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Orientable small covers over products of a prism with a simplex

Yanchang Chen, Yanying Wang

Abstract

In this paper, we determine the number of D-J equivalence classes of all orientable small covers over products of a prism with a simplex.

1 Introduction

A small cover, defined by Davis and Januszkiewicz in [6], is a smooth closed manifold M^n with a locally standard $(\mathbb{Z}_2)^n-$action such that its orbit space is a simple convex polytope. For instance, the real projective space $\mathbb{R}P^n$ with a natural $(\mathbb{Z}_2)^n-$action is a small cover over an n-simplex. This gives a direct connection between equivariant topology and combinatorics and makes it possible to study the topology of small covers through the combinatorial structure of quotient spaces.

In recent years, several studies have attempted to enumerate the number of equivalence classes of all small covers over a specific polytope. Garrison and Scott used a computer program to calculate the number of homeomorphism classes of all small covers over a dodecahedron [7]. The number of homeomorphism classes of small covers over cubes has also been counted [5, 9]. In [2], Cai, Chen and Lü calculated the number of equivariant homeomorphism classes of small covers over prisms. Choi determined the number of equivariant homeomorphism classes of small covers over cubes [3].

There are few results about orientable small covers. From [8], There exist orientable small covers over every simple convex 3-polytope. There also exist non-orientable small covers over every simple convex 3-polytope, except the

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3-simplex. An orientable 3-dimensional small cover corresponds to a 4-colored simple convex 3-polytope and the existence of an orientable small cover over every simple convex 3-polytope is closely related to the four color theorem (see [1] for the four color theorem). In [4], Choi calculated the number of D-J equivalence classes of orientable small covers over cubes. The objective of this paper is to give a general formula to calculate the number of D-J equivalence classes of all orientable small covers over $P^3(m) \times \Delta^n$ (see Theorem 3.1), where by $P^3(m)$ and Δ^n we denote an m-sided prism (i.e., the product of an m-gon and the interval I) and an n-simplex respectively.

The paper is organized as follows. In Section 2, we review the basic theory about orientable small covers. In Section 3, we determine the number of D-J equivalence classes of orientable small covers over $P^3(m) \times \Delta^n$.

2 Preliminaries

A convex polytope P^n of dimension n is said to be simple if every vertex of P^n is the intersection of exactly n facets (i.e. faces of dimension $(n-1)$) [10]. An n-dimensional smooth closed manifold M^n is said to be a small cover if it admits a smooth $(\mathbb{Z}_2)^n-$action such that the action is locally isomorphic to a standard action of $(\mathbb{Z}_2)^n$ on $\mathbb{R}^n$ and the orbit space $M^n/(\mathbb{Z}_2)^n$ is a simple convex polytope of dimension n.

Let P^n be a simple convex polytope of dimension n and $\mathcal{F}(P^n) = \{F_1, \cdots, F_\ell\}$ be the set of facets of P^n. Suppose that $\pi : M^n \to P^n$ is a small cover over P^n. Then there are ℓ connected submanifolds $\pi^{-1}(F_1), \cdots, \pi^{-1}(F_\ell)$. Each submanifold $\pi^{-1}(F_i)$ is fixed pointwise by a $\mathbb{Z}_2-$subgroup $\mathbb{Z}_2(F_i)$ of $(\mathbb{Z}_2)^n$, so that each facet F_i corresponds to the $\mathbb{Z}_2-$subgroup $\mathbb{Z}_2(F_i)$. Obviously, the $\mathbb{Z}_2-$subgroup $\mathbb{Z}_2(F_i)$ actually agrees with an element ν_i in $(\mathbb{Z}_2)^n$ as a vector space. For each face F of codimension u, since P^n is simple, there are u facets $F_{i_1}, \cdots, F_{i_u}$ such that $F = F_{i_1} \cap \cdots \cap F_{i_u}$. Then, the corresponding submanifolds $\pi^{-1}(F_{i_1}), \cdots, \pi^{-1}(F_{i_u})$ intersect transversally in the $(n-u)$-dimensional submanifold $\pi^{-1}(F)$, and the isotropy subgroup $\mathbb{Z}_2(F)$ of $\pi^{-1}(F)$ is a subtorus of rank u and is generated by $\mathbb{Z}_2(F_{i_1}), \cdots, \mathbb{Z}_2(F_{i_u})$ (or is determined by $\nu_{i_1}, \cdots, \nu_{i_u}$ in $(\mathbb{Z}_2)^n$) [6].

Consider a map $\lambda : \mathcal{F}(P^n) \longrightarrow (\mathbb{Z}_2)^n$ which satisfies the *non-singularity condition*: $\lambda(F_{i_1}), \cdots, \lambda(F_{i_n})$ are a basis of $(\mathbb{Z}_2)^n$ whenever the intersection $F_{i_1} \cap \cdots \cap F_{i_n}$ is non-empty. We call λ a *characteristic function.* If we regard each nonzero vector of $(\mathbb{Z}_2)^n$ as being a color, then the characteristic function λ means that each facet is colored by a color. Here we also call λ a $(\mathbb{Z}_2)^n$-coloring on P^n.

In fact, Davis and Januszkiewicz gave a reconstruction process of a small cover by using a $(\mathbb{Z}_2)^n$-coloring $\lambda : \mathcal{F}(P^n) \longrightarrow (\mathbb{Z}_2)^n$. Let $\mathbb{Z}_2(F_i)$ be the subgroup of $(\mathbb{Z}_2)^n$ generated by $\lambda(F_i)$. Given a point $p \in P^n$, by $F(p)$ we denote the minimal face containing p in its relative interior. Assume $F(p) = F_{i_1} \cap \cdots \cap F_{i_u}$ and $\mathbb{Z}_2(F(p)) = \bigoplus_{j=1}^{u} \mathbb{Z}_2(F_{i_j})$. Note that $\mathbb{Z}_2(F(p))$ is a u-dimensional subgroup of $(\mathbb{Z}_2)^n$. Let $M(\lambda)$ denote $P^n \times (\mathbb{Z}_2)^n / \sim$, where $(p, g) \sim (q, h)$ if $p = q$ and $g^{-1}h \in \mathbb{Z}_2(F(p))$. The free action of $(\mathbb{Z}_2)^n$ on $P^n \times (\mathbb{Z}_2)^n$ descends to an action on $M(\lambda)$ with quotient P^n. Thus $M(\lambda)$ is a small cover over P^n [6].

