EXISTENCE RESULTS FOR RANDOM NEUTRAL FUNCTIONAL INTEGRODIFFERENTIAL INCLUSIONS

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Abstract

In this paper we prove the existence of random solutions for neutral functional integrodifferential inclusions with lower semicontinuous and nonconvex-valued right-hand side. Then we establish a sufficient condition for the existence of viable solutions to yield the existence of random viable solutions to neutral functional integrodifferential inclusions defined in a separable Banach space.

1 Introduction

There are two typical methods in proving the existence of random solutions of differential inclusions; in the first one, the measurability of solutions with respect to a random parameter is proved step by step ([9], [10], [11]), in the second one, random fixed point theorems are used ([12]).

The purpose of this paper is to investigate the existence of solutions for random neutral functional integrodifferential inclusions involving a nonconvex valued orientor field. Following [11], we first prove the existence of random solutions for a class of partial neutral functional differential inclusions in $\mathbb{R}^n$ governed by a lower semicontinuous, nonconvex valued orientor field. The proof is based on the "measurable selection method" which makes use of an earlier deterministic result obtained by Benchohra and Ntouyas ([2]).

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Next, using a random fixed point principle for multivalued mappings, we derive the existence of the desired random solution to neutral functional integrodifferential inclusions in a separable Banach space, with viability condition from the existence of the deterministic solution. The idea of applying the random fixed point principle due to Rybinski in the space of derivatives of the solutions belongs to Engl ([5]) and it was already used for obtaining similar results for random functional and neutral functional differential inclusions ([13], [6])). In this way our results extend those in [6, 11, 13] to the case of neutral functional integrodifferential inclusions.

The paper is organized as follows: definitions, notations and basic results are given in the next section and the main results are presented in Section 3.

2 Notations and preliminary results

Let $(E, \|.\|)$ be a separable Banach space. If $x \in E$, the distance from the point x to the set $A \subseteq E$ will be denoted by $d(x, A)$. $\mathcal{P}(E)$ will stand for the set of all subsets of E.

For any topological space S, the script $B(S)$ will stand for the σ-field of Borel subsets of S.

If $I = [a, b]$ is a real interval, let $C(I, E)$ be the Banach space of continuous functions $x(.) : I \to E$ with the norm $\|x(.)\|_\infty = \sup\{\|x(t)\| : t \in I\}$. By $C^1(I, E)$ we denote the space of continuously differentiable mappings $x(.) : I \to E$ endowed with norm $\|x(.)\|_{C_1} = \|x(.)\|_\infty + \|x'(.)\|_\infty$ and by $L^p(I, E)$ we denote the Lebesgue-Bochner space with the norm

$$\|x(.)\|_{L^p} = \left(\int_I \|x(t)\|^p \mathrm{d}t \right)^{\frac{1}{p}}.$$

Consider the integral operator $\Gamma(.) : L^p(I, E) \to C(I, E)$ defined by the Bochner integral

$$\Gamma(y(.))(t) = \int_a^t y(s)\mathrm{d}s.$$

By $W^p(I, E)$ we denote a subspace of $C(I, E)$ composed of the elements $x(.) = x_1(.) + \Gamma(y(.))$ where $x_1(.) : I \to E$ is a constant mapping and $y(.) \in L^p(I, E)$. Clearly, every $x(.) \in W^p(I, E)$ is differentiable almost everywhere with $x'(.) \in L^p(I, E)$ and $x(.) = x(a) + \Gamma(x'(.))$. On this space we consider the norm

$$\|x(.)\|_{W^p} = \|x(a)\| + \|x'(.)\|_{L^p}.$$

Note that $W^p(I, E)$ is a Banach space with respect to this norm.

Let (Ω, Σ, μ) be a σ-finite measure space (not necessarily complete) and $L^1(\Omega, E)$ be the space of integrable functions $f(.) : \Omega \to E$ equipped with the norm $\|f(.)\|_1 = \int_\Omega \|f(\omega)\| d\mu(\omega)$.

Recall that a function $f : \Omega \times E \to E$ is said to be Caratheodory if $\omega \to f(\omega, x)$ is measurable for any $x \in E$ and $x \to f(\omega, x)$ is continuous for any $\omega \in \Omega$.

