathcal{C}(\rho)$, where $\mathcal{B}_0$ = the relatively compact Borel sets of T. We use property (9), pag. 549 of [8], asserting that the space $\mathcal{U}$ of all vector fields off the form $f = \sum_{i=1}^{n} \varphi_{A_i} x_i$ with $A_i \in \mathcal{B}_0$ and $x_i \in \mathcal{A}$ is dense in $\mathcal{L}_{||\cdot||_p}(\mathcal{E}, \mathcal{A})$.

Because of (ii), we have, for any $B \in \mathcal{B}_0$, $\rho(B) = \mu(B)^{1/p} < \mu(A)^{1/p} < \infty$ for any $\mathcal{K} \ni A \supset B$, hence $\mathcal{C}(\rho) \supset \mathcal{B}_0$, and this implies $\mathcal{U} \subset \mathcal{E}_\rho(\mathcal{A})$ and the last space must be also dense in $\mathcal{L}_{||\cdot||_p}(\mathcal{E}, \mathcal{A})$.

4. OPERATORS WITH SPECIAL PROPERTIES

Again we are within the Framework described at the end of paragraph 2.

Definition 4. A linear functional $H : \mathcal{L}_\rho \to K$ will be called positive if $H(f) \geq 0$ for any $f \geq 0$ in $\mathcal{L}_\rho$.

It is clear that, for positive H, one has $H(f) \geq H(g)$, whenever $f \geq g$ in $\mathcal{L}_\rho$.

Proposition 5. *Assume ρ $R-F$. Then any positive functional $H : \mathcal{L}_\rho \to K$ is continuous.*

Proof : Let us accept the existence of a positive functional $H : \mathcal{L}_\rho \to K$ which is not continuous. We shall arrive at a contradiction.

One can find a sequence $(f_n)_n \subset \mathcal{L}_\rho$, such that $\rho|f_n| \leq 1$ and $|H(f_n)| > n \cdot 2^n$ for any n. Let us define $f : T \to \overline{\mathcal{R}_+}$, via

$$f(t) = \sum_{n=1}^{\infty} \frac{1}{2^n} |f_n(t)|.$$

Because ρ $R - F$ one has

$$\rho|f| = \rho(f) \leq \sum_{n=1}^{\infty} \frac{1}{2^n} \rho|f_n| \leq \sum_{n=1}^{\infty} \frac{1}{2^n} = 1$$

and this implies $f(t) < \infty$ μ - a.e.. This enables us to define $u : T \to \mathbb{R}_+$ via

$$u(t) = \begin{cases} f(t), & \text{if } f(t) < \infty \\ 0, & \text{if } f(t) = \infty \end{cases}$$

hence $\mathcal{L}_\rho \ni u = f$ μ - a.e. and $\rho(u) \leq 1$. Because, for any $n \in \mathbb{N}^*$, one has μ-a.e. $u \geq \frac{1}{2^n}|f_n|$ it follows that

$$H(u) \geq H\left(\frac{1}{2^n}|f_n|\right) = \frac{1}{2^n} H(|f_n|) \geq n$$

i.e. $H(u) = \infty$, contradiction. See also Proposition 8, pag. 259 in [9]. ■

Definition 6. A linear operator $V : \mathcal{L}_\rho(\mathcal{E}, \mathcal{A}) \to F$ will be called dominated if there exists a positive functional $H : \mathcal{L}_\rho \to K$ such that $||V(f)|| \leq H(|f|)$ for any $f \in \mathcal{L}_\rho(\mathcal{E}, \mathcal{A})$. In this case we shall say that *V is dominated by H (or H dominates V).*

Proposition 7. *Assume $\rho\, R - F$. Then any dominated operator $V : \mathcal{L}_\rho(\mathcal{E}, \mathcal{A}) \to F$ is continuous.*

Proof: If V is dominated by H, let $(f_n)_n \subset \mathcal{L}_\rho(\mathcal{E}, \mathcal{A})$ be a sequence such that $f_n \overset{n}{\to} 0$, i.e. $|f_n| \overset{n}{\to} 0$ in $\mathcal{L}_\rho$. Because H is continuous (Proposition 5), it follows that $||V(f_n)|| \le H(|f_n|) \overset{n}{\to} 0$. ■

<u>Caution</u> : *From now on, we shall assume (P_1) throughout this paragraph.*
Let $V : \mathcal{L}_\rho(\mathcal{E}, \mathcal{A}) \to F$ be linear and continuous. We define :

$$\begin{aligned} ||V||_1 &\overset{def}{=} sup\,\{||V(f)||\,|f \in \mathcal{E}_\rho(A)\ , \rho|f| \le 1\} \\ |||V||| &\overset{def}{=} sup\left\{\sum_{i=1}^{n} ||V\left(\varphi_{A_i} x_i\right)||\,|f = \sum_{i=1}^{n} \varphi_{A_i} x_i \in \mathcal{E}_\rho(\mathcal{A}), \rho|f| \le 1\right\}. \end{aligned}$$

See also [9], pag. 256-257 (definitions of $\|U\|_A$, $|||U|||_A$), [8], pag. 572 and [3], Theorem 4.8, pag. 110, together with the comment at pag. 119.