Two small covers M_1 and M_2 over P^n are said to be weakly equivariantly homeomorphic if there is an automorphism $\varphi : (\mathbb{Z}_2)^n \to (\mathbb{Z}_2)^n$ and a homeomorphism $f : M_1 \to M_2$ such that $f(t \cdot x) = \varphi(t) \cdot f(x)$ for every $t \in (\mathbb{Z}_2)^n$ and $x \in M_1$. If φ is an identity, then M_1 and M_2 are equivariantly homeomorphic. Following [6], two small covers M_1 and M_2 over P^n are said to be Davis-Januszkiewicz equivalent (or simply, D-J equivalent) if there is a weakly equivariant homeomorphism $f : M_1 \to M_2$ covering the identity on P^n.

By $\Lambda(P^n)$ we denote the set of all $(\mathbb{Z}_2)^n$-colorings on P^n. Then we have

Theorem 2.1([6]). *All small covers over P^n are given by $\{M(\lambda) | \lambda \in \Lambda(P^n)\}$ up to D-J equivalence.*

In fact, for each small cover M^n over P^n, there is a $(\mathbb{Z}_2)^n$-coloring λ with an equivariant homeomorphism $M(\lambda) \longrightarrow M^n$ covering the identity on P^n. Nakayama and Nishimura found an orientability condition for a small cover [8].

Theorem 2.2. *For a basis $\{e_1, \cdots, e_n\}$ of $(\mathbb{Z}_2)^n$, a homomorphism $\varepsilon : (\mathbb{Z}_2)^n \longrightarrow \mathbb{Z}_2 = \{0, 1\}$ is defined by $\varepsilon(e_i) = 1 (i = 1, \cdots, n)$. A small cover $M(\lambda)$ over a simple convex polytope P^n is orientable if and only if there exists a basis $\{e_1, \cdots, e_n\}$ of $(\mathbb{Z}_2)^n$ such that the image of $\varepsilon\lambda$ is $\{1\}$.*

We call a $(\mathbb{Z}_2)^n$-coloring which satisfies the orientability condition in Theorem 2.2 an orientable coloring of P^n. We know the existence of orientable small cover over $P^3(m) \times \Delta^n$ by existence of orientable colorings and determine the number of D-J equivalence classes.

By $O(P^n)$ we denote the set of all orientable colorings on P^n. There is a natural action of $GL(n, \mathbb{Z}_2)$ on $O(P^n)$ defined by the correspondence $\lambda \longmapsto \sigma \circ \lambda$, and the action on $O(P^n)$ is free. Assume that $F_1, \cdots, F_n$ of $\mathcal{F}(P^n)$ meet at one vertex p of P^n. Let $e_1, \cdots, e_n$ be the standard basis of $(\mathbb{Z}_2)^n$. Write $B(P^n) = \{\lambda \in O(P^n) | \lambda(F_i) = e_i, i = 1, \cdots, n\}$. It is easy to check that $B(P^n)$ is the orbit space of $O(P^n)$ under the action of $GL(n, \mathbb{Z}_2)$.

Remark 1. In fact, we have $B(P^n) = \{\lambda \in O(P^n) | \lambda(F_i) = e_i, i = 1, \cdots, n$ and for $n + 1 \leq j \leq \ell, \lambda(F_j) = e_{j_1} + e_{j_2} + \cdots + e_{j_{2h_j+1}}, 1 \leq j_1 < j_2 < \cdots < j_{2h_j+1} \leq n\}$. Below we show that $\lambda(F_j) = e_{j_1} + e_{j_2} + \cdots + e_{j_{2h_j+1}}$ for $n + 1 \leq j \leq \ell$. If $\lambda \in O(P^n)$, there exists a basis $\{e'_1, \cdots, e'_n\}$ of $(\mathbb{Z}_2)^n$

such that for $1 \le i \le \ell,\ \lambda(F_i) = e'_{i_1} + \cdots + e'_{i_{2f_i+1}}, 1 \le i_1 < \cdots < i_{2f_i+1} \le n$. Since $\lambda(F_i) = e_i, i = 1, \cdots, n$, then $e_i = e'_{i_1} + \cdots + e'_{i_{2f_i+1}}$. So we obtain that for $n+1 \le j \le \ell$, there aren't $j_1, \cdots, j_{2k}$ such that $\lambda(F_j) = e_{j_1} + \cdots + e_{j_{2k}}, 1 \le j_1 < \cdots < j_{2k} \le n$.

Two orientable small covers $M(\lambda_1)$ and $M(\lambda_2)$ over P^n are D-J equivalent if and only if there is $\sigma \in GL(n, \mathbb{Z}_2)$ such that $\lambda_1 = \sigma \circ \lambda_2$. So the number of D-J equivalence classes of orientable small covers over P^n is $|B(P^n)|$. We shall particularly be concerned with the case in which the simple convex polytope is $P^3(m) \times \Delta^n$.

3 The number of D-J equivalence classes

In this section, we calculate the number of D-J equivalence classes of all orientable small covers over $P^3(m) \times \Delta^n$.

To be convenient, we introduce the following marks. By s'_1 and s'_2 we denote the top and bottom facets of $P^3(m)$ respectively, and by $a'_1, \cdots, a'_m$ we denote all sided facets of $P^3(m)$ in their general order. For an n-simplex Δ^n, by $b'_1, \cdots, b'_{n+1}$ we denote all facets of Δ^n. Set $\mathcal{F}' = \{s_1 = s'_1 \times \Delta^n, s_2 = s'_2 \times \Delta^n, a_i = a'_i \times \Delta^n | 1 \le i \le m\}$ and $\mathcal{F}'' = \{b_j = P^3(m) \times b'_j | 1 \le j \le n+1\}$. Then $\mathcal{F}(P^3(m) \times \Delta^n) = \mathcal{F}' \bigcup \mathcal{F}''$.

Next we give a criterion for a map $\lambda : \mathcal{F}(P^3(m) \times \Delta^n) \longrightarrow (\mathbb{Z}_2)^{n+3}$ to be a characteristic function. The non-singularity condition of the characteristic function means the following:

(1) $\{\lambda(s_1), \lambda(a_1), \lambda(a_2), \lambda(b_1), \cdots, \lambda(b_n)\}$ is a basis of $(\mathbb{Z}_2)^{n+3}$.

(2) $\lambda(b_{n+1})$ satisfies that $\{\lambda(b_{n+1}), \lambda(b_{k_1}), \cdots, \lambda(b_{k_{n-1}}), \lambda(s_1), \lambda(a_1), \lambda(a_2)\}$ is a basis of $(\mathbb{Z}_2)^{n+3}$, where $k_1 < k_2 < \cdots < k_{n-1}$ and $k_1, \cdots, k_{n-1} \in \{1,\ 2,\ \cdots,\ n\}$.

