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GROUP RINGS, HECKE ORDERS, QUASI-HEREDITARY ORDERS, CELLULAR ORDERS AND DEFORMATIONS *

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Dedicated to Professor Mirela Ştefănescu on the occasion of her 60th birthday

1 Introduction

The purpose of these notes is to explain the important notions of

Green orders, quasi-hereditary orders, cellular orders, Hecke-orders and deformations of blocks with cyclic defect

through a thorough examination of the integral group ring $\mathbb{Z}S_3$ of the symmetric group on three letters and its Hecke order. This is done in detail in Section 2.
In Section 3 the basic definition of the Hecke order of a BN-pair of rank two is given, and the Hecke order of the Dihedral group of order $2 \cdot p^n$ for an odd prime is described.
The Green orders are defined in Section 4, and the blocks of cyclic defect are described as examples of 1-dimensional Green orders. Two dimensional examples are the Hecke orders of the Dihedral groups of order $2 \cdot p^n$ for p an odd prime.
In the Sections 5, 6 and 7 the concepts of quasi-hereditary orders, separable and maximal deformations and cellular orders are discussed. In particular, we give definitions which seem to be better suited for the applications than the classical ones. For example, with the definition of Graham and Lehrer of cellular algebras [GrLe; 96], the integral group ring of the Dihedral groups

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of order $2 \cdot p^n$ for $p > 3$ would not be cellular. With our definition, which coincides with the one of Graham and Lehrer in the splitting case, these rings are cellular.
I would like to thank Steffen König for supplying references for the examples of cellular and quasi-hereditary orders.

2 The group ring of the symmetric group S_3 and its Hecke order as example of quasi-hereditary orders and Green orders

We shall here demonstrate the various concepts which arise in the structure of integral group rings at the example of the symmetric group on three letters.

2.1 The Hecke order and the integral group ring of the symmetric group on three letters

The classical Hecke order $\mathcal{H}_{S_3}$ of the symmetric group $\mathcal{S}_3$ is a deformation of the integral group ring of $\mathcal{S}_3$. It is defined as follows – here q is some positive real number, the index generator:

Definition 2.1 *Let A_q be the algebra over $\mathbb{C}$ generated by two elements $\{a, b\}$ which satisfy the following relations:*

1. ***Quadratic relations:***

$$x^2 = (q \times x - 1) \cdot x + q \times x \cdot 1 \text{ for } x = a\,,\, b\,.$$

2. ***Homogeneous relations:***

$$a \cdot b \cdot a = b \cdot a \cdot b\,.$$

Specializing $q = 1$, the algebra A_q is the complex group algebra of $\mathcal{S}_3$. The algebra A_q is described as follows:

1. It has a two-dimensional representation

$$a \longrightarrow \begin{pmatrix} -1 & 0 \\ -1 & q \end{pmatrix} \text{ and } b \longrightarrow \begin{pmatrix} -1 & 1+q+q^2 \\ 0 & q \end{pmatrix}$$

   – let us put $[3] = 1 + q + q^2$ –

2. and two one-dimensional representations, index representation: $ind(a) = q$ and $ind(b) = q$, and the sign representation: $sgn(a) = sgn(b) = -1$.

These representations show that for every positive real number the algebras A_q are all semi-simple of dimension 6. This is not so if q is specialized to a negative number. Examples are given in Subsection 2.7, Case 1 and Case 2.
We should point out that these are not the classical representations as in the book of Curtis and Reiner [CR; 87], Theorem 67.14. The classical representations are not at all suited to find the structure of Λ_3 in Lemma 2.2.
Instead of considering all the algebras $\{A_q \, : \, q \in \mathbb{R}^+\}$ one could also consider q as a variable and study the $\mathbb{Z}[q]$-algebra $\mathcal{H}^+_{S_3} := \mathbb{Z}[q] < a\,,\, b >$ (cf. Definition 2.1). We shall come back to this definition later in Subsection 2.7.
Usually though, one defines the $\mathbb{Z}[q,q^{-1}]$-algebra $\mathcal{H}_{S_3} := \mathbb{Z}[q,q^{-1}] < a\,,\, b >$ as Hecke algebra in order to have a and b invertible. But that is not the only reason. We need that q is invertible, in order to find a "nice" description as matrices (cf. Lemma 2.2).
We identify the Hecke-algebra with the $\mathbb{Z}[q,q^{-1}]$-algebra generated by the representing matrices of a and b inside

$$\begin{pmatrix} \mathbb{Q}(q) & \mathbb{Q}(q) \\ \mathbb{Q}(q) & \mathbb{Q}(q) \end{pmatrix} \oplus \mathbb{Q}(q)^- \oplus \mathbb{Q}(q)^q$$

