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Independence Number and Minimum Degree for the Existence of (g, f, n)-Critical Graphs

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Abstract

Let G be a graph, and let g, f be two integer-valued functions defined on $V(G)$ with $0 \leq g(x) \leq f(x)$ for each $x \in V(G)$. Then a spanning subgraph F of G is called a (g, f)-factor if $g(x) \leq d_F(x) \leq f(x)$ holds for each $x \in V(G)$. A graph G is said to be (g, f, n)-critical if $G - N$ has a (g, f)-factor for each $N \subseteq V(G)$ with $|N| = n$. In this paper, we obtain an independence number and minimum degree condition for a graph G to be a (g, f, n)-critical graph. Moreover, it is showed that the result in this paper is best possible in some sense.

1 Introduction

Many physical structures can conveniently be modelled by networks. Examples include a communication network with the nodes and links modelling cities and communication channels, respectively, or a railroad network with nodes and links representing railroad stations and railways between two stations, respectively. Factors and factorizations in networks are very useful in combinatorial design, network design, circuit layout, and so on. It is well known that a network can be represented by a graph. Vertices and edges of the graph correspond to nodes and links between the nodes, respectively. Henceforth we use the term *graph* instead of *network*.

Key Words: graph, independence number, minimum degree, (g, f)-factor, (g, f, n)-critical graph

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All graphs considered in this paper will be finite and undirected graphs without loops or multiple edges. Let G be a graph. We denote by $V(G)$ and $E(G)$ the set of vertices and the set of edges, respectively. For $x \in V(G)$, the degree of x and the set of vertices adjacent to x in G are denoted by $d_G(x)$ and $N_G(x)$, respectively. We write $N_G[x]$ for $N_G(x) \cup \{x\}$. The independence number and the minimum degree of G are denoted by $\alpha(G)$ and $\delta(G)$, respectively. For any subset $S \subseteq V(G)$, we denote by $G[S]$ the subgraph of G induced by S, and $G - S = G[V(G) \setminus S]$. If S and T are disjoint subsets of $V(G)$, then $e_G(S,T)$ denotes the number of edges that join a vertex in S and a vertex in T.

Let g, f be two integer-valued functions defined on $V(G)$ with $0 \leq g(x) \leq f(x)$ for each $x \in V(G)$. Then a spanning subgraph F of G is called a (g,f)-factor if $g(x) \leq d_F(x) \leq f(x)$ holds for each $x \in V(G)$. Let a and b be two integers with $0 \leq a \leq b$. If $g(x) = a$ and $f(x) = b$ for each $x \in V(G)$, then a (g,f)-factor is called an $[a,b]$-factor. A graph G is said to be (g,f,n)-critical if $G - N$ has a (g,f)-factor for each $N \subseteq V(G)$ with $|N| = n$. If $g(x) = a$ and $f(x) = b$ for each $x \in V(G)$, then a (g,f,n)-critical graph is called an (a,b,n)-critical graph. If $a = b = k$, then an (a,b,n)-critical graph is simply called a (k,n)-critical graph. If $k = 1$, then a (k,n)-critical graph is simply called an n-critical graph.

Many authors have investigated (g,f)-factors [1,2,8] and $[a,b]$-factors [4,12]. O. Favaron [3] studied the properties of n-critical graphs. G. Liu and Q. Yu [10] studied the characterization of (k,n)-critical graphs. G. Liu and J. Wang [9] gave the characterization of (a,b,n)-critical graph with $a < b$. S. Zhou [15,16,17,19,20] gave some sufficient conditions for graphs to be (a,b,n)-critical graphs. J. Li [5,6] gave three sufficient conditions for graphs to be (a,b,n)-critical graphs. A necessary and sufficient condition for a graph to be (g,f,n)-critical was given by Li and Matsuda [7]. Zhou [13,14,18] obtained some sufficient conditions for graphs to be (g,f,n)-critical graphs. Liu [11] showed a binding number and minimum degree condition for a graph to be (g,f,n)-critical.

The following results on (a,b,n)-critical graphs and (g,f,n)-critical graphs are known.