Let $F(.) : \Omega \to \mathcal{P}(E)$ with nonempty, closed values. $F(.)$ is said to be (weakly) measurable if any of the following equivalent conditions holds:
i) for any open subset $U \subseteq E$, $\{\omega \in \Omega : F(\omega) \cap U \neq \emptyset\} \in \Sigma$;
ii) for all $x \in E$, $\omega \to d(x, F(\omega))$ is measurable.
If, in addition, μ is complete, then the statements i) and ii) above are equivalent to any of the following
iii) Graph$(F(.)) := \{(\omega, x) \in \Omega \times E : x \in F(\omega)\} \in \Sigma \otimes B(E)$ (graph measurability).
iv) for any closed subset $C \subseteq E$, $\{\omega \in \Omega : F(\omega) \cap C \neq \emptyset\} \in \Sigma$ (strong measurability).

By S^1_F we will denote the set of Bochner integrable selections of $F(.)$, i.e.

$$S^1_F = \{f(.) \in L^1(\Omega) : \; f(\omega) \in F(\omega) \; \mu - \text{a.e.}\}.$$

In what follows we will need the following lemmas.

Lemma 2.1. ([9]) *Let (Ω, Σ, μ) a σ-finite measure space, Y a locally compact separable metric space and Z a metric space. Then $f : \Omega \times Y \to Z$ is a Caratheodory function if and only if $\omega \to g(\omega)(.) := f(\omega, .)$ is measurable as a mapping from Ω to the space $C(Y, Z)$ endowed with the compact-open topology.*

Lemma 2.2. ([5]) *Let $F(.) : \Omega \to \mathcal{P}(E)$ be weakly measurable and $f(.) : \Omega \to E$ be measurable. Then the function $\omega \to d(f(\omega), F(\omega))$ is measurable.*

Lemma 2.3. ([13]) *Let X be a Polish space. Assume that $F(., .) : \Omega \times X \to \mathcal{P}(E)$ is weakly measurable and $f(.) : \Omega \to X$ is measurable. Then the multi-valued map $G(.) : \Omega \to \mathcal{P}(E)$, $G(\omega) = F(\omega, f(\omega))$ is weakly measurable.*

Lemma 2.4. ([13]) *Let $G(.) : I \to \mathcal{P}(E)$ be a weakly measurable set-valued map with closed values and such that*

$$\int_I d(0, G(t))^p \mathrm{d}t < \infty.$$

Then the set $M = \{y(.) \in L^p(I, E) : \; y(t) \in G(t) \text{ a.e. } (I)\}$ is nonempty and for every $v(.) \in L^p(I, E)$ one has

$$d(v, M) = \left(\int_I d(v(t), G(t))^p \mathrm{d}t \right)^{\frac{1}{p}}.$$

Let Y, Z be two Hausdorff topological spaces and let $F(.) : Y \to \mathcal{P}(Z)$ be a multifunction with nonempty, closed values. $F(.)$ is lower semicontinuous (l.s.c) if the set $F^{-1}(U) := \{y \in Y :\ F(y) \cap U \neq \emptyset\}$ is open in Y for any open subset U in Z.

Let $I = [t_0, T]$ be a real interval, $0 < \Delta < T - t_0$ and let λ be the Lebesgue measure on I. Consider the following partial neutral functional integrodifferential inclusion

$$\frac{d}{dt}[x(t) - f(t, x_t(.))] \in F\left(t, x_t(.), \int_{t_0}^{t} k(t, s, x_s(.))\, \mathrm{d}s\right) \quad \lambda - \text{a.e.}, \tag{1}$$

$$x(.)|_{[t_0-\Delta, t_0]} = x_0(.), \tag{2}$$

where $F(.,.) : I \times C([t_0 - \Delta, t_0], \mathbb{R}^n) \times \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$, $k(.,.,.) : I \times I \times C([t_0 - \Delta, t_0], \mathbb{R}^n) \to \mathbb{R}^n$, $f(.) : I \times C([t_0 - \Delta, t_0], \mathbb{R}^n) \to \mathbb{R}^n$, $x_0(.) : [t_0 - \Delta, t_0] \to \mathbb{R}^n$ is a given continuous function. For any $x(.) \in C([t_0 - \Delta, T], \mathbb{R}^n)$ and $t \in I$, $x_t(.) : [t_0 - \Delta, t_0] \to \mathbb{R}^n$ is a continuous function defined by $x_t(s) = x(t+s-t_0)$. Hence $x_t(.)$ describes the history of the state $x(.)$ from time $t - \Delta$ up to the present t.