(3) $\lambda(s_2)$ satisfies that $\{\lambda(s_2), \lambda(a_1), \lambda(a_2), \lambda(b_{h_1}), \cdots, \lambda(b_{h_n})\}$ is a basis of $(\mathbb{Z}_2)^{n+3}$, where $h_1 < h_2 < \cdots < h_n$ and $h_1, \cdots, h_n \in \{1, 2, \cdots, n+1\}$.

(4) $\lambda(a_3),\ \cdots,\ \lambda(a_m)$ satisfy that both $\{\lambda(a_l),\ \lambda(a_{l-1}),\ \lambda(s_k), \lambda(b_{h_1}), \cdots, \lambda(b_{h_n})\}$ and $\{\lambda(a_m),\ \lambda(a_1),\ \lambda(s_k),\ \lambda(b_{h_1}), \cdots,\ \lambda(b_{h_n})\}$ are bases of $(\mathbb{Z}_2)^{n+3}$, where $l = 3, \cdots, m,\ k = 1, 2,\ \ h_1 < h_2 < \cdots < h_n$ and $h_1, \cdots, h_n \in \{1,\ 2,\ \cdots, n$
$+ 1\}$.

Theorem 3.1. *By $\mathbb{N}$ we denote the set of natural numbers. Let a, b, c, d, e be functions from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$ with the following properties:*

(1) $a(j, n) = 2^n a(j-1, n) + 2^{2n+1} a(j-2, n)$ *with* $a(1, n) = 1, a(2, n) = 2^n$*;*

(2) $b(j, n) = b(j-1, n) + 2^{n+1} b(j-2, n)$ *with* $b(1, n) = b(2, n) = 1$*;*

(3) $c(j,n) = 2c(j-1,n)+2^{n+1}c(j-2,n)-(2^{n+1}+2)c(j-3,n)-(2^{n+1}-1)c(j-4,n)+2^{n+1}c(j-5,n)$ *with* $c(1,n) = c(2,n) = 1, c(3,n) = 3, c(4,n) = 2^{n+1}+3, c(5,n) = 3\cdot 2^{n+1}+5$;

(4) $d(j,n) = 2^{n-1}d(j-1,n)+2^{2n}d(j-2,n)$ *with* $d(1,n) = 1, d(2,n) = 2^{n-1}$;

(5) $e(j,n) = 2^{n}e(j-1,n)+2^{2n}e(j-2,n)-3\cdot 2^{3n-2}e(j-3,n)-3\cdot 2^{4n-4}e(j-4,n) + 2^{5n-3}e(j-5,n)$ *with* $e(1,n) = 1, e(2,n) = 2^{n-1}, e(3,n) = 3\cdot 2^{2n-2}, e(4,n) = 7\cdot 2^{3n-3}, e(5,n) = 17\cdot 2^{4n-4}$.

Then the number of D-J equivalence classes of orientable small covers over $P^3(m)\times\Delta^n$ *is*

(1) $a(m-1,n)+2b(m-1,n)+c(m-1,n)+(1+(-1)^m)[(2^n-1)(2^{n+1})^{\frac{m}{2}-1}+(2^n)^{\frac{m}{2}}+1]$ *for* n *even,*

(2) $2^{n-1}[a(m-1,n)+2d(m-1,n)+e(m-1,n)]+2b(m-1,n)+c(m-1,n)+(2^{n+1}+1)\cdot\frac{1+(-1)^m}{2}$ *for* n *odd.*

Proof. Let $e_1,\cdots,e_{n+3}$ be the standard basis of $(\mathbb{Z}_2)^{n+3}$, then $(\mathbb{Z}_2)^{n+3}$ contains $2^{n+3}-1$ nonzero elements (or $2^{n+3}-1$ colors). We choose s_1, a_1, a_2 from $\mathcal{F}'$ and $b_1,\cdots,b_n$ from $\mathcal{F}''$, then $s_1, a_1, a_2, b_1,\cdots,b_n$ meet at one vertex of $P^3(m)\times\Delta^n$ and

$$B(P^3(m)\times\Delta^n) = \{\lambda\in O(P^3(m)\times\Delta^n)|\lambda(s_1) = e_1, \lambda(a_1) = e_2, \lambda(a_2) = e_3, \lambda(b_i) = e_{i+3}, 1\le i\le n\}.$$

The calculation of $|B(P^3(m)\times\Delta^n)|$ is divided into two cases: (I) n even, (II) n odd.

(I) n **even**

Write

$$B_0(P^3(m)\times\Delta^n) = \{\lambda\in B(P^3(m)\times\Delta^n)|\lambda(b_{n+1}) = e_4+\cdots+e_{n+3}+e_1\},$$
$$B_1(P^3(m)\times\Delta^n) = \{\lambda\in B(P^3(m)\times\Delta^n)|\lambda(b_{n+1}) = e_4+\cdots+e_{n+3}+e_2\},$$
$$B_2(P^3(m)\times\Delta^n) = \{\lambda\in B(P^3(m)\times\Delta^n)|\lambda(b_{n+1}) = e_4+\cdots+e_{n+3}+e_3\},$$
$$B_3(P^3(m)\times\Delta^n) = \{\lambda\in B(P^3(m)\times\Delta^n)|\lambda(b_{n+1}) = e_4+\cdots+e_{n+3}+e_1+e_2+e_3\}.$$

By Remark 1, we have $|B(P^3(m)\times\Delta^n)| = \sum\limits_{i=0}^{3}|B_i(P^3(m)\times\Delta^n)|$. Then, our argument proceeds as follows.

Case 1. Calculation of $|B_0(P^3(m)\times\Delta^n)|$.

By Remark 1, we have $\lambda(s_2) = e_1, e_1+e_2+e_3$. Write

$$B_0^0(P^3(m)\times\Delta^n) = \{\lambda\in B_0(P^3(m)\times\Delta^n)|\lambda(s_2) = e_1\},$$

$B_0^1(P^3(m)\times\Delta^n)=\{\lambda\in B_0(P^3(m)\times\Delta^n)|\lambda(s_2)=e_1+e_2+e_3\}$.

Then, we have $|B_0(P^3(m)\times\Delta^n)|=|B_0^0(P^3(m)\times\Delta^n)|+|B_0^1(P^3(m)\times\Delta^n)|$.

(1.1) Calculation of $|B_0^0(P^3(m)\times\Delta^n)|$.