where $\mathbb{Q}(q)^-$ is the sign representation and $\mathbb{Q}(q)^q$ is the index representation.
First we consider the $\mathbb{Z}[q]$-algebra generated by the one-dimensional representations, this is the "Hecke-algebra" $\mathcal{H}^+_2$ of the cyclic group of order 2. Here the elements a and b have the same image. Under the sign representation the element $a+1$ is mapped to zero, and under the index representation, the element $a-q$ is mapped to zero. Hence we have a commutative diagram with exact rows and columns, which describes $\mathcal{H}^+_2$ as a pullback. We put $q+1 = [2]$.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & (a+1)\cdot\mathcal{H}^+_2 \arrow[r] \arrow[d] & {[2]}\cdot\mathbb{Z}[q] \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & (a-q)\cdot\mathcal{H}^+_2 \arrow[r] \arrow[d] & \mathcal{H}^+_2 \arrow[r] \arrow[d] & \mathbb{Z}[q]^q \arrow[r] \arrow[d] & 0\,. \\
0 \arrow[r] & (-[2])\cdot\mathbb{Z}[q] \arrow[r] \arrow[d] & \mathbb{Z}[q]^- \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd} \tag{1}$$

The Hecke algebra over $\mathbb{Z}[q,q^{-1}]$ is then defined as $\mathcal{H}_2 = \mathbb{Z}[q,q^{-1}] \otimes_{\mathbb{Z}[q]} \mathcal{H}^+_2$, which has a similar pull-back structure. In order to describe the ring Λ_3 generated by the two-dimensional representation, we have to do some calculations *

$$a \cdot b = \begin{pmatrix} 1 & -[3] \\ 1 & -[3] + q^2 \end{pmatrix} \text{ and } b \cdot a = \begin{pmatrix} 1 - [3] & [3] \cdot q \\ -q & q^2 \end{pmatrix}.$$

$$\text{We calculate} \qquad a + e = \begin{pmatrix} 0 & 0 \\ -1 & [2] \end{pmatrix},\ b - a = \begin{pmatrix} 0 & [3] \\ 1 & 0 \end{pmatrix}$$

$$\text{and} \qquad b + e = \begin{pmatrix} 0 & [3] \\ 0 & [2] \end{pmatrix} \text{ with } e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

$$\text{Finally we have} \qquad (a + e) \cdot (b + e) = \begin{pmatrix} 0 & 0 \\ 0 & q \end{pmatrix}.$$

We shall write e_{ij} , $1 \le i, j \le 2$ for the matrix identities. Looking at the matrix $(a + 1) \cdot (b + 1)$ we see that $q \cdot e_{22} \in \Lambda_3$. We now use the fact that q is invertible in $\mathbb{Z}[q, q^{-1}]$ to conclude that $e_{22} \in \Lambda_3$, and hence also $e_{11} \in \Lambda$; moreover, $e_{11} \cdot (b - a) = ([3]) \cdot e_{12} \in \Lambda_3$ and finally $(b - a) \cdot e_{11} = e_{21} \in \Lambda_3$. Thus we have shown

Lemma 2.2 *The $\mathbb{Z}[q, q^{-1}]$-algebra generated by the two-dimensional representation is*

$$\Lambda_3 := \begin{pmatrix} \mathbb{Z}[q, q^{-1}] & [3] \cdot \mathbb{Z}[q, q^{-1}] \\ \mathbb{Z}[q, q^{-1}] & \mathbb{Z}[q, q^{-1}] \end{pmatrix}.$$

We point out that the result is definitely false if q is not invertible. We have another two-dimensional representation, where one does not need that q is invertible: We conjugate – from the left – the representing matrices a and b in the two-dimensional representation by the matrix $k := \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$. Using "Maple" we obtain

$$a_1 := k \cdot a \cdot k^{-1} = \begin{bmatrix} -1 & 0 \\ q & q \end{bmatrix} \text{ and } b_1 := k \cdot b \cdot k^{-1} = \begin{bmatrix} q \cdot [2] & [3] \\ -q^2 & -1 - q^2 \end{bmatrix}.$$

We then get

$$a_1 \cdot b_1 = \begin{bmatrix} -q \cdot [2] & -[3] \\ q^2 & q^2 \end{bmatrix}, b_1 \cdot a_1 = \begin{bmatrix} q^3 & q \cdot [3] \\ q^2 - q - q^3 & q \cdot (-1 - q^2) \end{bmatrix} \text{ and}$$

$$a_1 \cdot b_1 \cdot a_1 = \begin{bmatrix} q^3 & -[3] \cdot q \\ -q^2 \cdot [2] & q^3 \end{bmatrix}.$$