Definition 2.5. *By a solution of* (1)-(2) *we mean a continuous function* $x(.) : [t_0 - \Delta, T] \to \mathbb{R}^n$ *such that* $t \to x(t) - f(t, x_t)$ *is absolutely continuous on* I, $x(.)|_{[t_0-\Delta,t_0]} = x_0(.)$ *and the inclusion (1) holds a.e. on I.*

Assume that the following sets of hypotheses hold.
H(f): $f(.,.) : I \times C([t_0 - \Delta, t_0], \mathbb{R}^n) \to \mathbb{R}^n$ is a function such that

1. $f(.,.)$ is completely continuous.
2. There exist $c_1 \in [0, 1)$ and $c_2 \geq 0$ such that

$$\|f(t, u)\| \leq c_1 \|u(.)\|_\infty + c_2 \ \ \forall (t, u(.)) \in I \times C([t_0 - \Delta, t_0], \mathbb{R}^n).$$

H(k): For $k(.,.,.) : I \times I \times C([t_0 - \Delta, t_0], \mathbb{R}^n) \to \mathbb{R}^n$ there exists a function $\alpha(.) \in L^2(I, \mathbb{R}_+)$ such that

$$\left| \int_{t_0}^{t} k(t, s, u)\, \mathrm{d}s \right| \leq \alpha(t) \|u(.)\|_\infty \text{ for each } (t, u(.)) \in I \times C([t_0 - \Delta, T], \mathbb{R}^n).$$

H(F): $F(.,.,.) : I \times C([t_0 - \Delta, t_0], \mathbb{R}^n) \times \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ is a set-valued map with nonempty, compact values such that

1. $F(.,.,.)$ is jointly measurable.

2. $(u, v) \to F(t, u, v)$ is lower semicontinuous for a.e. $t \in I$.

3. For every $\rho > 0$, there exists a map $h_\rho \in L^1(I, \mathbb{R}_+)$ such that

$$\|F(t, u, v)\| := sup\{\|z\| : \ z \in F(t, u, v)\} \leq h_\rho(t) \quad \lambda - \text{a.e.}$$

for any $u(.) \in C([t_0 - \Delta, t_0], \mathbb{R}^n)$ with $\|u(.)\|_\infty \leq \rho$ and $v \in \mathbb{R}^n$.

4. There exist $\varphi(.) \in L^2(I, \mathbb{R}_+)$ and a continuous and increasing function $\psi : \mathbb{R}_+ \to (0, \infty)$ such that

$$\|F(t, u, v)\| \leq \varphi(t)\psi(\|u(.)\|_\infty + \|v\|) \quad \text{a.e. on } I$$

$$\psi(\alpha(t)\|u(.)\|_\infty) \leq \alpha(t)\psi(\|u(.)\|_\infty) \quad \forall t \in I$$

$$\int_{t_0}^{T} \varphi(s)(1 + \alpha(s)) \, \mathrm{d}s < \int_c^\infty \frac{ds}{\psi(s)}$$

for all $(u(.), v) \in C([t_0 - \Delta, t_0], \mathbb{R}^n) \times \mathbb{R}^n$, where

$$c = \frac{1}{1 - c_1}[(1 + c_1)\|x_0(.)\|_\infty + 2c_2].$$

In the next section we need the following theorem.

Theorem 2.6. ([2]) *Assume that hypotheses* $\mathbf{H(f)}$, $\mathbf{H(k)}$ *and* $\mathbf{H(F)}$ *hold. Then the Cauchy problem* (1)-(2) *admits at least one solution.*

3 Main results

In this section we prove two existence theorems for random neutral functional integrodifferential inclusions. One is about partial neutral functional differential inclusions defined on $C([t_0-\Delta, t_0], \mathbb{R}^n)$ and the other for neutral functional differential inclusions defined on a certain subset of $C([t_0 - \Delta, T], E)$, namely $W^p([t_0 - \Delta, T], E)$. This leads us to what is known in applied mathematics as "viability theory".

