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A MODIFIED ITERATIVE PROCESS FOR COMMON FIXED POINTS OF TWO FINITE FAMILIES OF NONEXPANSIVE MAPPINGS

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Abstract

In this paper, we introduce an iterative process for approximating common fixed points of two finite families of nonexpansive mappings in Banach spaces. Our process contains some iterative processes being used for the purpose. We prove some weak and strong convergence theorems for this iterative process. Our results generalize and improve some results in contemporary literature.

1 Introduction

Let K be a nonempty closed convex subset of a real normed linear space E, and $T : K \to K$ a mapping. T is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in K$. Throughout this paper, $\mathbb{N}$ denotes the set of natural numbers and $J = \{1, 2, ..., N\}$, the set of first N natural numbers. Denote by $F(T)$ the set of fixed points of T, that is, $F(T) = \{x \in K : Tx = x\}$ and by $F := (\cap_{j \in J} F(T_j)) \cap (\cap_{j \in J} F(S_j))$, the set of common fixed points of two families $\{S_j : j \in J\}$ and $\{T_j : j \in J\}$. In what follows we fix $x_0 \in K$ as a starting point of a process unless stated otherwise, and take $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ sequences in $(0, 1)$.

Mann iterative process for common fixed points of a finite family of mappings $\{T_j : j \in J\}$ is as follows:

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n) T_n x_{n-1}, \; n \in \mathbb{N} \tag{1.1}$$

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where $T_n = T_{n(\mathrm{mod}\, N)}$ and the mod N function takes values in J.
Concerning the common fixed points of the finite family $\{T_j : j \in J\}$, Xu and Ori [8] introduced the following implicit iterative process:

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n) T_n x_n, \quad n \in \mathbb{N} \tag{1.2}$$

where $T_n = T_{n(\mathrm{mod}\, N)}$.
Zhao et al. [9] introduced the following implicit iterative process for the same purpose.

$$x_n = \alpha_n x_{n-1} + \beta_n T_n x_{n-1} + \gamma_n T_n x_n, \;\; n \in \mathbb{N} \tag{1.3}$$

where $T_n = T_{n(\mathrm{mod}\, N)}$.
Plubtieng et al. [5] defined an implicit iterative process for two finite families of nonexpansive mappings $\{T_j : j \in J\}$ and $\{S_j : j \in J\}$ as follows:

$$\begin{aligned} x_n &= \alpha_n x_{n-1} + (1 - \alpha_n) T_n y_n, \\ y_n &= \beta_n x_n + (1 - \beta_n) S_n x_n, \;\; n \in \mathbb{N} \end{aligned} \tag{1.4}$$

where $T_n = T_{n(\mathrm{mod}\, N)}, S_n = S_{n(\mathrm{mod}\, N)}$.

Our purpose in this paper is to present an iterative process for two finite families of nonexpansive mappings $\{S_j : j \in J\}$ and $\{T_j : j \in J\}$ as follows:

$$\begin{aligned} x_1 &= \alpha_1 x_0 + \beta_1 S_1 x_0 + \gamma_1 T_1 x_1, \\ x_2 &= \alpha_2 x_1 + \beta_2 S_2 x_1 + \gamma_2 T_2 x_2, \\ &\;\;\vdots \\ x_N &= \alpha_N x_{N-1} + \beta_N S_N x_{N-1} + \gamma_N T_N x_N, \\ x_{N+1} &= \alpha_{N+1} x_N + \beta_{N+1} S_{N+1} x_N + \gamma_{N+1} T_{N+1} x_{N+1}, \\ &\;\;\vdots \end{aligned}$$

which can be written in compact form as:

$$x_n = \alpha_n x_{n-1} + \beta_n S_n x_{n-1} + \gamma_n T_n x_n, \;\; n \in \mathbb{N} \tag{1.5}$$

where $T_n = T_{n(\mathrm{mod}\, N)}$ and $S_n = S_{n(\mathrm{mod}\, N)}$.

The process (1.5) reduces to (1.3) when $S_i = T_i$ for all $i \in J$, to the iterative process (1.2) when $S_i = I$ for all $i \in J$ and to the Mann iterative process (1.1) when $T_i = I$ for all $i \in J$. Moreover, our process (1.5) is simpler than (1.4) from computational point of view.

Using process (1.5), we prove some weak and strong convergence theorems for approximating common fixed points of two finite families of mappings in a uniformly convex Banach space. We not only extend and improve the corresponding results of Chidume and Shahzad [2] and Zhao et al.[9] but also give some other results.