Further calculations show:

$$a_1 + e = \begin{bmatrix} 0 & 0 \\ q & [2] \end{bmatrix}, b_1 + e = \begin{bmatrix} [3] & [3] \\ -q^2 & -q^2 \end{bmatrix},$$

*We identify the elements a and b with the representing matrices.

$$x_1 := (a_1 + e) \cdot (b_1 + e) = \begin{bmatrix} 0 & 0 \\ q & q \end{bmatrix} \text{ finally } e_{2,2} := a_1 + e - x_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

For a later comment we note

$$b_1 - a_1 = \begin{bmatrix} [3] & [3] \\ -[2] \cdot q & -[3] \end{bmatrix} \text{ compared with } b - a = \begin{bmatrix} 0 & [3] \\ 1 & 0 \end{bmatrix}.$$

This shows

Lemma 2.3 † *The matrices a_1 and b_1 generate over $\mathbb{Z}[q]$ the ring*

$$\Lambda_1 := \begin{pmatrix} \mathbb{Z}[q] & [3] \cdot \mathbb{Z}[q] \\ q \cdot \mathbb{Z}[q] & \mathbb{Z}[q] \end{pmatrix}.$$

We note that this is the ring generated over $\mathbb{Z}[q]$ and not over $\mathbb{Z}[q, q^{-1}]$. So it would be NATURAL TO USE THIS RING to describe $\mathcal{H}_{S_3}$. The point is that this is not advisable, since the amalgamation between Λ_1 and the one-dimensional representations is now quite involved. In the exact sequence 2 (cf. below) the kernel is generated by $b - a$, which generates in a natural way the "radical" of Λ_3, and the radical quotient is easily described; the situation is very involved for the ideal generated by $b_1 - a_1$ in Λ_1. This is the reason why the representations a and b seem to be the more natural ones.
We now have to find the "amalgamation" in $\mathcal{H}_{S_3}$ between Λ_3 and $\mathcal{H}_2$. Since a and b act in the same way on $\mathcal{H}_2$, it is easily seen that we have a short exact sequence

$$0 \longrightarrow (b - a) \cdot \mathcal{H}_{S_3} \longrightarrow \mathcal{H}_{S_3} \longrightarrow \mathcal{H}_2 \longrightarrow 0. \tag{2}$$

Next we have to find out the kernel of the projection $\mathcal{H}_{S_3}$ onto Λ_3. For this again we have to do some calculations:

$$a := \begin{bmatrix} -1 & 0 \\ -1 & q \end{bmatrix}, \ b := \begin{bmatrix} -1 & [3] \\ 0 & q \end{bmatrix},$$

$$c := q^{-1} \cdot a \cdot b = \begin{bmatrix} 1/q & -[3]/q \\ 1/q & -[2]/q \end{bmatrix} \text{ and } e := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

We then calculate

$$c^2 := \begin{bmatrix} 1/q^2 + -[3]/q^2 & -[3]/q^2 + -[3] \cdot -[2]/q^2 \\ 1/q^2 + -[2]/q^2 & -[3]q^2 + (-[2])^2/q^2 \end{bmatrix},$$

† Similar calculations were done some years ago for the symmetric group on three letters by the author with the assistance of his assistant, the Apl. Prof. Dr. W. Kimmerle.

$$e + c := \begin{bmatrix} 1/q+1 & -[3]/q \\ 1/q & -[2]/q+1 \end{bmatrix} \text{ and}$$

$$K := e + c + c^2 =$$

$$\begin{bmatrix} 1/q+1+1/q^2+-[3]/q^2 & -[3]/q+-[3]/q^2+(-[3])\cdot(-[2])/q^2 \\ 1/q+1/q^2+-[2]/q^2 & -[2]/q+1+-[3]/q^2+(-[2])^2/q^2 \end{bmatrix}.$$

We now conclude $K = 0$. Note that

$$1 + q^{-1} \cdot a \cdot b + q^{-2} \cdot (a \cdot b)^2 = 1 + q^{-1} \cdot b \cdot a + q^{-2} \cdot (b \cdot a)^2 \text{ in } \mathcal{H}_{S_3}\,.$$