By Remark 1, we have that $\lambda(a_m)=e_3+e_{k_1}+\cdots+e_{k_i}, 1\le k_1<\cdots<k_i\le n+3, k_1\ne 3,\cdots,k_i\ne 3, i$ even and $0\le i\le n+2$. Set $B_0^{0,0}(P^3(m)\times\Delta^n)=\{\lambda\in B_0^0(P^3(m)\times\Delta^n)|\lambda(a_{m-1})=e_2+e_{f_1}+\cdots+e_{f_j}, 1\le f_1<\cdots<f_j\le n+3, f_1\ne 2,3,\cdots,f_j\ne 2,3, j$ even and $0\le j\le n+1\}$ and $B_0^{0,1}(P^3(m)\times\Delta^n)=B_0^0(P^3(m)\times\Delta^n)-B_0^{0,0}(P^3(m)\times\Delta^n)$. Take an orientable coloring λ in $B_0^{0,0}(P^3(m)\times\Delta^n)$. Then $\lambda(a_{m-2}),\lambda(a_m)\in\{e_3+e_{k_1}+\cdots+e_{k_i}, 1\le k_1<\cdots<k_i\le n+3, k_1\ne 3,\cdots,k_i\ne 3, i$ even and $0\le i\le n+2\}$. In this case, we see that the values of λ restricted to a_{m-1} and a_m have 2^{2n+1} possible choices. Thus, $|B_0^{0,0}(P^3(m)\times\Delta^n)|=2^{2n+1}|B_0^0(P^3(m-2)\times\Delta^n)|$. Take an orientable coloring λ in $B_0^{0,1}(P^3(m)\times\Delta^n)$. Then $\lambda(a_{m-1})=e_3+e_{k_1}+\cdots+e_{k_i}, 1\le k_1<\cdots<k_i\le n+3, k_1\ne 3,\cdots,k_i\ne 3, i$ even and $0\le i\le n+2$. In this case, if we fix any value of $\lambda(a_{m-1})$, then it is easy to see that $\lambda(a_m)$ has 2^n possible values. Thus $|B_0^{0,1}(P^3(m)\times\Delta^n)|=2^n|B_0^0(P^3(m-1)\times\Delta^n)|$. Further, we have that

$$|B_0^0(P^3(m)\times\Delta^n)|=2^n|B_0^0(P^3(m-1)\times\Delta^n)|+2^{2n+1}|B_0^0(P^3(m-2)\times\Delta^n)|.$$

A direct observation shows that $|B_0^0(P^3(2)\times\Delta^n)|=1$ and $|B_0^0(P^3(3)\times\Delta^n)|=2^n$. Thus, $|B_0^0(P^3(m)\times\Delta^n)|=a(m-1,n)$.

(1.2) Calculation of $|B_0^1(P^3(m)\times\Delta^n)|$.

In this case, $\lambda(a_m)=e_3,\lambda(a_{m-1})=e_2,\cdots,\lambda(a_{m-2i})=e_3,\lambda(a_{m-2i-1})=e_2,\cdots$. Thus, $|B_0^1(P^3(m)\times\Delta^n)|=\frac{1+(-1)^m}{2}$.

So, we have $|B_0(P^3(m)\times\Delta^n)|=a(m-1,n)+\frac{1+(-1)^m}{2}$.

Case 2. Calculation of $|B_1(P^3(m)\times\Delta^n)|$.

In this case, we have $\lambda(s_2)=e_1+e_{l_1}+\cdots+e_{l_i}, 2\le l_1<\cdots<l_i\le n+3, i$ even and $0\le i\le n+2$. Write

$B_1^0(P^3(m)\times\Delta^n)=\{\lambda\in B_1(P^3(m)\times\Delta^n)|\lambda(s_2)=e_1\}$,

$B_1^1(P^3(m)\times\Delta^n)=\{\lambda\in B_1(P^3(m)\times\Delta^n)|\lambda(s_2)=e_1+e_{f_1}+\cdots+e_{f_j}, 4\le f_1<\cdots<f_j\le n+3, j$ even and $2\le j\le n\}$,

$B_1^2(P^3(m)\times\Delta^n)=\{\lambda\in B_1(P^3(m)\times\Delta^n)|\lambda(s_2)=e_1+e_2+e_{f_1}+\cdots+e_{f_j}, 4\le$
$f_1<\cdots<f_j\le n+3, j$ odd and $1\le j\le n\}$,

$B_1^3(P^3(m)\times\Delta^n)=\{\lambda\in B_1(P^3(m)\times\Delta^n)|\lambda(s_2)=e_1+e_3+e_{f_1}+\cdots+e_{f_j}, 4\le$

$$f_1 < \cdots < f_j \leq n+3, j \text{ odd and } 1 \leq j \leq n\},$$

$$B_1^4(P^3(m)\times\Delta^n) = \{\lambda \in B_1(P^3(m)\times\Delta^n)|\lambda(s_2) = e_1+e_2+e_3+e_{f_1}+\cdots+ e_{f_j}, 4 \leq f_1 < \cdots < f_j \leq n+3, j \text{ even and } 0 \leq j \leq n\}.$$

Then, we have $|B_1(P^3(m)\times\Delta^n)| = \sum_{i=0}^{4} |B_1^i(P^3(m)\times\Delta^n)|$.

(2.1) Calculation of $|B_1^0(P^3(m)\times\Delta^n)|$.

By Remark 1, we have $\lambda(a_m) = e_3 + e_{k_1} + \cdots + e_{k_i}, 1 \leq k_1 < \cdots < k_i \leq n+3, k_1 \neq 3, \cdots, k_i \neq 3, i$ even and $0 \leq i \leq n+2$. Set $B_1^{0,0}(P^3(m)\times\Delta^n) = \{\lambda \in B_1^0(P^3(m)\times\Delta^n)|\lambda(a_{m-1}) = e_2\}$ and $B_1^{0,1}(P^3(m)\times\Delta^n) = B_1^0(P^3(m)\times\Delta^n) - B_1^{0,0}(P^3(m)\times\Delta^n)$. Take an orientable coloring λ in $B_1^{0,0}(P^3(m)\times\Delta^n)$. We then have $\lambda(a_{m-2}), \lambda(a_m) \in \{e_3 + e_{k_1} + \cdots + e_{k_i}, 1 \leq k_1 < \cdots < k_i \leq n+3, k_1 \neq 3, \cdots, k_i \neq 3, i$ even and $0 \leq i \leq n+2\}$, so $|B_1^{0,0}(P^3(m)\times\Delta^n)| = 2^{n+1}|B_1^0(P^3(m-2)\times\Delta^n)|$. Take an orientable coloring λ in $B_1^{0,1}(P^3(m)\times\Delta^n)$. We have $\lambda(a_{m-1}) = e_3 + e_{k_1} + \cdots + e_{k_i}, 1 \leq k_1 < \cdots < k_i \leq n+3, k_1 \neq 3, \cdots, k_i \neq 3, i$ even and $0 \leq i \leq n+2$, but $\lambda(a_m)$ has only one possible value whichever possible value of $\lambda(a_{m-1})$ is chosen, so $|B_1^{0,1}(P^3(m)\times\Delta^n)| = |B_1^0(P^3(m-1)\times\Delta^n)|$. We see that $|B_1^0(P^3(2)\times\Delta^n)| = |B_1^0(P^3(3)\times\Delta^n)| = 1$. Thus, $|B_1^0(P^3(m)\times\Delta^n)| = b(m-1,n)$.