2 Preliminaries

Let E be a Banach space, K a nonempty closed convex subset of E and $\{S_j : j \in J\}$ and $\{T_j : j \in J\}$ be two finite families of nonexpansive mappings. Let $\{x_n\}$ be defined by (1.5). Define a mapping $W_1 : K \to K$ by $W_1 x = \alpha_1 x_0 + \beta_1 S_1 x_0 + \gamma_1 T_1 x$ for all $x \in K$ where $\alpha_1 + \beta_1 + \gamma_1 = 1$. Existence of x_1 is guaranteed if W_1 has a fixed point. Now for any $x, y \in K$, we have

$$\begin{aligned} \|W_1 x - W_1 y\| &= \gamma_1 \|T_1 x - T_1 y\| \\ &\leq \gamma_1 \|x - y\|. \end{aligned}$$

Since $\gamma_1 < 1, W_1$ is a contraction. By Banach contraction principle, W_1 has a unique fixed point. Thus the existence of x_1 is established. Similarly, the existence of $x_2, x_3, ...$ is established. Thus the iteration process (1.5) is well-defined.

We also give an example to show that there do exist two families of nonexpansive mappings with a common fixed point.

Example 1. *Define $T_n : K \to K$ and $S_n : K \to K$ as*

$$T_n x = \frac{2x + n - 1}{2n}$$

and

$$S_n x = \frac{n^2 - 2x + 1}{2n^2}$$

for all $n \in \mathbb{N}$. Then both T_n and S_n are nonexpansive families and $F := (\cap_{j \in J} F(T_j)) \cap (\cap_{j \in J} F(S_j)) = \{\frac{1}{2}\}$.

Let K be a nonempty closed subset of a real Banach space E. $T : K \to K$ is said to be semicompact if for any bounded sequence $\{x_n\} \subset K$ with $\lim_{n \to \infty} \|x_n - Tx_n\| = 0$, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_j}\}$ converges strongly to $p \in K$.

A mapping $T : K \to K$ with $F(T) \neq \varnothing$ is said to satisfy the Condition (A) [7] if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0, f(t) > 0$ for all $t > 0$ such that

$$\|x - Tx\| \geq f(d(x, F(T)))$$

for all $x \in K$, where $d(x, F(T)) = \inf\{\|x - q\| : q \in F(T)\}$.

Two mappings $T, S : K \to K$ with $F^* := F(T) \cap F(S) \neq \varnothing$ are said to satisfy the Condition (A') [3] if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0, f(t) > 0$ for all $t > 0$ such that

$$\text{either } \|x - Tx\| \geq f(d(x, F^*)) \text{ or } \|x - Sx\| \geq f(d(x, F^*))$$

for all $x \in K$, where $d(x, F^*) = \inf\{\|x - q\| : q \in F^*\}$.

Let $\{T_j : j \in J\}$ be a finite family of nonexpansive mappings of K with nonempty fixed points set $F(T_j)$. Then $\{T_j : j \in J\}$ is said to satisfy the Condition (B) on K [2] if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(t) > 0$ for all $t > 0$ such that

$$\max_{j \in J} \|x - T_j x\| \geq f\left(d\left(x, F(T_j)\right)\right)$$

for all $x \in K$.

We can modify this definition for two finite families of mappings as follows. Let $\{T_j : j \in J\}$ and $\{S_j : j \in J\}$ be two finite families of nonexpansive mappings of K with nonempty fixed points set F. These families are said to satisfy Condition (B') on K if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(t) > 0$ for all $t > 0$ such that

$$\text{either } \max_{j \in J} \|x - T_j x\| \geq f\left(d\left(x, F\right)\right) \text{ or } \max_{j \in J} \|x - S_j x\| \geq f\left(d\left(x, F\right)\right)$$

for all $x \in K$. The Condition (B') reduces to the Condition (A') when $T_1 = T_2 = ... = T_N = T$ and $S_1 = S_2 = ... = S_N = S$, and to the Condition (B) when $S_j = T_j$ for all $j \in J$.

Note that the Condition (A) is weaker than both the semicompactness of the mapping $T : K \to K$ and the compactness of its domain K , see [7]. Thus the Condition (A') is weaker than both the semicompactness of the mappings T, $S : K \to K$ and the compactness of their domain K so that Condition (B') is weaker than both the semicompactness of $\{T_j : j \in J\}$ and $\{S_j : j \in J\}$ and the compactness of their domain K.