Zhou [20] obtained the following result on neighborhoods of independent sets for graphs to be (a,b,n)-critical graphs.

Theorem 1. [20] *Let a, b and n be nonnegative integers with $1 \leq a < b$, and let G be a graph of order p with $p \geq \frac{(a+b)(a+b-2)}{b} + n$. Suppose that*

$$|N_G(X)| > \frac{(a-1)p + |X| + bn - 1}{a+b-1}$$

for every non-empty independent subset X of $V(G)$, and

$$\delta(G) > \frac{(a-1)p+a+b+bn-2}{a+b-1}.$$

Then G is an (a,b,n)-critical graph.

Li [6] obtained the following results for the existence of (a,b,n)-critical graphs.

Theorem 2. [6] *Let a,b,m and n be integers such that $1 \leq a < b$, and let G be a graph of order m with $m \geq \frac{(a+b)(k(a+b)-2)}{b} + n$. If $\delta(G) \geq (k-1)a+n$, and*

$$|N_G(x_1) \cup N_G(x_2) \cup \cdots \cup N_G(x_k)| \geq \frac{am+bn}{a+b}$$

for any independent subset $\{x_1, x_2, \cdots, x_k\}$ of $V(G)$, where $k \geq 2$, then G is an (a,b,n)-critical graph.

Theorem 3. [6] *Let a,b,m and n be integers such that $1 \leq a < b$, and let G be a graph of order m with $m \geq \frac{(a+b)(a+b+k-3+(a-2)(k-2))+1}{b} + n$. If $\delta(G) \geq (k-1)a+n$, and*

$$\max\{d_G(x_1), d_G(x_2), \cdots, d_G(x_k)\} \geq \frac{am+bn}{a+b}$$

for any independent subset $\{x_1, x_2, \cdots, x_k\}$ of $V(G)$, where $k \geq 2$, then G is an (a,b,n)-critical graph.

Zhou [13] gave a binding number condition for a graph to be a (g,f,n)-critical graph.

Theorem 4. [13] *Let G be a graph of order p, and let a,b and n be nonnegative integers such that $1 \leq a < b$, and let g and f be two integer-valued functions defined on $V(G)$ such that $a \leq g(x) < f(x) \leq b$ for each $x \in V(G)$. If the binding number $bind(G) > \frac{(a+b-1)(p-1)}{(a+1)p-(a+b)-bn+2}$ and $p \geq \frac{(a+b-1)(a+b-2)}{a+1} + \frac{bn}{a}$, then G is a (g,f,n)-critical graph.*

In this paper, we discuss an independence number and minimum degree condition for graphs to be (g,f,n)-critical graphs. The main result will be given in the following section.

2 The Main Result and Its Proof

In this section, we firstly give our main result on (g,f,n)-critical graphs.

Theorem 5. *Let G be a graph, and let a, b, n be nonnegative integers with $0 \leq a < b$. Let g, f be two integer-valued functions defined on $V(G)$ such that $a \leq g(x) < f(x) \leq b$ for each $x \in V(G)$. If G satisfies*

$$\alpha(G) \leq \frac{4(a+1)(\delta(G)-b+2)-4bn}{b^2},$$

then G is a (g, f, n)-critical graph.

In Theorem 5, if $n = 0$, then we get the following corollary.

Corollary 1. *Let G be a graph, and let a and b be nonnegative integers with $a < b$. Let g, f be two integer-valued functions defined on $V(G)$ such that $a \leq g(x) < f(x) \leq b$ for each $x \in V(G)$. If G satisfies*

$$\alpha(G) \leq \frac{4(a+1)(\delta(G)-b+2)}{b^2},$$

then G has a (g, f)-factor.

In Theorem 5, if $g(x) \equiv a$ and $f(x) \equiv b$, then we obtain the following corollary.

Corollary 2. *Let G be a graph, and let a, b, n be nonnegative integers with $0 \leq a < b$. If G satisfies*

$$\alpha(G) \leq \frac{4(a+1)(\delta(G)-b+2)-4bn}{b^2},$$

then G is an (a, b, n)-critical graph.