Here we use the disjoint representation

$$f = \sum_{i=1}^{n} \varphi_{A_i} x_i$$

with $A_i \in \mathcal{C}(\rho)$ mutually disjoint. Because, for such representation, one has

$$||V(f)|| = ||\sum_{i=1}^{n} V(\varphi_{A_i} x_i)|| \le \sum_{i=1}^{n} ||V(\varphi_{A_i} x_i)||$$

it follows that

$$\begin{cases} \|V\|_1 \le & \|V\|_0 \\ \|V\|_1 \le & |||V||| \end{cases} \tag{1}$$

and, for any such $f = \sum_{i=1}^{n} \varphi_{A_i} x_i$:

$$\begin{cases} ||\sum_{i=1}^{n} V\left(\varphi_{A_i} x_i\right)|| = ||V(f)|| \le ||V||_1 \rho|f| \\ ||\sum_{i=1}^{n} V\left(\varphi_{A_i} x_i\right)|| \le |||V|||\rho|f| \end{cases} \tag{1'}$$

There exist linear and continuous operators V with $|||V||| = \infty$, as the following example shows.

Example 8 (see [9], pag. 256-257). *An operator V with $|||V||| = \infty$*

We shall work in the *unicity case* $\mathcal{C}(K)$, hence we have the space $\mathcal{L}_\rho = \mathcal{L}_\rho(K,\mu)$. We consider $1 < p < \infty$ and let us take $\rho = ||\cdot||_p$, hence $\mathcal{L}_\rho = \mathcal{L}^p(\mu)$, where μ is the Lebesgue measure on $T = [0,1]$. At the same time, let us take $F = L^p(\mu)$.

Finally, $V : \mathcal{L}^p(\mu) \to L^p(\mu)$ will be given via $V(f) = \widetilde{f}$ which is continuous ($||V(f)|| = \rho|f|$).

For any $n \in \mathbb{N}^*$, let $A_i = [\frac{i-1}{n}, \frac{i}{n})$, $i = 1,2,...,n-1$; $A_n = [\frac{n-1}{n}, \frac{n}{n} = 1]$ and $x_i = 1$, $i = 1,2,...,n$. Hence, we consider $f \equiv 1$ on $[0,1]$ with $\rho|f| = ||f||_p = 1$.

For any $i = 1,2,...,n$, $V(\varphi_{A_i} x_i) = V(\varphi_{A_i}) = \widetilde{\varphi_{A_i}}$, hence

$$||V(\varphi_{A_i} x_i)||_p = ||\widetilde{\varphi_{A_i}}||_p = \mu(A_i)^{\frac{1}{p}} = n^{-1/p}$$

$$\sum_{i=n}^{n} ||V(\varphi_{A_i} x_i)||_p = n \cdot n^{-1/p} = n^{1-\frac{1}{p}} \xrightarrow{n} \infty$$

hence $|||V||| = \infty$. ■

For linear and continuous functionals, the situation is totally different, namely the inequality $||V||_1 \leq |||V|||$ in (1) becomes equality.

Theorem 9. *Let $V : \mathcal{L}_\rho(\mathcal{E},\mathcal{A}) \to K$ be a linear and continuous functional. Then*

$$||V||_1 = |||V|||.$$

Proof: Let $f = \sum_{i=1}^{n} \varphi_{A_i} x_i \in \mathcal{E}_\rho(\mathcal{A})$ with $\rho|f| \leq 1$ (disjoint representation). For any $i = 1,2,...,n$, let $\theta_i \in K$ with $|\theta_i| = 1$ such that

$$|V(\varphi_{A_i} x_i)| = \theta_i V(\varphi_{A_i} x_i) = V(\varphi_{A_i} \theta_i x_i)$$

(we have $\theta_i x_i \in \mathcal{A}$ = vector space).

For $g = \sum_{i=1}^{n} \varphi_{A_i} \theta_i x_i \in \mathcal{E}_\rho(\mathcal{A})$ one has $|g| = |f|$, hence $\rho|g| \leq 1$ and

$$\sum_{i=1} |V(\varphi_{A_i} x_i)| = \sum_{i=1}^{n} V(\varphi_{A_i} \theta_i x_i) = |V(g)| \leq ||V||_1.$$

Because f is arbitrary, we obtain $|||V||| \leq ||V||_1$, hence $|||V||| = ||V||_1$. See also Proposition 5, pag. 256 in [9], and Proposition 28.36, pag. 252 in [8]. ■