Next, we state the following useful lemmas.

Lemma 1. *[1, 4] Let E be a uniformly convex Banach space, let K be a nonempty closed convex subset of E, and let $T : K \to K$ be a nonexpansive mapping. Then $I - T$ is demiclosed at zero.*

Lemma 2. *[6] Let E be a uniformly convex Banach space and let a, b be two constants with $0 < a < b < 1$. Suppose that $\{t_n\} \subset [a, b]$ is a real sequence and $\{x_n\}, \{y_n\}$ are two sequences in E. Then the conditions*

$$\lim_{n\to\infty} \|t_n x_n + (1 - t_n) y_n\| = d, \quad \limsup_{n\to\infty} \|x_n\| \leq d, \quad \limsup_{n\to\infty} \|y_n\| \leq d$$

imply that $\lim_{n\to\infty} \|x_n - y_n\| = 0$, where $d \geq 0$ is a constant.

3 Main results

3.1 Strong convergence results

Lemma 3. *Let E be a real uniformly convex Banach space and K be a nonempty closed convex subset of E. Let $\{T_j : j \in J\}$ and $\{S_j : j \in J\}$ be two finite families of nonexpansive mappings on K with nonempty fixed points set F. Let $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ be three real sequences satisfying $\alpha_n+\beta_n+\gamma_n = 1$, $0 < a \leq \alpha_n, \beta_n, \gamma_n \leq b < 1$. From arbitrary $x_0 \in K$, define a sequence $\{x_n\}$ by* (1.5)*. Then* $\lim_{n\to\infty} \|x_n - p\|$ *exists for each $p \in F$.*

Proof. Let $p \in F$. It follows from (1.5) that

$$\begin{aligned} \|x_n - p\| &\leq \alpha_n \|x_{n-1} - p\| + \beta_n \|S_n x_{n-1} - p\| + \gamma_n \|T_n x_n - p\| \\ &\leq \alpha_n \|x_{n-1} - p\| + \beta_n \|x_{n-1} - p\| + \gamma_n \|x_n - p\| \\ &= (\alpha_n + \beta_n) \|x_{n-1} - p\| + \gamma_n \|x_n - p\| . \end{aligned}$$

and this implies that

$$(1 - \gamma_n) \|x_n - p\| \leq (\alpha_n + \beta_n) \|x_{n-1} - p\| .$$

Since $1 - \gamma_n > 0$ for all $n \in \mathbb{N}$ and $1 - \gamma_n = \alpha_n + \beta_n$, we have

$$\|x_n - p\| \leq \|x_{n-1} - p\| .$$

By induction, $|\|x_n - p\|| \leq \|x_0 - p\|$ so $\{\|x_n - p\|\}$ is a bounded increasing sequence of real numbers. Thus $\lim_{n\to\infty} \|x_n - p\|$ exists for each $p \in F$. This completes the proof. ∎

Lemma 4. *Let E be a real uniformly convex Banach space and K be a nonempty closed convex subset of E. Let $\{T_j : j \in J\}$ and $\{S_j : j \in J\}$ be two finite families of nonexpansive mappings on K with nonempty fixed points set F. Let $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ be three real sequences satisfying $\alpha_n+\beta_n+\gamma_n = 1$, $0 < a \leq \alpha_n, \beta_n, \gamma_n \leq b < 1$. From arbitrary $x_0 \in K$, define a sequence $\{x_n\}$ by* (1.5)*. Then* $\lim_{n\to\infty} \|x_n - T_j x_n\| = \lim_{n\to\infty} \|x_n - S_j x_n\| = 0$ *for any $j \in J$.*

Proof. First we prove that

$$\lim_{n\to\infty} \|x_n - T_n x_n\| = 0 = \lim_{n\to\infty} \|x_n - S_n x_n\| \tag{3.1}$$

Let $p \in F$. Then $\lim_{n\to\infty} \|x_n - p\|$ exists by above lemma. Suppose that

$\lim_{n\to\infty}\|x_n - p\| = d$. Then

$$\begin{aligned}
\lim_{n\to\infty}\|x_n - p\| &= \lim_{n\to\infty}\|\alpha_n(x_{n-1}-p) + \beta_n(S_n x_{n-1}-p) + \gamma_n(T_n x_n - p)\| \\
&= \lim_{n\to\infty}\left\| \begin{array}{c} (1-\gamma_n)\left[\dfrac{\alpha_n}{1-\gamma_n}(x_{n-1}-p) + \dfrac{\beta_n}{1-\gamma_n}(S_n x_{n-1}-p)\right] \\ +\gamma_n(T_n x_n - p) \end{array} \right\| \\
&= d. \qquad (3.2)
\end{aligned}$$