(2.2) Calculation of $|B_1^1(P^3(m)\times\Delta^n)|$.

In this case, no matter which value of $\lambda(s_2)$ is chosen, we have $\lambda(a_m) = e_3 + e_{k_1} + \cdots + e_{k_i}, 1 \leq k_1 < \cdots < k_i \leq n+3, k_1 \neq 3, \cdots, k_i \neq 3, i$ even and $0 \leq i \leq n+2$. Set $B_1^{1,0}(P^3(m)\times\Delta^n) = \{\lambda \in B_1^1(P^3(m)\times\Delta^n)|\lambda(a_{m-1}) = e_2\}$ and $B_1^{1,1}(P^3(m)\times\Delta^n) = B_1^1(P^3(m)\times\Delta^n) - B_1^{1,0}(P^3(m)\times\Delta^n)$. Take an orientable coloring λ in $B_1^{1,0}(P^3(m)\times\Delta^n)$. Then $\lambda(a_{m-2}), \lambda(a_m) \in \{e_3 + e_{k_1} + \cdots + e_{k_i}, 1 \leq k_1 < \cdots < k_i \leq n+3, k_1 \neq 3, \cdots, k_i \neq 3, i$ even and $0 \leq i \leq n+2\}$. Thus, $|B_1^{1,0}(P^3(m)\times\Delta^n)| = 2^{n+1}|B_1^1(P^3(m-2)\times\Delta^n)|$. Take an orientable coloring λ in $B_1^{1,1}(P^3(m)\times\Delta^n)$. We have $\lambda(a_{m-1}) = e_3+e_{k_1}+\cdots+e_{k_i}, 1 \leq k_1 < \cdots < k_i \leq n+3, k_1 \neq 3, \cdots, k_i \neq 3, i$ even and $0 \leq i \leq n+2$, but $\lambda(a_m)$ has no possible values whichever possible value of $\lambda(a_{m-1})$ is chosen. Further, we have $|B_1^1(P^3(m)\times\Delta^n)| = 2^{n+1}|B_1^1(P^3(m-2)\times\Delta^n)|$. We see that $|B_1^1(P^3(2)\times\Delta^n)| = 2^{n-1}-1$ and $|B_1^1(P^3(3)\times\Delta^n)| = 0$. So, $|B_1^1(P^3(m)\times\Delta^n)| = \frac{1+(-1)^m}{2}\cdot(2^{n-1}-1)(2^{n+1})^{\frac{m}{2}-1}$.

(2.3) Calculation of $|B_1^2(P^3(m)\times\Delta^n)|$.

Similarly to above (2.2), $|B_1^2(P^3(m)\times\Delta^n)| = \frac{1+(-1)^m}{2}\cdot 2^{n-1}(2^{n+1})^{\frac{m}{2}-1}$.

(2.4) Calculation of $|B_1^3(P^3(m)\times\Delta^n)|$.

Similarly to above (2.2), $|B_1^3(P^3(m)\times\Delta^n)| = \frac{1+(-1)^m}{2}\cdot 2^{n-1}(2^n)^{\frac{m}{2}-1}$.

(2.5) Calculation of $|B_1^4(P^3(m)\times\Delta^n)|$.

Similarly to above (2.2), $|B_1^4(P^3(m)\times\Delta^n)| = \frac{1+(-1)^m}{2}\cdot 2^{n-1}(2^n)^{\frac{m}{2}-1}$.

Thus, $|B_1(P^3(m)\times\Delta^n)| = b(m-1,n)+\frac{1+(-1)^m}{2}[(2^n-1)(2^{n+1})^{\frac{m}{2}-1}+(2^n)^{\frac{m}{2}}]$.

Case 3. Calculation of $|B_2(P^3(m)\times\Delta^n)|$.

In this case, we have $\lambda(s_2) = e_1+e_{l_1}+\cdots+e_{l_i}, 2\le l_1<\cdots<l_i\le n+3, i$ even and $0\le i\le n+2$. If we interchange e_2 and e_3, then the problem is reduced to above Case 2, so $|B_2(P^3(m)\times\Delta^n)| = b(m-1,n)+\frac{1+(-1)^m}{2}[(2^n-1)(2^{n+1})^{\frac{m}{2}-1}+(2^n)^{\frac{m}{2}}]$.

Case 4. Calculation of $|B_3(P^3(m)\times\Delta^n)|$.

By Remark 1, we have $\lambda(s_2) = e_1, e_1+e_2+e_3$. Write

$B_3^0(P^3(m)\times\Delta^n) = \{\lambda\in B_3(P^3(m)\times\Delta^n)|\lambda(s_2) = e_1\}$,

$B_3^1(P^3(m)\times\Delta^n) = \{\lambda\in B_3(P^3(m)\times\Delta^n)|\lambda(s_2) = e_1+e_2+e_3\}$.

Then, we have $|B_3(P^3(m)\times\Delta^n)| = |B_3^0(P^3(m)\times\Delta^n)|+|B_3^1(P^3(m)\times\Delta^n)|$.

(4.1) Calculation of $|B_3^0(P^3(m)\times\Delta^n)|$.