Is is now easily seen that this is the only relation in Λ_3. We put

$$Tr_3 = 1 + q^{-1} \cdot a \cdot b + q^{-2} \cdot (a \cdot b)^2\,.$$

Then we have the Hecke algebra $\mathcal{H}_{S_3}$ defined by the following commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & Tr_3 \cdot \mathcal{H}_{S_3} \arrow[r, "\simeq"] \arrow[d] & (1+q^{-1}a^2+q^{-2}a^4)\cdot\mathcal{H}_2 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & (b-a)\cdot\mathcal{H}_{S_3} \arrow[r] \arrow[d] & \mathcal{H}_{S_3} \arrow[r] \arrow[d] & \mathcal{H}_2 \arrow[r] \arrow[d] & 0\,. \\
0 \arrow[r] & (b-a)\cdot\Lambda_3 \arrow[r] \arrow[d] & \Lambda_3 \arrow[r] \arrow[d] & \mathbb{Z}[q,q^{-1}]^-/([3]) \times \mathbb{Z}[q,q^{-1}]^q/([3]) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Here $\mathbb{Z}[q,q^{-1}]/<[3]>^- \simeq \mathbb{Z}[\theta_3]$ with a and b acting as the sign representation where θ_3 is a primitive third root ofunity. Moreover, we have an isomorphism $\mathbb{Z}[q,q^{-1}]/([3])^q \simeq \mathbb{Z}[\theta_3]$ with a and b acting as the index representation.
Now we have a closer look at the ideal $(1+q^{-1}a^2+q^{-2}a^4)\cdot\mathcal{H}_2$. Using the relation $a^2 = (q-1)\cdot a + q$, one finds out

$$(1 + q^{-1}a^2 + q^{-2}a^4) = q^{-2} \cdot [3] \cdot a^2\,.$$

Hence $(1+q^{-1}a^2+q^{-2}a^4)\cdot\mathcal{H}_2 = [3]\cdot\mathcal{H}_2$, making again extensive use of the fact that q is invertible.
Let us turn briefly to the integral group ring $\mathbb{Z}S_3$ of the symmetric group on three letters,

$$S_3 := <\alpha\,,\, \beta\, :\, a^2\,,\, \beta^2\,,\, \alpha\cdot\beta\cdot\alpha = \beta\cdot\alpha\cdot\beta>\ .$$

Since reduction modulo $< q-1 >$ is in general not compatible with taking quotients, one has to be a bit careful. The calculations are straightforward and so we omit them. The group ring $\mathbb{Z}S_2$ of the cyclic group $S_2 :=< \alpha >$ of order 2 is a pull-back:

$$\begin{tikzcd} \mathbb{Z}S_2 \arrow[r, "mod(a-1)"] \arrow[d, "mod(a+1)"'] & \mathbb{Z} \arrow[d, "mod(2)"] \\ \mathbb{Z} \arrow[r, "mod(2)"] & \mathbb{F}_2 \end{tikzcd} .$$

If we define

$$\Gamma_3 := \begin{pmatrix} \mathbb{Z} & 3\cdot\mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix},$$

then the group ring of S_3 is given by the pull-back

$$\begin{tikzcd} \mathbb{Z}S_3 \arrow[r, "mod(\alpha-\beta)"] \arrow[d, "mod(1+\alpha\cdot\beta+(\alpha\cdot\beta)^2)"'] & \mathbb{Z}S_2 \arrow[d, "mod(3)"] \\ \Gamma_3 \arrow[r] & \mathbb{F}_3S_2 \simeq \mathbb{F}_3^+ \times \mathbb{F}_3^- \end{tikzcd} .$$

2.2 The Hecke order and the integral group ring of the symmetric group on three letters

We shall first give a more intrinsic description of $\mathbb{Z}S_3$ and of $\mathcal{H}_{S_3}$. By R we denote either $\mathbb{Z}$ or $\mathbb{Z}[q,q^{-1}]$, for a rational prime number p we write $[p] = p$ in case $R = \mathbb{Z}$ and $[p] = 1 + q + q^2 + \cdots + q^{p-1}$ for $R = \mathbb{Z}[q,q^{-1}]$. Moreover, we use the notation

$$\left(R \overset{[p]}{\longrightarrow} R\right) = \{(r, r+[p]\cdot s \,:\, r\,,\, s \in R\}\,.$$

Then we can write the Hecke algebra $\mathcal{H}_{S_3}$ and the integral group ring $\mathbb{Z}S_3$ in the unified form:

(3)

$$\mathcal{O} := \begin{tikzcd} (R^q) \arrow[d, dash, "{[p_1]}"'] \arrow[dr, dash, "{[p_2]}"] & \\ \begin{pmatrix} R & R \\ <[p_1]> & R \end{pmatrix} \arrow[r, dash, "{[p_1]}"] & (R^-) \end{tikzcd} ,$$

where R^q denotes either the trivial representation or the index representation and R^- denotes the sign representation; moreover, $[p_1] = [3]$ and $[p_2] = [2]$.