Consider the following random version of inclusion (1)-(2)

$$\frac{d}{dt}\left[x(\omega, t) - f(\omega, t, x_t(\omega, .))\right] \in F\left(\omega, t, x_t(\omega, .), V(\omega, x(\omega, .))(t)\right) \ \mu \times \lambda - \text{a.e.} \tag{3}$$

$$x(\omega, .)|_{[t_0-\Delta, t_0]} = x_0(\omega, .) \ \forall \omega \in \Omega \tag{4}$$

where $F(., ., ., .) : \Omega \times I \times C([t_0 - \Delta, t_0], \mathbb{R}^n) \times \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$, $f(., ., .) : \Omega \times I \times C([t_0-\Delta, t_0], \mathbb{R}^n) \to \mathbb{R}^n$, $x_0(., .) : \Omega\times[t_0-\Delta, t_0] \to \mathbb{R}^n$ is a given Caratheodory map, $V(.,.) : \Omega \times C([t_0 - \Delta, T], \mathbb{R}^n) \to C([t_0, T], \mathbb{R}^n)$ is a random integral operator defined by

$$V(\omega, x)(t) = \int_{t_0}^{t} k(\omega, t, s, x_s)\, \mathrm{d}s.$$

with $k(.,.,.,.) : \Omega \times I \times I \times C([t_0 - \Delta, t_0], \mathbb{R}^n) \to \mathbb{R}^n$. If $x(.,.) : \Omega \times [t_0 - \Delta, T] \to \mathbb{R}^n$ is continuous in the second variable, then the "segment process" $x_t(\omega, .) : [t_0 - \Delta, t_0] \to \mathbb{R}^n$ is a continuous function defined by $x_t(\omega, s) = x(\omega, t + s - t_0)$.

Definition 3.7. *A solution to the random neutral functional integrodifferential inclusion* (3)-(4) *is a stochastic process $x(.,.) : \Omega \times [t_0 - \Delta, T] \to \mathbb{R}^n$ with continuous paths (i.e., for all $t \in [t_0 - \Delta, T]$, $x(., t)$ is measurable and for all $\omega \in \Omega$, $x(\omega, .) \in C([t_0 - \Delta, T], \mathbb{R}^n)$) such that $t \to x(\omega, t) - f(\omega, t, x_t(\omega, .))$ is absolutely continuous on I, $x(\omega, .)|_{[t_0 - \Delta, t_0]} = x_0(\omega, .)$ for all $\omega \in \Omega$ and the inclusion* (3) *holds a.e. on $\Omega \times I$.*

We will need the following sets of hypotheses on the data.

$\mathbf{H_r(f)}$: $f(.,.,.) : \Omega \times I \times C([t_0 - \Delta, t_0], \mathbb{R}^n) \to \mathbb{R}^n$ is a function such that

1. $f(.,.,.)$ is completely continuous and $f(., t, u)$ is measurable for all $(t, u(.)) \in I \times C([t_0 - \Delta, t_0], \mathbb{R}^n)$.
2. There exist $c_1 : \Omega \to [0, 1)$ and $c_2 : \Omega \to [0, \infty)$ both measurable such that for almost all $\omega \in \Omega$, $t \in I$ and $u(.) \in C([t_0 - \Delta, t_0], \mathbb{R}^n)$ one has

$$\|f(\omega, t, u)\| \le c_1(\omega)\|u(.)\|_\infty + c_2(\omega).$$

$\mathbf{H_r(k)}$: $k(.,.,.,.) : \Omega \times I \times I \times C([t_0 - \Delta, t_0], \mathbb{R}^n) \to \mathbb{R}^n$ is a map such that

1. $\omega \to k(\omega, t, s, u)$ is measurable $\forall (t, s, u) \in I \times I \times C([t_0 - \Delta, t_0], \mathbb{R}^n)$.
2. There exists a function $\alpha : \Omega \times I \to \mathbb{R}_+$ with $\alpha(., t)$ measurable and $\alpha(\omega, .) \in L^2(I, \mathbb{R}_+)$ such that for $(\omega, t, u(.)) \in \Omega \times I \times C([t_0 - \Delta, T], \mathbb{R}^n)$

$$\left| \int_{t_0}^{t} k(\omega, t, s, u)\, \mathrm{d}s \right| \le \alpha(\omega, t)\|u(.)\|_\infty.$$