Accepting (P_2), the set $B \cap \mathcal{E}_\rho(\mathcal{A})$ is dense in $B = \{f \in \mathcal{L}_\rho(\mathcal{E},\mathcal{A}) \mid \rho|f| \leq 1\}$ as we have seen and taking the supremum of $||V(f)||$ on $B \cap \mathcal{E}_\rho(\mathcal{A})$ we get another case of equality in (1), namely

Theorem 10. *Assume (P_2).*

1. Let $V : \mathcal{L}_\rho(\mathcal{E},\mathcal{A}) \to F$ be a linear and continuous operator. Then

$$||V||_0 = ||V||_1 \leq |||V|||$$

2. Let $V : \mathcal{L}_\rho(\mathcal{E},\mathcal{A}) \to K$ be a linear and continuous functional. Then

$$||V||_0 = ||V||_1 = |||V|||.$$

Our next goal is to study the connection between $||| \cdot |||$ and domination.

Theorem 11. *Assume $\rho\ R-F$. Let $V : \mathcal{L}_\rho(\mathcal{E},\mathcal{A}) \to F$ be a dominated operator. Then $|||V||| < \infty$.*

Proof : Let V be dominated by H.

For any $f = \sum_{i=1}^{n} \varphi_{A_i} x_i$ with $\rho|f| \leq 1$ (disjoint representation), one has

$$\sum_{i=1}^{n} ||V(\varphi_{A_i} x_i)|| \leq \sum_{i=1}^{n} H(|\varphi_{A_i} x_i|) \tag{2}$$

Because A_i are disjoint, one has $|f| = \sum_{i=1}^{n} \varphi_{A_i} |x_i|$, hence

$$\begin{aligned}\sum_{i=1}^{n} H(|\varphi_{A_i} x_i|) = \sum_{i=1}^{n} H(\varphi_{A_i} |x_i|) = H\left(\sum_{i=1}^{n} \varphi_{A_i} |x_i|\right) = \\ = H(|f|) \leq \|H\|_0\, \rho|f| \leq \|H\|_0\end{aligned} \tag{3}$$

(H is continuous, because $\rho\, R-F$, Proposition 5). Using (2) and (3) and taking supremum (f is arbitrary), one gets

$$|||V||| \leq ||H||_0 < \infty. \quad \blacksquare$$

It is natural to ask whether the converse of Theorem 11 is true. A partial converse is true, in the *unicity case* $\mathcal{C}(E)$. We shall write $|x|$ instead of $||x||$ for $x \in E$ and $|y|$ instead of $||y||$ for $y \in F$.

Theorem 12. *Assume (P_2), ρ is of absolutely continuous type and we are in the unicity case $\mathcal{C}(E)$. Let $V : \mathcal{L}_\rho(E,\mu) \to F$ be linear and continuous and such that $|||V||| < \infty$. Then V is dominated.*

Proof : 1) According to the integral Representation Theorem 3, let $m \in M_F(\mathcal{A}, \rho)$ represent V. Here we are in the unicity case $\mathcal{C}(E)$ with $\mathcal{A} \equiv E$, so, for any $A \in \mathcal{C}(\rho)$, one has $m(A) : E \to F$, given via $m(A)(x) = V(\varphi_A x)$.

It is seen that $m(A) \in \mathcal{L}(E, F)$ for any $A \in \mathcal{C}(\rho)$, because

$$|m(A)(x)| = |V(\varphi_A x)| \le ||V||_0 \rho|\varphi_A x| \le ||V||_0 \cdot \rho(A)|x|$$

hence (Theorem 10)

$$||m(A)||_0 \le ||V||_0 \rho(A) \le |||V|||\rho(A).$$

For the additive measure $m : \mathcal{C}(\rho) \to \mathcal{L}(E, F)$, one can construct <u>*the variation*</u>, <u>*of m*</u> which is the additive measure $\nu : \mathcal{C}(\rho) \to \overline{\mathbb{R}_+}$, given via

$$\nu(A) = sup\{\sum_{i \in I} ||m(A_i)||_0\}$$

where the supremum is computed for all possible finite partitions of A. More precisely, one consider all finite families $(A_i)_{i \in I}$ of mutually disjoint $A_i \in \mathcal{C}(\rho)$ such that $\bigcup_{i \in I} A_i = A$ and one computes the supremum of all sums $\sum_{i \in I} ||m(A_i)||_0$ (see [1]).

We shall see that ν is finite, more precisely, we have, for any $A \in \mathcal{C}(\rho)$:

$$\nu(A) \le |||V|||\rho(A) \tag{4}$$

Indeed, let $A \in \mathcal{C}(\rho)$ and $\epsilon > 0$. For any mutually disjoint $A_1, A_2, ..., A_n \in \mathcal{C}(\rho)$ with $\bigcup_{i=1}^{n} A_i = A$, let $x_1, x_2, ...x_n$ in E such that $|x_i| \le 1$ and

$$\|m(A_i)\|_0 \le |m(A_i)(x_i)| + \frac{\epsilon}{n}, \ i = 1, 2, ..., n \tag{5}$$

We have $x = \sum_{i=1}^{n} \varphi_{A_i} x_i \in \mathcal{E}_\rho(E)$. Because $x(t) = 0$ for $t \notin A$ and $x(t) = x_i$ for $t \in A_i$, we have $|x| \le \varphi_A$, hence $\rho|x| \le \rho(A)$.

