An. Şt. Univ. Ovidius Constanţa Vol. 20(3), 2012, 33–36

An Extension of Nice Bases on Ulm Subgroups of Primary Abelian Groups

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Abstract

Suppose A is an Abelian p-group and α is an ordinal such that $A/p^\alpha A$ is a direct sum of countable groups. It is shown that A has a nice basis if, and only if, $p^\alpha A$ has a nice basis. This strengthens an earlier result of ours in Bull. Allah. Math. Soc. (2011) proved when $\alpha < \omega^2$.

1 Introduction

Everywhere in the text of the present paper, let it be agreed that α is an ordinal and A is an additive Abelian p-group with *p^α-power subgroup* $p^\alpha A$ defined as follows - $p^0 A = A$ if $\alpha = 0$, $pA = \{pa : a \in A\}$, $p^\alpha A = \cap_{\beta<\alpha} p^\beta A$ if $\alpha - 1$ does not exist and $p^\alpha A = p(p^{\alpha-1}A)$ otherwise, and also called *α-th Ulm subgroup* of A. A problem of interest is the following:

Problem. Suppose the factor-group $A/p^\alpha A$ is a totally projective group. Does it follow that A has the property $\mathcal{P}$ (or belongs to the group class $\mathcal{K}$) if, and only if, the same holds for $p^\alpha A$?

In [1] and [4] we established in the affirmative a solution to this question when $\alpha = \omega$ and $\mathcal{K}$ coincides with the class of all groups equipped with a nice basis – recall that a group is said to have *a nice basis* if it is the countable ascending union of nice direct sums of cyclic groups. In particular, if these

Key Words: Direct sums of countable groups, Groups with nice bases, Primary Abelian groups, Totally projective quotients, Ulm subgroups, Ulm factors.
2010 Mathematics Subject Classification: 20K10.
Received: July 26, 2011.
Accepted: October 13, 2011.

direct sums are bounded, the group has *a bounded nice basis*. Furthermore, we improve in [2] that achievement to ordinals α strictly less than the ordinal ω^2. In the next section this will be extended to any ordinal α not exceeding the first uncountable ordinal Ω - see also [3] where there are some technical errors corrected here.

2 Main Result

We are now ready to prove below the following statement.

Theorem 2.1. *Suppose the quotient $A/p^\alpha A$ is totally projective for some $\alpha \leq \Omega$. Then A has a nice basis if, and only if, $p^\alpha A$ has a nice basis.*

Proof. If we write $A = \cup_{n<\omega} A_n$, where $A_n \subseteq A_{n+1} \leq A$ and all A_n are nice in A direct sums of cyclic groups, it easily follows that $p^\alpha A = \cup_{n<\omega}(A_n \cap p^\alpha A)$, where all $A_n \cap p^\alpha A$ are direct sums of cyclic groups being subgroups of A_n, and also are obviously nice in $p^\alpha A$. This completes the necessity.

As for the sufficiency, we first observe that $A/p^\alpha A$ is a direct sum of countable groups, say $A/p^\alpha A = \oplus_{i\in I}(A_i/p^\alpha A)$, where each $A_i \leq A$ and $A_i/p^\alpha A$ is countable. Moreover, write $A_i/p^\alpha A = \cup_{k<\omega}(C_{ik}/p^\alpha A)$, where every $C_{ik}/p^\alpha A$ is finite for any $k < \omega$, whence $C_{ik} = F_{ik} + p^\alpha A$ with finite F_{ik}; for simpleness denote $F_{ik} = F_i$. Thus $A = \sum_{i\in I} A_i$ and $A_i = \cup_{k<\omega} C_{ik}$. Let $exp(C_{ik}/p^\alpha A) = p^{t_i}$ for some $t_i \in \mathbb{N}$. For an arbitrary $n < \omega$, let

$$I_n = \{i \in I : t_i \leq n, p^{t_i} F_i \subseteq B_n\},$$

where $p^\alpha A = \cup_{n<\omega} B_n$ with $B_n \subseteq B_{n+1} \leq p^\alpha A$ being direct sums of cyclic groups and also being nice in $p^\alpha A$, and hence nice in A. Define

$$K_n = B_n + \langle f_{i_n} | i_n \in I_n, f_{i_n} \in F_{i_n} \setminus p^\alpha A\rangle$$

and next we intend to check that the requirements for a nice basis about $\{K_n\}_{n<\omega}$ are satisfied on A. Specifically, the sequence $\{K_n\}_{n<\omega}$ forms a nice basis for A by showing that

(1) $A = \cup_{n<\omega} K_n$ with $K_n \subseteq K_{n+1} \leq A$;
(2) Each K_n is nice in A;
(3) Each K_n is a direct sum of cyclic groups.

Indeed, we observe that $p^n K_n \subseteq B_n$ is a direct sum of cyclic groups, whence so is K_n, so that (3) holds.