3. $$\lim_{\substack{|\bar{t} - t| \to 0 \\ \|\overline{u}(.) - u(.)\|_\infty \to 0}} \left[\int_{t}^{\bar{t}} \|k(\omega, \bar{t}, s, \overline{u}_s)\| \mathrm{d}s + \int_{t_0}^{t} \|k(\omega, \bar{t}, s, \overline{u}_s) - k(\omega, t, s, u_s)\| \mathrm{d}s \right] = 0$$

$\mathbf{H_r(F)}$: $F(.,.,.,.) : \Omega \times I \times C([t_0 - \Delta, t_0], \mathbb{R}^n) \times \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ is a set-valued map with nonempty, compact values such that

1. $F(.,.,.,.)$ is jointly measurable.
2. $(u,v) \to F(\omega,t,u,v)$ is lower semicontinuous for a.e. $(\omega,t) \in \Omega \times I$.
3. For every $\rho > 0$ there exists a map $h_\rho : \Omega \times I \to \mathbb{R}_+$ with $h_\rho(.,t)$ measurable and $h_\rho(\omega,.) \in L^1(I,\mathbb{R}_+)$ such that
$$\|F(\omega,t,u,v)\| := sup\{\|z\| : \ z \in F(\omega,t,u,v)\} \leq h_\rho(\omega,t) \quad \lambda - \text{a.e.}$$
for all $\omega \in \Omega$, $u(.) \in C([t_0-\Delta,t_0],\mathbb{R}^n)$ with $\|u(.)\|_\infty \leq \rho$ and $v \in \mathbb{R}^n$.
4. There exist $\varphi(.,.) : \Omega \times I \to \mathbb{R}_+$ with $\varphi(.,t)$ measurable and $\varphi(\omega,.) \in L^2(I,\mathbb{R}_+)$ and a continuous and increasing function $\psi : \mathbb{R}_+ \to (0,\infty)$ such that
$$\|F(\omega,t,u,v)\| \leq \varphi(\omega,t)\psi(\|u(.)\|_\infty + \|v\|) \quad \text{a.e. on } I$$
$$\psi(\alpha(\omega,t)\|u(.)\|_\infty) \leq \alpha(\omega,t)\psi(\|u(.)\|_\infty) \quad \forall t \in I$$
$$\int_{t_0}^{T} \varphi(\omega,s)(1+\alpha(\omega,s))\ \mathrm{d}s < \int_{c(\omega)}^{\infty} \frac{ds}{\psi(s)}$$
for all $(\omega,u(.),v) \in \Omega \times C([t_0-\Delta,t_0],\mathbb{R}^n) \times \mathbb{R}^n$, where
$$c(\omega) = \frac{1}{1-c_1(\omega)}\left[(1+c_1(\omega))\|x_0(\omega,.)\|_\infty + 2c_2(\omega)\right].$$

Theorem 3.8. *Assume that hypotheses* $\mathbf{H_r(f)}$*,* $\mathbf{H_r(k)}$ *and* $\mathbf{H_r(F)}$ *hold. Then the Cauchy problem* (3)-(4) *admits at least one solution.*

Proof. Let $I_1 = [t_0-\Delta,T]$ and consider the functions $p : \Omega \times I_1 \times C(I_1,\mathbb{R}^n) \times L^1(I,\mathbb{R}^n) \to \mathbb{R}^n$ and $q : \Omega \times C(I_1,\mathbb{R}^n) \times L^1(I,\mathbb{R}^n) \to \mathbb{R}$ defined respectively by
$$p(\omega,t,x(.),g(.)) = \begin{cases} x(t) - x_0(\omega,t), & \text{if } t \in [t_0-\Delta,t_0] \\ x(t) - x_0(\omega,t_0) + f(\omega,t_0,x_0(\omega,.)) & \\ \qquad -f(\omega,t,x_t(.)) - \int_{t_0}^{t} g(s)\ \mathrm{d}s, & \text{if } t \in I \end{cases}$$
$$q(\omega,x(.),g(.)) = d\left(g(.), S^1_{F(\omega,.,x_.,V(\omega,x)(.))}\right).$$