Let g, f be two nonnegative integer-valued functions defined on $V(G)$ with $g(x) < f(x)$ for each $x \in V(G)$. If $S, T \subseteq V(G)$, then we define $f(S) = \sum_{x \in S} f(x)$, $g(T) = \sum_{x \in T} g(x)$ and $d_{G-S}(T) = \sum_{x \in T} d_{G-S}(x)$. If S and T are disjoint subsets of $V(G)$ define

$$\delta_G(S, T) = f(S) + d_{G-S}(T) - g(T),$$

and if $|S| \geq n$ define

$$f_n(S) = \max\{f(U) : U \subseteq S \text{ and } |U| = n\}. \tag{1}$$

Li and Matsuda [7] obtained a necessary and sufficient condition for a graph to be a (g, f, n)-critical graph, which is very useful in the proof of Theorem 5.

Theorem 6. [7] *Let G be a graph, $n \geq 0$ an integer, and let g and f be two integer-valued functions defined on $V(G)$ such that $g(x) < f(x)$ for each $x \in V(G)$. Then G is a (g, f, n)-critical graph if and only if for any $S \subseteq V(G)$ with $|S| \geq n$*

$$\delta_G(S, T) \geq f_n(S),$$

where $T = \{x : x \in V(G) \setminus S, d_{G-S}(x) \leq g(x) - 1\}$.

Proof of Theorem 5. We prove the theorem by contradiction. Suppose that a graph G satisfies the assumption of Theorem 5, but is not a (g, f, n)-critical graph. Then by Theorem 6, there exists a subset S of $V(G)$ with $|S| \geq n$ such that

$$\delta_G(S, T) \leq f_n(S) - 1, \tag{2}$$

where $T = \{x : x \in V(G) \setminus S,\ d_{G-S}(x) \leq g(x) - 1\}$. We firstly prove the following claim.

Claim 1. $d_{G-S}(x) \leq g(x) - 1 \leq b - 2$ for each $x \in T$.

Proof. According to the definition of T and the condition of the theorem, Claim 1 clearly holds. This completes the proof of Claim 1.

If $T = \emptyset$, then by (1) and (2), $f(S) - 1 \geq f_n(S) - 1 \geq \delta_G(S, T) = f(S)$, which is a contradiction. Hence, $T \neq \emptyset$. Define

$$h = \min\{d_{G-S}(x) | x \in T\}.$$

In view of Claim 1, we obtain

$$0 \leq h \leq b - 2. \tag{3}$$

Let x_1 be a vertex in T such that $d_{G-S}(x_1) = h$. Then we obtain $\delta(G) \leq d_G(x_1) \leq d_{G-S}(x_1) + |S| = h + |S|$. Thus

$$|S| \geq \delta(G) - h. \tag{4}$$

Since $a \leq g(x) < f(x) \leq b$ for each $x \in V(G)$, it follows from (1) and (2) that

$$\delta_G(S, T) \leq f_n(S) - 1 \leq bn - 1 \tag{5}$$

and

$$\delta_G(S, T) = f(S) + d_{G-S}(T) - g(T) \geq (a + 1)|S| + d_{G-S}(T) - (b - 1)|T|,$$

so that

$$bn - 1 \geq (a + 1)|S| + d_{G-S}(T) - (b - 1)|T|. \tag{6}$$

We now consider the subgraph $G[T]$ of G induced by T. Set $T_1 = G[T]$. Let x_1 be a vertex with minimum degree in T_1 and $M_1 = N_{T_1}[x_1]$. Moreover, for $i \geq 2$, let x_i be a vertex with minimum degree in $T_i = G[T] - \bigcup_{1\leq j<i} M_j$ and $M_i = N_{T_i}[x_i]$. We denote by m_i the cardinality of M_i. We continue these procedures until we reach the situation in which $T_i = \emptyset$ for some i, say for $i = r+1$. Then from the above definition we know that $\{x_1, x_2, \cdots, x_r\}$ is an independent set of G. Since $T \neq \emptyset$, we have $r \geq 1$.