In view of (5), one has (see $(1')$)

$$\sum_{i=1}^{n} ||m(A_i)||_0 \le \sum_{i=1}^{n} |m(A_i)(x_i)| + \epsilon = \sum_{i=1}^{n} |V(\varphi_{A_i} x_i)| + \epsilon \le$$
$$\le |||V|||\rho|x| + \epsilon \le |||V|||\rho(A) + \epsilon.$$

Taking the supremum according to all partitions $(A_i)_i$ we obtain

$$\nu(A) \leq |||V|||\rho(A) + \epsilon$$

and ϵ being arbitrary, we obtain (4).

2) Let $\mathcal{E}_\rho$ be the vector subspace of $\mathcal{L}_\rho$ consisting in all simple functions of the form

$$\varphi = \sum_{i=1}^{n} \varphi_{A_i} c_i$$

with $A_i \in \mathcal{C}(\rho)$ and $c_i \in K$ (one can consider A_i to be mutually disjoint).

We can define the linear map $P : \mathcal{E}_\rho \to K$ via

$$P(\varphi) = \sum_{i=1}^{n} \nu(A_i) c_i$$

where φ is as above (the definition is coherent, not depending upon the representation of f, general theory). It is clear that, for $\varphi \in \mathcal{E}_\rho$, one has $|\varphi| \in \mathcal{E}_p$ and we shall show that

$$P(|\varphi|) = \sum_{i=1}^{n} \nu(A_i) |c_i| \leq |||V||| \, \rho|\varphi| \tag{6}$$

thus showing that P is continuous on $\mathcal{E}_\rho$ with the topology of $\mathcal{L}_\rho$. One can see that $P(\varphi) \geq 0$ if $\varphi \geq 0$.

In order to show (6) in the non trivial case, we shall assume all $c_i \neq 0$. Let $\epsilon > 0$. For any $i = 1, 2, ..., n$, one can find a finite partition $(B_{ij})_j \subset \mathcal{C}(\rho)$ of A_i, with p_i elements such that

$$\nu(A_i) \leq \sum_j \|m(B_{ij})\|_0 + \frac{\epsilon}{2n|c_i|} \tag{7}$$

For any pair (i, j), let $x_{ij} \in E$ with $|x_{ij}| \leq 1$ such that

$$||m(B_{ij})||_0 \leq |m(B_{ij})(x_{ij})| + \frac{\epsilon}{2np_i|c_i|}. \tag{8}$$

Hence, for any $i = 1, 2, ...n$ (see (7) and (8)) :

$$\nu(A_i)|c_i| \leq \sum_j \|m(B_{ij})\|_0 |c_i| + \frac{\epsilon}{2n} \leq$$
$$\leq \sum_j |m(B_{ij})(x_{ij})| |c_i| + \frac{\epsilon}{2n} + \frac{\epsilon}{2n} = \sum_j |m(B_{ij})(x_{ij})| |c_i| + \frac{\epsilon}{n}.$$

Consequently

$$\sum_{i=1}^{n} \nu\left(A_i\right)\left|c_i\right| \leq \sum_{ij}\left|m\left(B_{ij}\right)\left(x_{\mathrm{ij}}\right)\right|\left|c_i\right| + \epsilon \tag{9}$$

We are led to consider the function

$$x = \sum_{ij} \varphi_{B_{ij}} c_i x_{ij} \in \mathcal{E}_\rho(E)$$

for which

$$\sum_{ij}\left|m\left(B_{ij}\right)\left(x_{ij}\right)\right|\left|c_i\right| = \sum_{ij}\left|m\left(B_{ij}\right)\left(c_{ij} x_{ij}\right)\right| = \sum_{ij}\left|V\left(\varphi_{B_{ij}} c_i x_{ij}\right)\right| \leq |||V|||\rho|x| \tag{10}$$

One has

$$|x| \leq |\varphi| \tag{11}$$

Indeed, if $t \notin \bigcup_{i=1}^{n} A_i = \bigcup_{ij} B_{ij}$, one has $x(t) = 0$, and if $t \in \bigcup_{ij} B_{ij}$, one finds an unique $t \in B_{ij}$, hence $t \in A_i$ and

$$|x(t)| = \left|c_i x_{ij}\right| \leq \left|c_i\right|.$$

Using (9), (10) and (11), one has

$$\sum_{i=1}^{n} \nu\left(A_i\right)\left|c_i\right| \leq \|V\| \rho|\varphi|$$

and (6) is proved.