Since $\omega \to x_0(\omega,.)$ is measurable and $f(.,.,.)$ is continuous we have that $\omega \to p(\omega,t,x(.),g(.))$ is measurable and $(t,x(.),g(.)) \to p(\omega,t,x(.),g(.))$ is continuous. Applying Lemma III-14 of Castaing and Valadier ([3]) we obtain that $(\omega, t, x(.), g(.)) \to p(\omega, t, x(.), g(.))$ is measurable. Let D be a dense subset of $[t_0 - \Delta, T]$ and define $p_1 : \Omega \times C(I_1, \mathbb{R}^n) \times L^1(I, \mathbb{R}^n) \to \mathbb{R}^n$ by

$$p_1(\omega, x(.), g(.)) = \sup_{t \in D} p(\omega, t, x(.), g(.)).$$

Then $(\omega, x(.), g(.)) \to p_1(\omega, x(.), g(.))$ is jointly measurable.

On the other hand, using Theorem 2.2 in [7] we have:

$$\begin{aligned} q(\omega, x(.), g(.)) &= \inf \left\{ \|g - h\|_{L^1} : \ h(.) \in S^1_{F(\omega,.,x_.,V(\omega,x)(.))} \right\} \\ &= \inf \left\{ \int_{t_0}^{T} \|g(s) - h(s)\| \, \mathrm{d}s : \ h(.) \in S^1_{F(\omega,.,x_.,V(\omega,x)(.))} \right\} \\ &= \int_{t_0}^{T} d(g(s), F(\omega, s, x_s, V(\omega, x)(s))) \, \mathrm{d}s. \end{aligned}$$

By the hypothesis $\mathbf{H_r(k)}$ 3, we have that

$$(\omega, s, x) \to V(\omega, x)(s) = \int_{t_0}^{s} k(\omega, s, u, x_u) \, \mathrm{d}u$$

is measurable in $\omega \in \Omega$ and continuous in $(s, x) \in I \times C([t_0 - \Delta, T], \mathbb{R}^n)$. Then $(\omega, s, x) \to (\omega, s, x_s, V(\omega, x)(s))$ is Caratheodory, hence jointly measurable. Using the measurability hypothesis on $F(., ., ., .)$ we obtain that $(\omega, s, x, z) \to d(z, F(\omega, s, x_s, V(\omega, x)(s)))$ is measurable in (ω, s, x) and continuous in $z \in \mathbb{R}^n$, so it is jointly measurable. Finally from Fubini's theorem we get that $(\omega, x(.), g(.)) \to q(\omega, x(.), g(.))$ is also measurable.

Now consider the multifunction $R(.) : \Omega \to \mathcal{P}(C(I_1, E) \times L^1(I, E))$ defined by

$$R(\omega) = \left\{ (x, g) \in C(I_1, E) \times L^1(I, E) : p_1(\omega, x(.), g(.)) = 0, q(\omega, x(.), g(.)) = 0 \right\}.$$

From Theorem 2.6 we have that for all $\omega \in \Omega$, $R(\omega) \neq \emptyset$. Using measurability of $p_1(., ., .)$ and $q(., ., .)$ we get that $Graph(R(.))$ is measurable.

Apply Theorem 3 of Saint-Beuve ([14]) to get $\lambda_1 : \Omega \to C(I_1, E)$ and $\lambda_2 : \Omega \to L^1(I_1, E)$ both measurable such that $(\lambda_1(\omega), \lambda_2(\omega)) \in R(\omega)$, μ-a.e. (Ω). Set

$$x(\omega, t) = \lambda_1(\omega)(t)$$

Also from Lemma 16 of Dunford and Schwartz ([4]) we get the existence of a function $g(., .) \in L^1(\Omega \times I, E)$ such that

$$g(\omega, t) = \lambda_2(\omega)(t) \quad \mu - \text{a.e.}.$$

By Lemma 2.1 $x(.,.)$ and $g(.,.)$ are Caratheodory functions and satisfy

$$x(\omega,t)-f(\omega,t,x_t(\omega,.))=x_0(\omega,t_0)-f(\omega,t_0,x_0(\omega,.))+\int_{t_0}^{t}g(\omega,s)\,\mathrm{d}s \quad \mu-\text{a.e.}, \forall t\in I$$

$$x(\omega,t)=x_0(\omega,t) \qquad \forall(\omega,t)\in\Omega\times[t_0-\Delta,t_0].$$