The following properties are easily verified ((7) and (8) are trivial; (9) follows because our choice of x_i implies that all vertices in M_i have degree at least $m_i - 1$ in T_i).

$$\alpha(G[T]) \geq r, \tag{7}$$

$$|T| = \sum_{1\leq i\leq r} m_i, \tag{8}$$

$$\sum_{1\leq i\leq r} (\sum_{x\in M_i} d_{T_i}(x)) \geq \sum_{1\leq i\leq r} (m_i^2 - m_i). \tag{9}$$

According to (9), we have

$$d_{G-S}(T) \geq \sum_{1\leq i\leq r} (m_i^2 - m_i) + \sum_{1\leq i<j\leq r} e_G(M_i, M_j) \geq \sum_{1\leq i\leq r} (m_i^2 - m_i). \tag{10}$$

By (7), the obvious inequality $\alpha(G) \geq \alpha(G[T])$ and the assumption $\alpha(G) \leq \frac{4(a+1)(\delta(G)-b+2)-4bn}{b^2}$, we obtain

$$r \leq \frac{4(a+1)(\delta(G)-b+2)-4bn}{b^2}. \tag{11}$$

In view of (6), (8), (10), (11) and the obvious inequality $m_i^2 - bm_i \geq -\frac{b^2}{4}$, we have

$$\begin{aligned}
bn - 1 &\geq (a+1)|S| + d_{G-S}(T) - (b-1)|T| \\
&\geq (a+1)|S| + \sum_{1\leq i\leq r} (m_i^2 - m_i) - (b-1)\sum_{1\leq i\leq r} m_i \\
&= (a+1)|S| + \sum_{1\leq i\leq r} (m_i^2 - bm_i) \\
&\geq (a+1)|S| - \frac{b^2 r}{4} \\
&\geq (a+1)|S| - \frac{b^2}{4} \cdot \frac{4(a+1)(\delta(G)-b+2)-4bn}{b^2} \\
&= (a+1)|S| - (a+1)(\delta(G)-b+2) + bn,
\end{aligned}$$

that is,

$$bn - 1 \geq (a+1)|S| - (a+1)(\delta(G)-b+2) + bn. \tag{12}$$

From (3), (4) and (12), we have

$$\begin{aligned} bn-1 &\geq (a+1)|S|-(a+1)(\delta(G)-b+2)+bn \\ &\geq (a+1)(\delta(G)-h)-(a+1)(\delta(G)-b+2)+bn \\ &= (a+1)(b-2-h)+bn \\ &\geq bn. \end{aligned}$$

This is a contradiction.

From the argument above, we deduce the contradictions. Hence, G is a (g, f, n)-critical graph.

Completing the proof of Theorem 5.

3 Remark

Let us show that the condition in Theorem 5 cannot be replaced by the condition that $\alpha(G) \leq \frac{4(a+1)(\delta(G)-b+2)-4bn}{b^2}+1$. Let $a=1$, $b=2$ and $n \geq 0$ be integers and $G = K_{a+n} \bigvee (b+1)K_1$. Obviously, we have $\alpha(G) = b+1 = \frac{4(a+1)(\delta(G)-b+2)-4bn}{b^2}+1$. Let $S = V(K_{a+n}) \subseteq V(G)$ and $T = V((b+1)K_1) \subseteq V(G)$, then $|S| = a+n > n$ and $|T| = b+1$. Since $a=1$, $b=2$ and $a \leq g(x) < f(x) \leq b$, then we have $g(x) = a$ and $f(x) = b$ for each $x \in V(G)$. Thus, we obtain

$$\begin{aligned} \delta_G(S,T) &= f(S)+d_{G-S}(T)-g(T) \\ &= b|S|+d_{G-S}(T)-a|T| \\ &= b(a+n)-a(b+1) \\ &= bn-a < bn = f_n(S). \end{aligned}$$

According to Theorem 6, G is not a (g, f, n)-critical graph. In the above sense, the condition in Theorem 5 is best possible.

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