In this case, we have $\lambda(a_m) = e_3, e_3+e_1+e_2$. Set $B_3^{0,0}(P^3(m)\times\Delta^n) = \{\lambda\in B_3^0(P^3(m)\times\Delta^n)|\lambda(a_{m-1}) = e_2\}$, $B_3^{0,1}(P^3(m)\times\Delta^n) = \{\lambda\in B_3^0(P^3(m)\times\Delta^n)|\lambda(a_{m-1}) = e_3, e_3+e_1+e_2\}$, and $B_3^{0,2}(P^3(m)\times\Delta^n) = \{\lambda\in B_3^0(P^3(m)\times\Delta^n)|\lambda(a_{m-1}) = e_2+e_{k_1}+\cdots+e_{k_i}, e_3+e_{k_1}+\cdots+e_{k_i}, 1\le k_1<\cdots<k_i\le n+3, k_1\ne 2,3,\cdots,k_i\ne 2,3, i$ even and $2\le i\le n+1\}$. Then $|B_3^0(P^3(m)\times\Delta^n)| = |B_3^{0,0}(P^3(m)\times\Delta^n)|+|B_3^{0,1}(P^3(m)\times\Delta^n)|+|B_3^{0,2}(P^3(m)\times\Delta^n)|$. An easy argument shows that $|B_3^{0,0}(P^3(m)\times\Delta^n)| = 2|B_3^0(P^3(m-2)\times\Delta^n)|$ and $|B_3^{0,1}(P^3(m)\times\Delta^n)| = |B_3^0(P^3(m-1)\times\Delta^n)|$, so

$$|B_3^0(P^3(m)\times\Delta^n)| = |B_3^0(P^3(m-1)\times\Delta^n)|+2|B_3^0(P^3(m-2)\times\Delta^n)|+|B_3^{0,2}(P^3(m)\times\Delta^n)|. \quad (2)$$

Set $B'(m,n) = \{\lambda\in B_3^{0,2}(P^3(m)\times\Delta^n)|\lambda(a_{m-2}) = e_3+e_1+e_2\}$. Then we see that

$$|B_3^{0,2}(P^3(m)\times\Delta^n)| = |B_3^{0,2}(P^3(m-1)\times\Delta^n)|+|B'(m,n)| \quad (3)$$

and

$$|B'(m,n)| = (2^{n+1}-2)|B_3^{0,2}(P^3(m-2)\times\Delta^n)|+(2^{n+1}-2)|B_3^0(P^3(m-4)\times\Delta^n)|+(2^{n+1}-2)|B_3^0(P^3(m-5)\times\Delta^n)|+|B'(m-2,n)| \quad (4)$$

Combining Eqs. (2), (3) and (4), we obtain

$|B_3^0(P^3(m)\times\Delta^n)| = 2|B_3^0(P^3(m-1)\times\Delta^n)|+2^{n+1}|B_3^0(P^3(m-2)\times\Delta^n)|-(2^{n+1}+2)|B_3^0(P^3(m-3)\times\Delta^n)|-(2^{n+1}-1)|B_3^0(P^3(m-4)\times\Delta^n)|+$

$2^{n+1}|B_3^0(P^3(m-5)\times\Delta^n)|$.

A direct observation gives that $|B_3^0(P^3(2)\times\Delta^n)| = |B_3^0(P^3(3)\times\Delta^n)| = 1, |B_3^0(P^3(4)\times\Delta^n)| = 3, |B_3^0(P^3(5)\times\Delta^n)| = 2^{n+1}+3$, and $|B_3^0(P^3(6)\times\Delta^n)| = 3\cdot 2^{n+1}+5$. Thus, we have $|B_3^0(P^3(m)\times\Delta^n)| = c(m-1,n)$.

(4.2) Calculation of $|B_3^1(P^3(m)\times\Delta^n)|$.

In this case, $\lambda(a_m) = e_3, \lambda(a_{m-1}) = e_2, \cdots, \lambda(a_{m-2i}) = e_3, \lambda(a_{m-2i-1}) = e_2, \cdots$. Thus, $|B_3^1(P^3(m)\times\Delta^n)| = \frac{1+(-1)^m}{2}$.

So, $|B_3(P^3(m)\times\Delta^n)| = c(m-1,n) + \frac{1+(-1)^m}{2}$.

(II) n **odd**

Write

$$B_0(P^3(m)\times\Delta^n) = \{\lambda\in B(P^3(m)\times\Delta^n)|\lambda(b_{n+1}) = e_4+\cdots+e_{n+3}\},$$
$$B_1(P^3(m)\times\Delta^n) = \{\lambda\in B(P^3(m)\times\Delta^n)|\lambda(b_{n+1}) = e_4+\cdots+e_{n+3}+e_1+e_2\},$$
$$B_2(P^3(m)\times\Delta^n) = \{\lambda\in B(P^3(m)\times\Delta^n)|\lambda(b_{n+1}) = e_4+\cdots+e_{n+3}+e_1+e_3\},$$
$$B_3(P^3(m)\times\Delta^n) = \{\lambda\in B(P^3(m)\times\Delta^n)|\lambda(b_{n+1}) = e_4+\cdots+e_{n+3}+e_2+e_3\}.$$

By Remark 1, we have that $|B(P^3(m)\times\Delta^n)| = \sum_{i=0}^{3}|B_i(P^3(m)\times\Delta^n)|$. Then, our argument proceeds as follows.

Case 1. Calculation of $|B_0(P^3(m)\times\Delta^n)|$.

In this case, we have $\lambda(s_2) = e_1+e_{l_1}+\cdots+e_{l_i}, 2\le l_1<\cdots<l_i\le n+3, i$ even and $0\le i\le n+2$. Write

$$B_0^0(P^3(m)\times\Delta^n) = \{\lambda\in B_0(P^3(m)\times\Delta^n)|\lambda(s_2) = e_1+e_{f_1}+\cdots+e_{f_j}, 4\le f_1<\cdots<f_j\le n+3, j \text{ even and } 0\le j\le n\},$$
$$B_0^1(P^3(m)\times\Delta^n) = \{\lambda\in B_0(P^3(m)\times\Delta^n)|\lambda(s_2) = e_1+e_2+e_{f_1}+\cdots+e_{f_j}, 4\le f_1<\cdots<f_j\le n+3, j \text{ odd and } 1\le j\le n\},$$
$$B_0^2(P^3(m)\times\Delta^n) = \{\lambda\in B_0(P^3(m)\times\Delta^n)|\lambda(s_2) = e_1+e_3+e_{f_1}+\cdots+e_{f_j}, 4\le f_1<\cdots<f_j\le n+3, j \text{ odd and } 1\le j\le n\},$$
$$B_0^3(P^3(m)\times\Delta^n) = \{\lambda\in B_0(P^3(m)\times\Delta^n)|\lambda(s_2) = e_1+e_2+e_3+e_{f_1}+\cdots+e_{f_j}, 4\le f_1<\cdots<f_j\le n+3, j \text{ even and } 0\le j\le n\}.$$

Then, we have $|B_0(P^3(m)\times\Delta^n)| = \sum_{i=0}^{3}|B_0^i(P^3(m)\times\Delta^n)|$.