Because ρ is of absolutely continuous type, $\mathcal{E}_\rho$ is dense in $\mathcal{L}_\rho$ (see [1]) and we can extend the linear and continuous $P : \mathcal{E}_\rho \to K$ to an unique linear and continuous $H : \mathcal{L}_\rho \to K$.

We show that H is positive. Indeed, if $0 \leq \varphi \in \mathcal{L}_\rho$, one can find an increasing sequence $(\varphi_n)_n \subset \mathcal{E}_\rho$ such that $0 \leq \varphi_n \uparrow \rho$ pointwise (general theory). Consequently , $\varphi - \varphi_n \downarrow 0$ pointwise and, $\rho(\varphi - \varphi_n) \overset{n}{\to} 0$ because ρ is of absolutely continuous type. So $\rho|\varphi - \varphi_n| \overset{n}{\to} 0$, hence $\varphi_n \overset{n}{\to} \varphi$ in $\mathcal{L}_\rho$, hence $H(\varphi_n) \to H(\varphi)$. But $H(\varphi_n) = P(\varphi_n) \geq 0$, hence $H(\varphi) \geq 0$.

3) We end the proof by showing that V is dominated by H.

Let $f = \sum_{i=1}^{n} \varphi_{A_i} x_i \in \mathcal{E}_\rho(E),\ A_i$ disjoint. We have

$$|V(f)| = \left|\sum_{i=1}^{n} V\left(\varphi_{A_i} x_i\right)\right| = \left|\sum_{i=1}^{n} m\left(A_i\right)\left(x_i\right)\right| \leq$$
$$\leq \sum_{i=1}^{n} \left|m\left(A_i\right)\left(x_i\right)\right| \leq \sum_{i=1}^{n} \left\|m\left(A_i\right)\right\|_0 \left|x_i\right| \leq \sum_{i=1}^{n} \nu\left(A_i\right)\left|x_i\right|.$$

We made use of the fact that $||m(A)||_0 \leq \nu(A)$ for any $A \in \mathcal{C}(\rho)$, (see [1]). Because $|f| = \sum_{i=1}^{n} \varphi_{A_i}(x_i)$, we have

$$\sum_{i=1}^{n} \nu\left(A_i\right)\left(x_i\right) = P(|f|) = H(|f|)$$

and we get

$$|V(f)| \leq H(|f|) \tag{12}$$

All it remains to be proved is that (12) remains valid for any $f \in \mathcal{L}_\rho(E, \mu)$. For such f, we use (P_2) to find a sequence $(f_n)_n \subset \mathcal{E}_\rho(E)$ such that $f_n \stackrel{n}{\to} f$ in $\mathcal{L}_\rho(E, \mu)$. One has pointwise

$$||f_n| - |f|| \leq |f_n - f| \text{ and } \rho\,||f_n| - |f|| \leq \rho\,|f_n - f| \stackrel{n}{\to} 0,$$

hence $|f_n| \stackrel{n}{\to} |f|$ in $\mathcal{L}_\rho$.

For any n one has (see (12))

$$|V(f_n)| \leq H(|f_n|)$$

and continuity reasons show that $|V(f)| \leq H(|f|)$ a.s.o. ∎

The reader can notice that $\nu =$ the variation of m is finite. So, the proof can be viewed as part of the theory of integration with respect to a measure with finite variation, but we did not use this idea.

5. OPERATORS GENERATED BY FIELDS OF OPERATORS

Again we are within the Framework described at the end of paragraph 2.

For any $t \in T$, put $G_t \stackrel{def}{=} \mathcal{L}(E_t, F)$. In case $F = K$, we have $G_t = E_t^{'}$. Let us write $\mathcal{G}_F = (G_t)_{t\in T}$. In case $F = K$, we shall write $\mathcal{E}^{'}$ insted of $\mathcal{G}_K = (E_t^{'})_{t\in T}$.

Consequently, we can consider the space of <u>*operator fields*</u>:

$$\mathcal{C}\left(\mathcal{G}_F\right) = \left\{U : T \to \bigcup_{t\in T} G_t | U(t) \in G_t \text{ for any } t \in T\right\}.$$

In case $F = K$, we have the space

$$\mathcal{C}(\mathcal{E}') = \left\{U : T \to \bigcup_{t\in T} E_t' | U(t) \in E_t' \text{ for any } t \in T\right\}.$$

For any $U \in \mathcal{C}(\mathcal{G}_F)$ and for any $x \in \mathcal{C}(\mathcal{E})$ one defines

$$Ux : T \to F,\ (Ux)(t) \stackrel{def}{=} U(t)(x(t)).$$

For any $y' \in F'$ and any $x \in \mathcal{C}(\mathcal{E})$, we shall write $y'(Ux)$ instead of $y' \circ Ux$. (we have $y'(Ux) : T \to K$).

Definition 13. Let $U \in \mathcal{C}(\mathcal{G}_F)$.