So $x(.,.)$ is the desired random trajectory which solves the problem (3)-(4). □

Remark 3.9. *If $F(.,.,.,.)$ and $f(.,.,.)$ are constant with respect to the random parameter, i.e. $F(\omega,t,u(.),v)=F_1(t,u(.),v)$ and $f(\omega,t,u(.))=f_1(t,u(.))$, then the above theorem yields the result of Benchohra and Ntouyas [2] (Theorem 5.1).*

In what follows we will work in a separable Banach space E. X will denote the space $W^p(I_1,E)$ and Y will denote the space $L^p(I_1,E)$, where $I_1=[t_0-\Delta,T]$. Let $k(.,.,.):\Omega\times I_1\times X\to E$ be such that for any $\omega\in\Omega$ and $x(.)\in X$, $t\to k(\omega,t,x)\in Y$. Let $V(.,.):\Omega\times X\to X$ be a random Volterra integral operator defined by $V(\omega,x)(t)=\int_{t_0-\Delta}^{t}k(\omega,s,x)ds$. Hence $V(\omega,x)=\Gamma(k(\omega,.,x))$. Consider the following random neutral functional integrodifferential inclusion with viability condition

$$\frac{d}{dt}x(\omega,t)\in F\left(\omega,t,\frac{d}{dt}x(\omega,.),V(\omega,x(\omega,.))\right) \quad \mu\times\lambda-\text{a.e.} \tag{5}$$

$$x(\omega,.)|_{[t_0-\Delta,t_0]}=x_0(\omega,.) \quad \forall\omega\in\Omega \tag{6}$$

$$x(\omega,.)\in L(\omega) \qquad \forall\omega\in\Omega \tag{7}$$

where $F(.,.,.,.):\Omega\times I\times Y\times X\to\mathcal{P}(E)$, $L(.):\Omega\to\mathcal{P}(X)$ and $x_0(.,.):\Omega\times[t_0-\Delta,t_0]\to E$ is a measurable function with $x_0(\omega,.)\in L(\omega)$ for each $\omega\in\Omega$.

Theorem 3.10. *Assume that $F(.,.,.,.):\Omega\times I\times Y\times X\to\mathcal{P}(E)$ and $L(.):\Omega\to\mathcal{P}(X)$ are weakly measurable, $F(.,.,.)$ has closed values and*

$$\int_{t_0-\Delta}^{T} d(0,F(\omega,t,y(.),z(.)))^p\,\mathrm{dt}<\infty$$

for every $(\omega,y(.),z(.))\in\Omega\times Y\times X$. If for every $\omega\in\Omega$ there exists a deterministic solution to the problem (5)-(7)*, then there exists a random solution to this problem.*

Proof. For $t \in [t_0 - \Delta, t_0]$, we extend $F(.,.,.)$ putting $F(\omega, t, y, z) = \frac{d}{dt} x_0(\omega, t)$. Note that $F(.,.,.)$ is weakly measurable. Define $G(.,.,.,.) : \Omega \times Y \times X \to \mathcal{P}(Y)$ letting

$$G(\omega, y, z) = \{u(.) \in Y : \ u(t) \in F(\omega, t, y, z) \text{ a.e.}(I_1)\}.$$

Using Lemma 2.4 we get that for every $(\omega, y(.), z(.))) \in \Omega \times Y \times X$ the sets $G(\omega, y, z)$ are nonempty and for every $v(.) \in Y$ we have

$$d(v, G(\omega, y, z)) = \left(\int_{t_0 - \Delta}^{T} d(v(t), F(\omega, t, y, z))^p \ \mathrm{dt} \right)^{\frac{1}{p}}.$$

By Lemma 2.2 the map $(\omega, t, y, z) \to d(v(t), F(\omega, t, y, z))^p$ is measurable and thus, by Fubini's theorem, the map $(\omega, y, z) \to d(v, G(\omega, y, z))$ is also measurable, hence $G(.,.,.)$ is weakly measurable (Theorem 3.3 in [8]).