Since T_n is a nonexpansive mapping for all n, we have $\|T_n x_n - p\| \le \|x_n - p\|$. Taking $\limsup$ on both sides of this inequality, we obtain

$$\limsup_{n\to\infty}\|T_n x_n - p\| \le \limsup_{n\to\infty}\|x_n - p\| = d. \qquad (3.3)$$

Now

$$\begin{aligned}
& \limsup_{n\to\infty}\left\|\frac{\alpha_n}{1-\gamma_n}(x_{n-1}-p) + \frac{\beta_n}{1-\gamma_n}(S_n x_{n-1}-p)\right\| \\
\le\ & \limsup_{n\to\infty}\left[\frac{\alpha_n}{1-\gamma_n}\|x_{n-1}-p\| + \frac{\beta_n}{1-\gamma_n}\|x_{n-1}-p\|\right] \\
=\ & \limsup_{n\to\infty}\left(\frac{\alpha_n+\beta_n}{1-\gamma_n}\right)\|x_{n-1}-p\| = d \qquad (3.4)
\end{aligned}$$

By using (3.2), (3.3), (3.4) and Lemma 2, we get

$$\lim_{n\to\infty}\left\|\frac{\alpha_n}{1-\gamma_n}(x_{n-1}-p) + \frac{\beta_n}{1-\gamma_n}(S_n x_{n-1}-p) - (T_n x_n - p)\right\| = 0.$$

This means that

$$\begin{aligned}
& \lim_{n\to\infty}\left\|\frac{\alpha_n}{1-\gamma_n}x_{n-1} + \frac{\beta_n}{1-\gamma_n}S_n x_{n-1} - T_n x_n\right\| \\
=\ & \lim_{n\to\infty}\left(\frac{1}{1-\gamma_n}\right)\|\alpha_n x_{n-1} + \beta_n S_n x_{n-1} - (1-\gamma_n)T_n x_n\| \\
=\ & 0.
\end{aligned}$$

Since $0 < a \le \gamma_n \le b < 1$, we have $1/(1-a) \le 1/(1-\gamma_n) \le 1/(1-b)$. Thus

$$\lim_{n\to\infty}\|x_n - T_n x_n\| = 0. \qquad (3.5)$$

Similarly, we have

$$\lim_{n\to\infty}\|x_n - p\| = \lim_{n\to\infty}\left\| \begin{array}{c} (1-\beta_n)\left[\dfrac{\alpha_n}{1-\beta_n}(x_{n-1}-p) + \dfrac{\gamma_n}{1-\beta_n}(T_n x_n - p)\right] \\ +\beta_n(S_n x_{n-1}-p) \end{array} \right\| = d.$$

Taking lim sup on both sides of $\|S_n x_{n-1} - p\| \leq \|x_{n-1} - p\|$, we obtain

$$\limsup_{n\to\infty} \|S_n x_{n-1} - p\| \leq \limsup_{n\to\infty} \|x_{n-1} - p\| = d.$$

Next,

$$\begin{aligned}
& \limsup_{n\to\infty} \left\| \frac{\alpha_n}{1-\beta_n} (x_{n-1} - p) + \frac{\gamma_n}{1-\beta_n} (T_n x_n - p) \right\| \\
\leq\ & \limsup_{n\to\infty} \left[\frac{\alpha_n}{1-\beta_n} \|x_{n-1} - p\| + \frac{\gamma_n}{1-\beta_n} \|x_n - p\| \right] \\
\leq\ & \limsup_{n\to\infty} \left(\frac{\alpha_n}{1-\beta_n} + \frac{\gamma_n}{1-\beta_n} \right) \|x_{n-1} - p\| \\
=\ & \limsup_{n\to\infty} \|x_{n-1} - p\| \\
=\ & d.
\end{aligned}$$

Applying Lemma 2 once again, we have

$$\begin{aligned}
& \lim_{n\to\infty} \left\| \frac{\alpha_n}{1-\beta_n} x_{n-1} + \frac{\gamma_n}{1-\beta_n} T_n x_n - S_n x_{n-1} \right\| \\
=\ & \lim_{n\to\infty} \left(\frac{1}{1-\beta_n} \right) \|\alpha_n x_{n-1} + \gamma_n T_n x_n - (1-\beta_n) S_n x_{n-1}\| \\
=\ & 0.
\end{aligned}$$