(1.1) Calculation of $|B_0^0(P^3(m)\times\Delta^n)|$.

Similarly to (1.1) in (I), $|B_0^0(P^3(m)\times\Delta^n)| = 2^{n-1}a(m-1,n)$.

(1.2) Calculation of $|B_0^1(P^3(m)\times\Delta^n)|$.

By Remark 1, we have $\lambda(a_m) = e_3 + e_{k_1} + \cdots + e_{k_i}, 1 \le k_1 < \cdots < k_i \le n+3, k_1 \neq 3, \cdots, k_i \neq 3, i$ even and $0 \le i \le n+2$. Set $B_0^{1,0}(P^3(m)\times\Delta^n) = \{\lambda \in B_0^1(P^3(m)\times\Delta^n) | \lambda(a_{m-1}) = e_2 + e_{g_1} + \cdots + e_{g_k}, 4 \le g_1 < \cdots < g_k \le n+3, k$ even and $0 \le k \le n\}$ and $B_0^{1,1}(P^3(m)\times\Delta^n) = B_0^1(P^3(m)\times\Delta^n) - B_0^{1,0}(P^3(m)\times\Delta^n)$. Take an orientable coloring λ in $B_0^{1,0}(P^3(m)\times\Delta^n)$. Then $\lambda(a_{m-2}), \lambda(a_m) \in \{e_3 + e_{k_1} + \cdots + e_{k_i}, 1 \le k_1 < \cdots < k_i \le n+3, k_1 \neq 3, \cdots, k_i \neq 3, i$ even and $0 \le i \le n+2\}$. In this case, we see that the values of λ restricted to a_{m-1} and a_m have 2^{2n} possible choices. Thus, $|B_0^{1,0}(P^3(m)\times\Delta^n)| = 2^{2n}|B_0^1(P^3(m-2)\times\Delta^n)|$. Take an orientable coloring λ in $B_0^{1,1}(P^3(m)\times\Delta^n)$. Then $\lambda(a_{m-1}) = e_3 + e_{k_1} + \cdots + e_{k_i}, 1 \le k_1 < \cdots < k_i \le n+3, k_1 \neq 3, \cdots, k_i \neq 3, i$ even and $0 \le i \le n+2$. In this case, if we fix any value of $\lambda(a_{m-1})$, then it is easy to see that $\lambda(a_m)$ has 2^{n-1} possible values. Thus $|B_0^{1,1}(P^3(m)\times\Delta^n)| = 2^{n-1}|B_0^1(P^3(m-1)\times\Delta^n)|$. A direct observation shows that $|B_0^1(P^3(2)\times\Delta^n)| = 2^{n-1}$ and $|B_0^1(P^3(3)\times\Delta^n)| = 2^{2n-2}$. So, $|B_0^1(P^3(m)\times\Delta^n)| = 2^{n-1}d(m-1,n)$.

(1.3) Calculation of $|B_0^2(P^3(m)\times\Delta^n)|$.

If we interchange e_2 and e_3, then the case is reduced to above (1.2) in (II), so $|B_0^2(P^3(m)\times\Delta^n)| = 2^{n-1}d(m-1,n)$.

(1.4) Calculation of $|B_0^3(P^3(m)\times\Delta^n)|$.

In this case, no matter which value of $\lambda(s_2)$ is chosen, we have $\lambda(a_m) = e_3 + e_{g_1} + \cdots + e_{g_k}, 2 \le g_1 < \cdots < g_k \le n+3, g_1 \neq 3, \cdots, g_k \neq 3, k$ even and $0 \le k \le n+1$. Set $B_0^{3,0}(P^3(m)\times\Delta^n) = \{\lambda \in B_0^3(P^3(m)\times\Delta^n) | \lambda(a_{m-1}) = e_2 + e_{h_1} + \cdots + e_{h_j}, 4 \le h_1 < \cdots < h_j \le n+3, j$ even and $0 \le j \le n\}$, $B_0^{3,1}(P^3(m)\times\Delta^n) = \{\lambda \in B_0^3(P^3(m)\times\Delta^n) | \lambda(a_{m-1}) = e_3 + e_{g_1} + \cdots + e_{g_k}, 2 \le g_1 < \cdots < g_k \le n+3, g_1 \neq 3, \cdots, g_k \neq 3, k$ even and $0 \le k \le n+1\}$, and $B_0^{3,2}(P^3(m)\times\Delta^n) = \{\lambda \in B_0^3(P^3(m)\times\Delta^n) | \lambda(a_{m-1}) = e_2 + e_1 + e_{h_1} + \cdots + e_{h_j}, e_3 + e_1 + e_{h_1} + \cdots + e_{h_j}, 4 \le h_1 < \cdots < h_j \le n+3, j$ odd and $0 \le j \le n\}$. Then $|B_0^3(P^3(m)\times\Delta^n)| = |B_0^{3,0}(P^3(m)\times\Delta^n)| + |B_0^{3,1}(P^3(m)\times\Delta^n)| + |B_0^{3,2}(P^3(m)\times\Delta^n)|$. An easy argument shows that $|B_0^{3,0}(P^3(m)\times\Delta^n)| = 2^{2n-1}|B_0^3(P^3(m-2)\times\Delta^n)|$ and $|B_0^{3,1}(P^3(m)\times\Delta^n)| = 2^{n-1}|B_0^3(P^3(m-1)\times\Delta^n)|$, so

$$\begin{aligned}|B_0^3(P^3(m)\times\Delta^n)| = 2^{n-1}|B_0^3(P^3(m-1)\times\Delta^n)| + 2^{2n-1}|B_0^3(P^3(m-2)\\ \times\Delta^n)| + |B_0^{3,2}(P^3(m)\times\Delta^n|. \qquad (2)\end{aligned}$$

Set $B''(m,n) = \{\lambda \in B_0^{3,2}(P^3(m)\times\Delta^n) | \lambda(a_{m-2}) = e_3 + e_2 + e_{h_1} + \cdots + e_{h_j}, 4 \le$

$h_1 < \cdots < h_j \le n+3, j$ odd and $0 \le j \le n\}$. Then we see that

$$|B_0^{3,2}(P^3(m)\times\Delta^n)| = 2^{n-1}|B_0^{3,2}(P^3(m-1)\times\Delta^n)|+|B''(m,n)| \quad (3)$$

and

$$|B''(m,n)| = 2^{2n-1}|B_0^{3,2}(P^3(m-2)\times\Delta^n)| + 2^{4n-3}|B_0^3(P^3(m-4)\times\Delta^n)|+2^{5n-4}|B_0^3(P^3(m-5)\times\Delta^n)|+2^{2n-2}|B''(m-2,n)| \quad (4)$$