We shall say that U <u>*is simply μ - measurable*</u> in case, for any $x \in \mathcal{A}$, the function Ux is μ- measurable.

We shall say that U <u>*is weakly μ - measurable*</u> in case, for any $x \in \mathcal{A}$ and any $y' \in F'$, the function $y'(Ux)$ is μ- measurable.

The definition given above, transfers from $\mathcal{A}$ to the whole $\mathcal{M}(\mathcal{A}, \mu)$ as the following result (see [8]) shows :

Theorem 14. *Let $U \in \mathcal{G}_F$.*

1. *U is simply μ - measurable if and only if Ux is μ - measurable, for any $x \in \mathcal{M}(\mathcal{A}, \mu)$.*

2. *U is weakly μ - measurable if and only if $y'(Ux)$ is μ - measurable, for any $x \in \mathcal{M}(\mathcal{A}, \mu)$ and any $y' \in F'$.*

It is clear that the following implication holds :

U simply μ - measurable $\Rightarrow U$ weakly μ - measurable. $(*)$

Of course, in case $F = K$, simply μ - measurable means weakly μ - measurable, but, generally speaking, the converse implication of $(*)$ is not true. We have the following result, showing a case when the converse of $(*)$ is true (the case $F = K$ is included), see [8].

Theorem 15. *Assume F is separable. Then an element $U \in \mathcal{G}_F$ is simply μ -measurable if and only if it is weakly μ - measurable.*

In the sequel we shall consider an operator field $U \in \mathcal{C}(\mathcal{G}_F)$ and we shall assume that U is <u>*simply μ - measurable*</u> and <u>*bounded*</u>, i.e.

$$M = \sup_{t \in T} ||U(t)||_0 < \infty.$$

As concerns ρ, we shall assume that $\rho\ R - F$, hence $L_\rho(F, \mu)$ is Banach (in case $F = K$, L_ρ is Banach, see [1] and [2]). Accepting these assumptions, we shall construct a linear and continuous operator

$$H_U : L_\rho(\mathcal{E}, \mathcal{A}) \to L_\rho(F, \mu).$$

a) Because U is simply μ - measurable, for any $x \in \mathcal{M}(\mathcal{A}, \mu)$, one has

$$Ux \in M_F(\mu) = \{f : T \to F \mid f \text{ is } \mu- \text{ measurable}\}.$$

Assuming, supplementarily, that $x \in \mathcal{L}_\rho(\mathcal{E}, \mathcal{A})$, we have, for any $t \in T$:

$$||Ux(t)|| = ||U(t)(x(t))|| \leq ||U(t)||_0 ||x(t)|| \leq M||x(t)||$$

i.e. $|Ux| \leq M|x|$.

Because $|Ux| \in M_+(\mu)$, one gets

$$\rho|Ux| \leq M\rho|x| < \infty \tag{13}$$

and (13) shows that $Ux \in \mathcal{L}_\rho(F, \mu)$.

Using again (13), if x and y are in $\mathcal{L}_\rho(\mathcal{E}, \mathcal{A})$ and $x(t) = y(t)$ μ - a.e., we get

$$\rho|U(x - y)| \leq M\ \rho|x - y| = 0$$

hence $Ux(t) = Uy(t)$ μ - a.e.. Then, it follows that for any $\widetilde{x} \in L_\rho(\mathcal{E}, \mathcal{A})$ and any $y \in \widetilde{x}, z \in \widetilde{x}$, one has $Uy = Uz$ μ - a.e. and one can define (without ambiguity)

$$H_U(\widetilde{x}) = \widetilde{Ux} \in L_\rho(F, \mu).$$

So, we have the linear map $H_U : L_\rho(\mathcal{E}, \mathcal{A}) \to L_\rho(F, \mu)$ which is continuous. Indeed, if $\widetilde{x} \in L_\rho(F, \mu)$ and $x \in \widetilde{x}$, we use (13) to see that

$$||H_U(\widetilde{x})|| = ||\widetilde{Ux}|| = \rho|Ux| \leq M\rho|x| = M||\widetilde{x}||.$$

At this moment we proved :

Theorem 16. *Let $U \in \mathcal{C}(\mathcal{G}_F)$ which is simply μ - measurable and such that $M = \sup_{t\in T} ||U(t)||_0 < \infty$. Assume that $\rho\, R - F$. Then we have the linear and continuous operator $H_U : L_\rho(\mathcal{E}, \mathcal{A}) \to L_\rho(F, \mu)$, given via $H_U(\widetilde{x}) = \widetilde{Ux}$ and $||H_U||_0 \leq M$.*

In the particular case $F = K$, one has $U \in \mathcal{C}(\mathcal{E}')$ and $H_U : L_\rho(\mathcal{E}, \mathcal{A}) \to L_\rho$.

Definition 17. We call H_U *the operator generated by U*.