As for (1), we see that for any $a \in A$ it is fulfilled that $a = a_{i_1} + \cdots + a_{i_s} = c_{i_1 k_1} + \cdots + c_{i_s k_s} = f_{i_1} + b_{n_1} + \cdots + f_{i_s} + b_{n_s}$. But the sequence $\{B_n\}$ is ascending, so that $b_{n_1} + \cdots + b_{n_s} \in B_m$ for some $m < \omega$. On the other hand, $f_{i_1} \in F_{i_1}, \cdots, f_{i_s} \in F_{i_s}$, where $i_1 \in I_1, \cdots, i_s \in I_s$. Since $\{I_n\}_{n<\omega}$ is an ascending sequence, we deduce that $i_1, \cdots, i_s \in I_r$ for some $r < \omega$. This gives that $a \in K_l$ for some $l < \omega$, as required.

Clearly, $B_n \subseteq B_{n+1}$ and, moreover, $I_n \subseteq I_{n+1}$. This immediately forces that $K_n \subseteq K_{n+1}$, as asserted. This ensures the validity of (1).

Finally, to establish point (2), we foremost observe that $K_n \cap p^\alpha A = B_n$, so that $K_n \cap p^\alpha A$ is nice in $p^\alpha A$. Indeed, in view of the modular law, $K_n \cap p^\alpha A = B_n + p^\alpha A \cap (\langle f_{i_n} | i_n \in I_n, f_{i_n} \in F_{i_n} \setminus p^\alpha A \rangle)$. But the intersection obviously lies in B_n, whence the wanted equality follows. In fact, $(A_i/p^\alpha A) \cap \sum_{j \neq i}(A_j/p^\alpha A) = 0$ precisely when $A_i \cap (\sum_{j\neq i} A_j) \subseteq p^\alpha A$. That is why, given arbitrary $x \in p^\alpha A \cap (\langle f_{i_n} | i_n \in I_n, f_{i_n} \in F_{i_n} \setminus p^\alpha A \rangle)$, we have $x = a_\alpha = u_1 f_{i_{n_1}} + \cdots + u_s f_{i_{n_s}}$, where $a_\alpha \in p^\alpha A$, only when $u_l f_{i_{n_l}} \in p^\alpha A$ for all $l \in [1, s]$. Therefore, p^{t_l}/u_l and hence $u_l f_{i_{n_l}} = v_l p^{t_l} f_{i_{n_l}} \in B_n$ because $p^{t_l} f_{i_{n_l}} \in B_n$.

On the other hand, $(K_n + p^\alpha A)/p^\alpha A = ((\langle f_{i_n} | i_n \in I_n, f_{i_n} \in F_{i_n} \setminus p^\alpha A \rangle + p^\alpha A)/p^\alpha A = (\sum_{i_n \in I_n} F_{i_n} + p^\alpha A)/p^\alpha A = (\sum_{i\in I_n} C_{ik})/p^\alpha A = (\sum_{i \in I_n} A_i)/p^\alpha A = \sum_{i\in I_n}(A_i/p^\alpha A) = \oplus_{i \in I_n}(A_i/p^\alpha A)$, hence it is nice in $A/p^\alpha A = \oplus_{i\in I}(A_i/p^\alpha A)$ as its direct summand. Finally, one can conclude that K_n is nice in A for every index n, thus obtaining (2) as stated. This completes the proof.

The same idea is applicable even for groups with bounded nice basis. So, we state without proof the following direct consequence.

Corollary 2.2. *Let the factor-group $A/p^\alpha A$ be totally projective for an ordinal $\alpha \leq \Omega$. Then A has a bounded nice basis if, and only if, $p^\alpha A$ has a bounded nice basis.*

We close the article with two problems.

Problem 1. Does it follow that the theorem remains true without the restriction $\alpha \leq \Omega$?

Problem 2. If G is a group of length α $(\leq \Omega)$ with the property $G \cong A/p^\alpha A$ for any group A such that A has a nice basis if, and only if, $p^\alpha A$ has a nice basis, does it follow that G is totally projective (a direct sum of countable groups)?

Correction: In ([1], p. 402, (3)) the equality "$p^n M_n = p^n N_n$" should be written and read as "$p^n M_n = p^n N_n + p^n \langle a_j : j \in J_n \rangle \subseteq N_n$".

Besides, we shall give one more detail in the proof of Theorem 2.2 from [1]. It concerns why $\langle a_j : j \in J_n \rangle \cap p^\omega A \subseteq N_n$. In fact, write $x = \varepsilon_1 a_{j_1} + \cdots + \varepsilon_s a_{j_s} \in p^\omega A$ which holds exactly when $\varepsilon_i a_{j_i} \in p^\omega A$ for all $i \in [1, s]$, i.e., when

$\varepsilon_i(a_{j_i} + p^\omega A) = p^\omega A$. Consequently, p^{e_i}/ε_i and thus $\varepsilon_i a_{j_i} = s_i p^{e_i} a_{j_i} \in N_n$ because $p^{e_i} a_{j_i} \in N_n$. Finally, $x \in N_n$ as required.

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