Since $0 < a \leq \beta_n \leq b < 1$, we have $1/(1-a) \leq 1/(1-\beta_n) \leq 1/(1-b)$. Thus,

$$\lim_{n\to\infty} \|x_n - S_n x_{n-1}\| = 0. \tag{3.6}$$

Moreover, we have

$$\lim_{n\to\infty} \|x_n - p\| = \lim_{n\to\infty} \left\| \begin{array}{c} (1-\alpha_n) \left[\dfrac{\beta_n}{1-\alpha_n} (S_n x_{n-1} - p) + \dfrac{\gamma_n}{1-\alpha_n} (T_n x_n - p) \right] \\ +\alpha_n (x_{n-1} - p) \end{array} \right\| = d$$

and

$$\begin{aligned}
& \limsup_{n\to\infty} \left\| \frac{\beta_n}{1-\alpha_n} (S_n x_{n-1} - p) + \frac{\gamma_n}{1-\alpha_n} (T_n x_n - p) \right\| \\
\leq\ & \limsup_{n\to\infty} \left[\frac{\beta_n}{1-\alpha_n} \|x_{n-1} - p\| + \frac{\gamma_n}{1-\alpha_n} \|x_n - p\| \right] \\
\leq\ & \limsup_{n\to\infty} \left(\frac{\beta_n + \gamma_n}{1-\alpha_n} \right) \|x_{n-1} - p\| \\
=\ & \limsup_{n\to\infty} \|x_{n-1} - p\| \\
=\ & d.
\end{aligned}$$

By Lemma 2,

$$\begin{aligned} & \lim_{n\to\infty}\left\|\frac{\beta_n}{1-\alpha_n}S_n x_{n-1}+\frac{\gamma_n}{1-\alpha_n}T_n x_n - x_{n-1}\right\| \\ = & \lim_{n\to\infty}\left(\frac{1}{1-\alpha_n}\right)\left\|\alpha_n x_{n-1}+\beta_n S_n x_{n-1}+\gamma_n T_n x_n - x_{n-1}\right\| \\ = & \ 0. \end{aligned}$$

Since $0 < a \leq \alpha_n \leq b < 1$, we have $1/(1-a) \leq 1/(1-\alpha_n) \leq 1/(1-b)$. Thus

$$\lim_{n\to\infty}\|x_n - x_{n-1}\| = 0. \tag{3.7}$$

Thus

$$\begin{aligned} \|x_n - S_n x_n\| &\leq \|x_n - S_n x_{n-1}\| + \|S_n x_{n-1} - S_n x_n\| \\ &\leq \|x_n - S_n x_{n-1}\| + \|x_{n-1} - x_n\| \end{aligned}$$

together with (3.6) and (3.7) implies that

$$\lim_{n\to\infty}\|x_n - S_n x_n\| = 0. \tag{3.8}$$

Now we show that, for any $j \in J$,

$$\lim_{n\to\infty}\|x_n - T_j x_n\| = 0 = \lim_{n\to\infty}\|x_n - S_j x_n\|.$$

From (3.7), we have $\lim_{n\to\infty}\|x_n - x_{n-1}\| = 0$, so that for any $j \in J$,

$$\lim_{n\to\infty}\|x_n - x_{n+j}\| = 0. \tag{3.9}$$

Since, for any $j \in J$, we have

$$\begin{aligned} \|x_n - T_{n+j}x_n\| &\leq \|x_n - x_{n+j}\| + \|x_{n+j} - T_{n+j}x_{n+j}\| + \|T_{n+j}x_{n+j} - T_{n+j}x_n\| \\ &\leq \|x_n - x_{n+j}\| + \|x_{n+j} - T_{n+j}x_{n+j}\| + \|x_n - x_{n+j}\| \\ &= 2\|x_n - x_{n+j}\| + \|x_{n+j} - T_{n+j}x_{n+j}\|, \end{aligned} \tag{3.10}$$

it follows from (3.5) and (3.9) that

$$\lim_{n\to\infty}\|x_n - T_{n+j}x_n\| = 0$$

for all $j \in J$. It follows that for any $j \in J$,

$$\lim_{n\to\infty}\|x_n - T_j x_n\| = 0. \tag{3.11}$$

Replacing T_{n+j} by S_{n+j} in the inequality (3.10),we get

$$\lim_{n\to\infty}\|x_n - S_j x_n\| = 0 \tag{3.12}$$

for any $j \in J$. ∎

Now we prove our strong convergence theorems as follows:

Theorem 1. *Let E be a real uniformly convex Banach space and K be a nonempty closed convex subset of E. Let $\{T_j : j \in J\}$ and $\{S_j : j \in J\}$ be two finite families of nonexpansive mappings of K that satisfy the Condition (B') and $F \neq \varnothing$. Let $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ be three real sequences satisfying $\alpha_n + \beta_n + \gamma_n = 1, 0 < a \leq \alpha_n, \beta_n, \gamma_n \leq b < 1$. Then the iterative process $\{x_n\}$ defined by (1.5) converges strongly to a common fixed point of $\{T_j : j \in J\}$ and $\{S_j : j \in J\}$.*

Proof. Let $p \in F$. As proved in Lemma 3, $\|x_n - p\| \leq \|x_{n-1} - p\|$ for all $n \in \mathbb{N}$. This implies that

$$d(x_n, F) \leq d(x_{n-1}, F).$$

Thus $\lim_{n\to\infty} d(x_n, F)$ exists. Since $\{T_j : j \in J\}$ and $\{S_j : j \in J\}$ satisfy Condition (B'), therefore

$$\text{either } f(d(x_n, F)) \leq \max_{j\in J} \|x_n - T_j x_n\| \text{ or } f(d(x_n, F)) \leq \max_{j\in J} \|x_n - S_j x_n\|.$$

It follows from (3.11) and (3.12) that $\lim_{n\to\infty} f(d(x_n, F)) = 0$. Since f is a nondecreasing function and $f(0) = 0$, so it follows that $\lim_{n\to\infty} d(x_n, F) = 0$.

Next, we show that $\{x_n\}$ is a Cauchy sequence in K. Let $\epsilon > 0$ be arbitrarily chosen. Since $\lim_{n\to\infty} d(x_n, F) = 0$, there exists a positive integer n_0 such that

$$d(x_n, F) < \frac{\epsilon}{4}, \quad \forall n \geq n_0.$$

In particular, $\inf\{\|x_{n_0} - p\| : p \in F\} < \frac{\epsilon}{4}$. Thus there must exist $p^* \in F$ such that

$$\|x_{n_0} - p^*\| < \frac{\epsilon}{2}.$$

Now, for all $m, n \geq n_0$, we have

$$\begin{aligned}
\|x_{n+m} - x_n\| &\leq \|x_{n+m} - p^*\| + \|x_n - p^*\| \\
&\leq 2\|x_{n_0} - p^*\| \\
&< 2\left(\frac{\epsilon}{2}\right) = \epsilon.
\end{aligned}$$

Hence $\{x_n\}$ is a Cauchy sequence in a closed subset K of a Banach space E and so it must converge to a point q in K. Now, $\lim_{n\to\infty} d(x_n, F) = 0$ gives that $d(q, F) = 0$. Thus we have $q \in F$. ∎

Although the following is a corollary to our Theorem 1, yet it is new in itself.

Theorem 2. *Let E be a real uniformly convex Banach space and K be a nonempty closed convex subset of E. Let $\{T_j : j \in J\}$ be nonexpansive self-mappings of K with $\cap_{j=1}^{N} F(T_j) \neq \varnothing$. Suppose that $\{T_j : j \in J\}$ satisfies the Condition (B). Let $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ be three real sequences satisfying $\alpha_n + \beta_n + \gamma_n = 1$, $0 < a \leq \alpha_n, \beta_n, \gamma_n \leq b < 1$. Then the iterative process $\{x_n\}$ defined by* (1.3) *converges strongly to a common fixed point of the mappings $\{T_j : j \in J\}$.*

Proof. Choose $S_j = T_j$ for all $j \in J$ in Theorem 1. ∎

Taking $S_j = I$ for all $j \in J$, we have the following corollary which handles the case of iterative process (1.2).

Corollary 1. (*Theorem* 3.2,*[2]*) *Let E be a real uniformly convex Banach space and K be a nonempty closed convex subset of E. Let $\{T_j : j \in J\}$ be nonexpansive self-mappings of K with $\cap_{j=1}^{N} F(T_j) \neq \varnothing$. Suppose that $\{T_j : j \in J\}$ satisfies the Condition (B). Let $\{\alpha_n\}$ be a real sequence satisfying $0 < a \leq \alpha_n \leq b < 1$. Then the iterative process $\{x_n\}$ defined by* (1.2) *converges strongly to a common fixed point of the mappings $\{T_j : j \in J\}$.*

The results using Mann iterative process (1.1) are covered by the following corollary by choosing $T_j = I$ for all $j \in J$.