We would like to close with

Example 18. This example is based upon Example 19 in [2], which we briefly review, in order to make the present paper self-contained.

The (locally) compact space T is $[0, 1]$ with its usual topology. We have $(T, \mathcal{T}, \mu)$, where $\mathcal{T}$ = the Lebesgue measurable sets of $[0, 1]$ and $\mu : \mathcal{T} \to \mathbb{R}_+$ is the Lebesgue measure. Take $\mathcal{E} = (E_t)_{t\in T}$ where

$$E_t = L^{1/t}(\mu) \stackrel{def}{=} L^{1/t},\ \text{if } 0 < t \leq 1 \text{ (write also } \mathcal{L}^{1/t} \stackrel{def}{=} \mathcal{L}^{1/t}(\mu))$$

$$E_0 = L^{\infty}(\mu) \stackrel{def}{=} L^{\infty},\ \text{if } t = 0 \text{ (write also } \mathcal{L}^{\infty} \stackrel{def}{=} \mathcal{L}^{\infty}(\mu))$$

and accept always the conventions $1/0 = \infty, 1/\infty = 0$.

So, in our schema from the Framework, paragraph 2, we have, for any $t \in T$, the measure space $(S_t,\ \Sigma_t,\ \mu_t)$, where $S_t = T,\ \Sigma_t = \mathcal{T},\ \mu_t = \mu$. At the same time, for any $t \in T$, we have the function norm $\rho_t = || \cdot ||_{1/t}$ with $\rho_t\ F$, thus obtaining the *Köthe spaces* $L_{\rho_t} = E_t,\ t \in [0, 1]$.

In order to complete the schema, we put into evidence the fundamental family $\mathcal{A}$ of continuous vector fields for this $\mathcal{C}(E)$.

Write

$$\mathcal{E}(\mathcal{T}) = \{f : T \to K | f \text{ is } \mathcal{T} - \text{simple}\} \text{ and } E(\mathcal{T}) = \left\{ \tilde{f} \,\middle|\, f \in \mathcal{E}(\mathcal{T}) \right\}$$

(one knows that $E(\mathcal{T})$ is dense in E_t for any $t \in [0, 1]$). So, we can construct, for any $\widetilde{f} \in E(\mathcal{T})$, the element $x(\widetilde{f}) \in \mathcal{C}(\mathcal{E})$, acting via

$$x(\widetilde{f})(t) = \widetilde{f} \in E_t = L^{1/t}$$

for any $t \in [0, 1]$.

Then

$$\mathcal{A} = \{x(\widetilde{f}) \mid \widetilde{f} \in E(\mathcal{T})\}$$

is a fundamental family of continuous vector fields for $\mathcal{C}(\mathcal{E})$, as shown in [2].

We completed the schema for obtaining

$$L_\rho(\mathcal{E}, \mathcal{A}) = L_\rho((\rho_t)_t, \mathcal{A}).$$

Let $h : [0,1] \to K$ be a Lebesgue measurable and bounded function. We shall use h to construct an operator field $U \in \mathcal{C}(\mathcal{E}^{'})$. For any $t \in [0,1]$, one has $h \in \mathcal{L}^{1/1-t}$, hence, for any $f \in \mathcal{L}^{1/t}$, one has $fh \in \mathcal{L}^1$.

Hölder's inequality says that

$$|\int fh d\mu| \leq ||f||_{1/t} \cdot ||h||_{1/1-t} \tag{14}$$

and taking $\sup\limits_{t \in T} |h(t)| = M < \infty$, one has $||h||_{1/1-t} \leq M$ for any $t \in [0,1]$.

This is obvious for $t = 1$, i.e. $||h||_\infty \leq M$ and, for $0 \leq t < 1$, one has

$$||h||_{1/1-t} = (\int |h|^{1/1-t}\, d\mu)^{1-t} \leq (M^{1/1-t})^{1-t} = M.$$

In view of (14) we get, for any $t \in [0,1]$ and any $f \in \mathcal{L}^{1/t}$

$$|\int fh d\mu| \leq M ||f||_{1/t} \tag{15}$$

It follows from (15) that, for any $t \in [0,1]$, one can define $U(t) : L^{1/t} \to K$, via

$$U(t)(\tilde{f}) = \int fh d\mu \tag{16}$$

where $f \in \widetilde{f}$ is arbitrarily taken. Also from (15), it follows that, for any $t \in T$ one has $U(t) \in E_t^{'}$ and $||U(t)||_0 \leq M$. At this moment, we got the bounded operator field $U \in \mathcal{C}(\mathcal{E}^{'})$, given by (16).

Now, we show that U is simply μ - measurable. Namely, we must show that for any $x(\widetilde{f}) \in \mathcal{A}$, the function $X : [0,1] \to K$ given via

$$X(t) = U(t)(x(\widetilde{f})(t))$$

is μ - measurable.