Consider the integral operator $\Gamma_0(.,.) : \Omega \times Y \to X$ defined by

$$\Gamma_0(\omega, y) = x_0(\omega, t_0 - \Delta) + \Gamma(y)$$

and define the set-valued map $H(.,.) : \Omega \times Y \to \mathcal{P}(Y)$ by

$$H(\omega, y) = G(\omega, y, V(\omega, \Gamma_0(\omega, y))).$$

Then $H(.,.)$ has closed values. On the other hand, we have that $V(.,.)$ and $\Gamma_0(.,.)$ are measurable. Since $G(.,.,.)$ is weakly measurable, by Lemma 2.3 it follows that $H(.,.)$ also is weakly measurable.

Let $M(.) : \Omega \to \mathcal{P}(Y)$ be defined by

$$M(\omega) = \{y(.) \in Y : \Gamma_0(\omega, y) \in L(\omega)\}$$

and note that $M(.)$ has measurable graph. Define $P_H(.) : \Omega \to \mathcal{P}(Y)$,

$$P_H(\omega) = \{y(.) \in Y : \ y \in H(\omega, y)\}.$$

By assumption, for every $\omega \in \Omega$ the set $M(\omega) \cap P_H(\omega)$ is nonempty. Invoking Rybinski's random fixed point principle (Proposition 1 in [13]) we obtain the existence of a measurable map $u(.) : \Omega \to Y$ such that $u(\omega) \in M(\omega) \cap P_H(\omega)$ $\mu -$ a.e.. Notice that for every $\omega \in \Omega$, $x(\omega, .)$ is a solution to (5)-(7) iff $\dfrac{d}{dt} x(\omega, .) \in M(\omega) \cap P_H(\omega)$. Thus we define $x(.,.) : [t_0 - \Delta, T] \times \Omega \to E$ by

$$x(\omega, t) = \Gamma_0(\omega, u(\omega))(t).$$

Clearly $x(., t)$ is measurable (Lemma 2.3) and this is the desired random viable trajectory.

□

Example 3.11. *The model of the endotoxin tolerance is a good example of uncertainty. The mathematical model ([15]) considers a non-autonomous first order differential system of two equations with two unknowns, i.e. the concentrations of a TNF-α pro-inflammatory cytokine and of the inhibitor of the cytokines, denoted by x and y, respectively. The system is:*

$$\begin{aligned} \frac{dx}{dt} &= A(t)D_1\frac{x^3+E_1^3}{x^3+1}\cdot\frac{1}{F_1y+1}-x \\ \frac{dy}{dt} &= A(t)F_2\frac{y^2+E_2^2}{y^2+1}-D_2y \end{aligned} \tag{8}$$

where $A(.)$ represents the LPS endotoxin concentration and $D_2(.)$ is the clearance rate of the inhibitor and is supposed to be a time-dependent function. All the variables and parameters are non negative. We may suppose for the beginning that D_2 is our uncertain variable.

Let U be the set of control parameter $D_2 \in [0.1, 4]$ and define the multifunction $G(.,.) : [t_0, T] \times \mathbb{R}^2 \to \mathcal{P}(\mathbb{R}^2)$ by

$$G(t,(x,y)) = \left\{ \left(\frac{x^3+E_1^3}{x^3+1}\cdot\frac{A(t)D_1}{F_1y+1}-x, A(t)F_2\frac{y^2+E_2^2}{y^2+1}-D_2y \right) :\ D_2 \in U \right\}.$$

We associate to the system (8) the following differential inclusion

$$\begin{aligned} &z'(t) \in G(t,z(t)) \\ &z(t_0) = z_0 \in \mathbb{R}^2 \end{aligned} \tag{9}$$

The reason to treat the uncertain variables in a non probabilistic way is that the analysis of the reachable set of inclusion (9) gives us informations about possible extreme values. Using differential inclusions as the main modeling tool, both the steady-state behaviour and also the transient behaviour of the system in question may be covered by the informations obtained from reachability computation in the set-valued context.

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