Combining Eqs. (2), (3) and (4), we obtain

$$|B_0^3(P^3(m)\times\Delta^n)| = 2^n|B_0^3(P^3(m-1)\times\Delta^n)| + 2^{2n}|B_0^3(P^3(m-2)\times\Delta^n)| - 3\cdot 2^{3n-2}|B_0^3(P^3(m-3)\times\Delta^n)| - 3\cdot 2^{4n-4}|B_0^3(P^3(m-4)\times\Delta^n)| + 2^{5n-3}|B_0^3(P^3(m-5)\times\Delta^n)|.$$

A direct observation gives that $|B_0^3(P^3(2)\times\Delta^n)| = 2^{n-1}, |B_0^3(P^3(3)\times\Delta^n)| = 2^{2n-2}, |B_0^3(P^3(4)\times\Delta^n)| = 3\cdot 2^{3n-3}, |B_0^3(P^3(5)\times\Delta^n)| = 7\cdot 2^{4n-4}$, and $|B_0^3(P^3(6)\times\Delta^n)| = 17\cdot 2^{5n-5}$. Thus, we have $|B_0^3(P^3(m)\times\Delta^n)| = 2^{n-1}e(m-1,n)$.

So, $|B_0(P^3(m)\times\Delta^n)| = 2^{n-1}[a(m-1,n)+2d(m-1,n)+e(m-1,n)]$.

Case 2. Calculation of $|B_1(P^3(m)\times\Delta^n)|$.

By Remark 1, we have $\lambda(s_2) = e_1, e_1+e_2+e_3$. Write

$B_1^0(P^3(m)\times\Delta^n) = \{\lambda \in B_1(P^3(m)\times\Delta^n)|\lambda(s_2) = e_1\}$,

$B_1^1(P^3(m)\times\Delta^n) = \{\lambda \in B_1(P^3(m)\times\Delta^n)|\lambda(s_2) = e_1+e_2+e_3\}$.

Then, we have $|B_1(P^3(m)\times\Delta^n)| = |B_1^0(P^3(m)\times\Delta^n)| + |B_1^1(P^3(m)\times\Delta^n)|$.

(2.1) Calculation of $|B_1^0(P^3(m)\times\Delta^n)|$.

Just as Case (2.1) in (I), $|B_1^0(P^3(m)\times\Delta^n)| = b(m-1,n)$.

(2.2) Calculation of $|B_1^1(P^3(m)\times\Delta^n)|$.

In this case, $\lambda(a_m) = e_3, \lambda(a_{m-1}) = e_2, \cdots, \lambda(a_{m-2i}) = e_3, \lambda(a_{m-2i-1}) = e_2, \cdots$. Thus, $|B_1^1(P^3(m)\times\Delta^n)| = \frac{1+(-1)^m}{2}$.

So, $|B_1(P^3(m)\times\Delta^n)| = b(m-1,n) + \frac{1+(-1)^m}{2}$.

Case 3. Calculation of $|B_2(P^3(m)\times\Delta^n)|$.

In this case, $\lambda(s_2) = e_1, e_1+e_2+e_3$. If we interchange e_2 and e_3, then the case is reduced to above Case 2 in (II), so $|B_2(P^3(m)\times\Delta^n)| = b(m-1,n) + \frac{1+(-1)^m}{2}$.

Case 4. Calculation of $|B_3(P^3(m)\times\Delta^n)|$.

We have $\lambda(s_2) = e_1+e_{l_1}+\cdots+e_{l_i}, 2 \le l_1 < \cdots < l_i \le n+3, i$ even and $0 \le i \le n+2$. Write

$$B_3^0(P^3(m)\times\Delta^n)=\{\lambda\in B_3(P^3(m)\times\Delta^n)|\lambda(s_2)=e_1\},$$
$$B_3^1(P^3(m)\times\Delta^n)=\{\lambda\in B_3(P^3(m)\times\Delta^n)|\lambda(s_2)=e_1+e_{l_1}+\cdots+e_{l_i},2\leq l_1<\cdots<l_i\leq n+3, i \text{ even and } 2\leq i\leq n+2\}.$$

Then, $|B_3(P^3(m)\times\Delta^n)|=|B_3^0(P^3(m)\times\Delta^n)|+|B_3^1(P^3(m)\times\Delta^n)|$.

(4.1) Calculation of $|B_3^0(P^3(m)\times\Delta^n)|$.

Just as Case (4.1) in (I), $|B_3^0(P^3(m)\times\Delta^n)|=c(m-1,n)$.

(4.2) Calculation of $|B_3^1(P^3(m)\times\Delta^n)|$.

No matter which value of $\lambda(s_2)$ is chosen, we have $\lambda(a_m)=e_3,\lambda(a_{m-1})=e_2,\cdots,\lambda(a_{m-2i})=e_3,\lambda(a_{m-2i-1})=e_2,\cdots$. Thus, $|B_3^1(P^3(m)\times\Delta^n)|=(2^{n+1}-1)\cdot\frac{1+(-1)^m}{2}$. So, $|B_3(P^3(m)\times\Delta^n)|=c(m-1,n)+(2^{n+1}-1)\cdot\frac{1+(-1)^m}{2}$.

The proof is completed. $\square$

Remark 2. Using the formula in Theorem 3.1, we may calculate the number $|B(P^3(m)\times\Delta^n)|$ of D-J equivalence classes of orientable small covers over $P^3(m)\times\Delta^n$ for several small m and $n=1,2$.

m	3	4	5	6
$\lvert B(P^3(m)\times\Delta^1)\rvert$	8	43	90	331
$\lvert B(P^3(m)\times\Delta^2)\rvert$	7	151	365	3537

Note that from Theorem 3.1 we can determine the number $|O(P^3(m)\times\Delta^n|$ of all orientable colorings on $P^3(m)\times\Delta^n$.

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College of Mathematics and Information Science
Henan Normal University
Xinxiang 453007, P. R. China
and
College of Mathematics and Information Science
Hebei Normal University
Shijiazhuang 050016, P. R. China e-mail:cyc810707@163.com

College of Mathematics and Information Science
Hebei Normal University
Shijiazhuang 050016, P. R. China e-mail:wyanying2003@yahoo.com.cn