According to the definition, $x(\widetilde{f})(t) = \widetilde{f}$ for any $t \in [0,1]$, hence

$$U(t)(x(\widetilde{f})(t)) = U(t)(\widetilde{f}) = \int fh d\mu$$

and the function X is *constant*, hence trivially μ - measurable.

We succeeded in proving that U is bounded and simply μ - measurable. Taking a function norm ρ on $(T, \mathfrak{T}, \mu)$ such that ρ $R{-}F$, we obtain the operator generated by U :

$$H_U : L_\rho((\rho_t)_t, \mathcal{A}) \to L_\rho.$$

We shall see the concrete action of H_U in a case described in Example 19 from [2].

Let us consider $1 \le p \le \infty$ and the function norm ρ on $(T, \mathfrak{T}, \mu)$ given by $\rho = ||\cdot||_p$. Let also $x : [0,1] \to E_t = L^{1/t}$, given by

$$x(t) = \widetilde{\varphi_{[0,t]}} \tag{17}$$

We have seen that $x \in \mathcal{L}_\rho((\rho_t)_t, \mathcal{A})$, hence $\widetilde{x} \in L_\rho((\rho_t)_t, \mathcal{A})$. We shall compute $H_U(\widetilde{x})$. According to the definition :

$$H_U(\widetilde{x}) = \widetilde{Ux}$$

for any $x \in \widetilde{x}$. Taking x as in (17), one has for any $t \in [0,1]$ (see (16)) :

$$Ux(t) = U(t)(x(t)) = U(t)\left(\widetilde{\varphi_{[0,t]}}\right) = \int h\varphi_{[0,t]}d\mu = \int_{[0,t]} hd\mu.$$

Conclusion : One has $H_U(\widetilde{x}) = \widetilde{y} \in L^p$ where a representative $y \in \widetilde{y}$ can be $y : [0,1] \to K$ given by

$$y(t) = \int_{[0,t]} hd\mu.$$

It will be of some interest to consider this example from the point of view generated by the Integral Representation Theorem (Theorem 3). More precisely, we shall see the action of the representing measure m for an operator identical to H_U.

The operator H_U is obtained in three steps as follows :

1) First we define $M_U : \mathcal{L}_\rho((\rho_t)_t, \mathcal{A}) \to \mathcal{L}_\rho$ (here $\rho_t = ||\cdot||_{1/t}$) via

$$M_U(x) = Ux.$$

More precisely, for any $t \in T = [0,1]$:

$$(Ux)(t) = U(t)(x(t)) = \int hf_t d\mu$$

where f_t is arbitrarily taken in $x(t) \in L_{\rho_t} = L^{1/t}$.

2) Next we define $N_U : \mathcal{L}_\rho((\rho_t)_t, \mathcal{A}) \to L_\rho$, via

$$N_U(x) = \widetilde{M_U(x)} = \widetilde{Ux}.$$

3) Finally, we obtain $H_U : L_\rho((\rho_t)_t, \mathcal{A}) \to L_\rho$, namely

$$H_U(\widetilde{x}) = N_U(x) = \widetilde{Ux}$$

for any $x \in \widetilde{x}$.

It is seen that one has $H_U = \widetilde{N_U}$ (see Preliminary Facts) and one can identify $H_U \equiv N_U$. Our goal is to compute the representing measure m for N_U (see Theorem 3). This can be done, of course, if assumption (P_2) holds, e.g. in case $\rho = ||\cdot||_p$, $1 \le p < \infty$ (see the Remark at the end of paragraph 3).

We have $m : \mathcal{C}(\rho) \to L(\mathcal{A}, L_\rho)$ given via

$$m(A)(x) = N_U(\varphi_A x)$$

for any $A \in \mathcal{C}(\rho)$ and any $x \in \mathcal{A}$. Hence

$$m(A)(x) = \widetilde{U\varphi_A x}.$$

For any $t \in T$, one has

$$(U\varphi_A x)(t) = \int hg_t \, d\mu$$

where $g_t \in \widetilde{(\varphi_A x)}(t) = \varphi_A(t)\widetilde{x(t)} \in L^{1/t}$.

Because $x \in \mathcal{A}$, one has $x = x(\widetilde{f})$, for some $f \in \mathcal{E}(\mathcal{T})$ and $x(t) = \widetilde{f} \in L^{1/t}$. So $g_t \in \varphi_A(t)\widetilde{f}$ and we can take $g_t = \varphi_A(t)f$ which gives

$$(U\varphi_A x)(t) = \varphi_A(t) \int hf d\mu.$$

It follows that a representative of $m(A)\,(x = x(f))$ is the function $U : T \to K$ given via

$$U(t) = (\int hf d\mu) \cdot \varphi_A(t).$$

Conclusion: For any $A \in \mathcal{C}(\rho)$ and any $x = x(\widetilde{f}) \in \mathcal{A}$, one has

$$m(A)(x) = (\int hf d\mu)\widetilde{\varphi_A}.$$

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