Corollary 2. *Let E be a real uniformly convex Banach space and K be a nonempty closed convex subset of E. Let $\{T_j : j \in J\}$ be nonexpansive self-mappings of K with $\cap_{j=1}^{N} F(T_j) \neq \varnothing$. Suppose that $\{T_j : j \in J\}$ satisfies the Condition (B). Let $\{\alpha_n\}$ be a real sequence satisfying $0 < a \leq \alpha_n \leq b < 1$. Then the Mann iterative process $\{x_n\}$ defined by* (1.1) *converges strongly to a common fixed point of the mappings $\{T_j : j \in J\}$.*

Note that the Condition (B') is weaker than both the compactness of K and the semicompactness of the nonexpansive mappings $\{T_j : j \in J\}$ and $\{S_j : j \in J\}$, therefore we already have the following theorem proved. However, for the sake of completeness, we include its proof in the following.

Theorem 3. *Let E be a real uniformly convex Banach space and K be a nonempty closed convex subset of E. Let $\{T_j : j \in J\}$ and $\{S_j : j \in J\}$ be two finite families of nonexpansive mappings of K with nonempty fixed points set F. Let $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ be three real sequences satisfying $\alpha_n + \beta_n + \gamma_n = 1$, $0 < a \leq \alpha_n, \beta_n, \gamma_n \leq b < 1$. Assume that either K is compact or one of the mappings in each of $\{T_j : j \in J\}$ and $\{S_j : j \in J\}$ is semicompact. Then the iterative process $\{x_n\}$ defined by* (1.5) *converges strongly to a point of F.*

Proof. For any $j \in J$, we first suppose that T_j and S_j are semicompact. By (3.11) and (3.12), we have

$$\lim_{n\to\infty} \|x_n - T_j x_n\| = \lim_{n\to\infty} \|x_n - S_j x_n\| = 0.$$

From the semicompactness of T_j and S_j, there exists a subsequence $\left\{x_{n_j}\right\}$ of $\{x_n\}$ such that $\left\{x_{n_j}\right\}$ converges strongly to a $q \in K$. Using (3.11) and (3.12), we have

$$\lim_{j\to\infty} \left\|x_{n_j} - T_j x_{n_j}\right\| = \|q - T_j q\| = 0 \text{ and } \lim_{j\to\infty} \left\|x_{n_j} - S_j x_{n_j}\right\| = \|q - S_j q\| = 0$$

for all $j \in J$. This implies that $q \in F$. Since $\lim_{n\to\infty} \left\|x_{n_j} - q\right\| = 0$ and $\lim_{n\to\infty} \|x_n - q\|$ exists for all $q \in F$ by Lemma 3, therefore

$$\lim_{n\to\infty} \|x_n - q\| = 0.$$

Next, assume the compactness of K, then again there exists a subsequence $\left\{x_{n_j}\right\}$ of $\{x_n\}$ such that $\left\{x_{n_j}\right\}$ converges strongly to a $q \in K$ and the proof follows the above lines. ∎

Remark 1. *In view of simplicity of the iterative process* (1.5) *as compared with* (1.4), *Theorem* 1 *and Theorem* 3 *improve Theorem* 3.3 *and Theorem* 3.4 *of [5] respectively and generalize the results generalized therein.*

3.2 Weak convergence results

Here we give weak convergence theorems for two finite families of nonexpansive mappings.

Theorem 4. *Let E be a real uniformly convex Banach space which satisfies Opial's condition and K be a nonempty closed convex subset of E. Let $\{T_j : j \in J\}$ and $\{S_j : j \in J\}$ be two finite families of nonexpansive mappings of K with nonempty fixed points set F. Let $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ be three real sequences satisfying $\alpha_n + \beta_n + \gamma_n = 1$, $0 < a \leq \alpha_n, \beta_n, \gamma_n \leq b < 1$. Then the iterative process $\{x_n\}$ defined by* (1.5) *converges weakly to a $q \in F$.*

Proof. Let $x^* \in F$. Then, as proved in Lemma 3, $\lim_{n\to\infty} \|x_n - x^*\|$ exists. We prove that $\{x_n\}$ has a unique weak subsequential limit in F. Since $\{x_n\}$ is bounded sequence in a uniformly convex Banach space E, there exist two convergent subsequences $\{x_{n_i}\}$ and $\{x_{n_j}\}$ of $\{x_n\}$. Let $z_1 \in K$ and $z_2 \in K$ be weak limits of the subsequences $\{x_{n_i}\}$ and $\{x_{n_j}\}$ respectively. By Lemma 4, $\lim_{n\to\infty} \|x_n - S_j x_n\| = 0$ and for every j, $I - S_j$ is demiclosed with respect to zero by Lemma 1, so we obtain $S_j z_1 = z_1$ for every j. Similarly, $T_j z_1 = z_1$ for every j. Again, in the same way, we can prove that $z_2 \in F$.

Next, we prove the uniqueness. For this, suppose that $z_1 \neq z_2$. Then, by the Opial's condition, we have

$$\begin{aligned}
\lim_{n\to\infty} \|x_n - z_1\| &= \lim_{n_i\to\infty} \|x_{n_i} - z_1\| \\
&< \lim_{n_i\to\infty} \|x_{n_i} - z_2\| \\
&= \lim_{n\to\infty} \|x_n - z_2\| \\
&= \lim_{n_j\to\infty} \|x_{n_j} - z_2\| \\
&< \lim_{n_j\to\infty} \|x_{n_j} - z_1\| \\
&= \lim_{n\to\infty} \|x_n - z_1\|,
\end{aligned}$$

which is a contradiction. Hence $\{x_n\}$ converges weakly to a point in F. ∎

As in the strong convergence case, although the following is a corollary to Theorem 4, yet it is new in itself.

Theorem 5. *Let E be a real uniformly convex Banach space which satisfies Opial's condition and K be a nonempty closed convex subset of E. Let $\{T_j : j \in J\}$ be a finite family of nonexpansive mappings of K with $\cap_{j=1}^{N} F(T_j) \neq \varnothing$. Let $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ be three real sequences satisfying $\alpha_n + \beta_n + \gamma_n = 1$, $0 < a \leq \alpha_n, \beta_n, \gamma_n \leq b < 1$. Then the iterative process $\{x_n\}$ defined by* (1.3) *converges weakly to a $q \in \cap_{j=1}^{N} F(T_j)$.*

We can compare this theorem with Theorem 2.3 of Zhao et al. [9]. Basically Zhao et al. proved the following.

Theorem 6. *(Theorem* 2.3*, [9]) Let E be a real uniformly convex Banach space which satisfies Opial's condition and K be a nonempty closed convex subset of E. Let $\{T_j : j \in J\}$ be a finite family of nonexpansive mappings of K with $\cap_{j=1}^{N} F(T_j) \neq \varnothing$. Let $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ be three real sequences in* $[0, 1]$ *satisfying $\alpha_n + \beta_n + \gamma_n = 1$, $0 < a \leq \gamma_n \leq b < 1, \alpha_n - \beta_n > c > 0$, where a, b, c are some constants. Then the iterative process $\{x_n\}$ defined by* (1.3) *converges weakly to a $q \in \cap_{j=1}^{N} F(T_j)$.*

Note that Zhao et al. imposed the condition $\alpha_n - \beta_n > c > 0$ which forces α_n to be greater than β_n whereas we do not impose any such condition on the parameters α_n, β_n in our Theorem 5.

We also have the following corollaries.

Corollary 3. *Let E be a real uniformly convex Banach space which satisfies Opial's condition and K be a nonempty closed convex subset of E. Let $\{T_j : j \in J\}$ be a finite family of nonexpansive mappings of K with $\cap_{j=1}^{N} F(T_j) \neq \varnothing$. Let $\{\alpha_n\}$ be a real sequence satisfying $0 < a \leq \alpha_n \leq b < 1$. Then the iterative process $\{x_n\}$ defined by* (1.2) *converges weakly to a $q \in \cap_{j=1}^{N} F(T_j)$.*

Corollary 4. *Let E be a real uniformly convex Banach space which satisfies Opial's condition and K be a nonempty closed convex subset of E. Let $\{T_j : j \in J\}$ be a finite family of nonexpansive mappings of K with $\cap_{j=1}^{N} F(T_j) \neq \varnothing$. Let $\{\alpha_n\}$ be a real sequence satisfying $0 < a \leq \alpha_n \leq b < 1$. Then the iterative process $\{x_n\}$ defined by* (1.1) *converges weakly to a $q \in \cap_{j=1}^{N} F(T_j)$.*

Remark 2. *Theorem* 4 *improves Theorem* 3.8 *of [5] in view of simplicity of the iterative process* (1.5) *as compared with* (1.4).

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