2.3 The cell structure of $\mathbb{Z}S_3$ and $\mathcal{H}_{S_3}$

We use the above notation for R and $[p]$ and define a chain of ideals as follows.

$J_0 = \overline{J_0} = P_0 := [2] \cdot [3] \cdot R^q$ is an R-free ideal in $\mathcal{O}$ with R-free quotient.

The ring $\mathcal{O}_1 := \mathcal{O}/J_0$ now has the following structure

$$\mathcal{O}_1 := \begin{tikzcd} \begin{pmatrix} R & R \\ <[p_1]> & R \end{pmatrix} \arrow[r, dash, "{[p_1]}"] & (R^-) \end{tikzcd} ; \tag{4}$$

here $[p_1] = [3]$.
$\mathcal{O}_1$ again has an ideal, which is R-free with R-free quotient,

$$\overline{J_1} := \begin{pmatrix} R & <[p_1]> \\ R & <[p_1]> \end{pmatrix} \text{ ; we put } P_1 := \begin{pmatrix} R \\ R \end{pmatrix} .$$

We denote by J_1 the preimage of $\overline{J_1}$ in $\mathcal{O}$. We now have

$$\mathcal{O}_2 = \overline{J_2} = P_2 := \mathcal{O}_1/\overline{J_1} = \mathcal{O}/J_1 = R^- .$$

So we have constructed what is called a Chain of cell ideals

$$J_0 \subset J_1 \subset \mathcal{O} ,$$

which has the following interesting property: The ring $\mathcal{O}$ admits an anti involution ι, which is the identity on R^q and on R^-, and which acts on

$$\begin{pmatrix} R & [p_1] \cdot R \\ R & R \end{pmatrix} \text{ by sending } \begin{pmatrix} a & [p_1] \cdot b \\ c & d \end{pmatrix} \text{ to } \begin{pmatrix} a & [p_1] \cdot c \\ b & d \end{pmatrix} .$$

We can now verify the most important property of cell ideals (cf. Definition 7.4): For $0 \leq i \leq 2$ we have

$$\overline{J_i} = P_i \otimes_{End_{\mathcal{O}}(P_i)} {}^{\iota}P_i\,.$$

We thus have proved

Lemma 2.4 *The Hecke algebra $\mathcal{H}_{S_3}$ and the integral group ring $\mathbb{Z}S_3$ of the symmetric group on three letters are* CELLULAR ORDERS *(cf. Definition 7.4).*

Any two anti-involutions ι_1 and ι_2 of a ring Λ differ by an automorphism ρ of Λ. Let P be a left Λ-module. If ρ is inner, then

$$P \otimes_{End_\Lambda(P)} {}^{\iota_1}P \simeq P \otimes_{End_\Lambda(P)} {}^{\iota_2}P$$

as two-sided Λ-modules. This need not be so if ρ is not inner. Let us demonstrate this with an EXAMPLE: Let $(R, <\pi>)$ be a local commutative principal ideal domain, and let

$$\Lambda := \begin{pmatrix} R & \pi \cdot R \\ R & R \end{pmatrix}, \omega := \begin{pmatrix} 0 & \pi \\ 1 & 0 \end{pmatrix} \text{ and } P := \begin{pmatrix} R \\ R \end{pmatrix}.$$

Conjugation with ω is an automorphism ρ of Λ, which is not inner. Then we have two anti-involutions

$$\iota_1 := \begin{pmatrix} a & \pi \cdot b \\ c & d \end{pmatrix} \longmapsto \begin{pmatrix} a & \pi \cdot c \\ b & d \end{pmatrix} \text{ and}$$

$$\iota_2 = \iota_1 \circ \rho := \begin{pmatrix} a & \pi \cdot b \\ c & d \end{pmatrix} \longmapsto \begin{pmatrix} d & \pi \cdot b \\ c & a \end{pmatrix}.$$

ι_1 and ι_2 are two non-conjugate anti-involutions, and we have

$$P \otimes_R {}^{\iota_1}P = \begin{pmatrix} R & \pi \cdot R \\ R & \pi \cdot R \end{pmatrix} \text{ but } P \otimes_R {}^{\iota_2}P = \begin{pmatrix} R & R \\ R & R \end{pmatrix},$$

and so the two-sided ideals are not isomorphic. They are isomorphic as left modules, but not as right modules.

Note 2.5 1. The group ring $\mathbb{Z}S_3$ has another $\mathbb{Z}$-linear anti-involution, induced from the anti-involution of S_3, namely $\iota_1 : x \longmapsto x^{-1}$ for $x \in S_3$. If one modifies the above ι with conjugation by the unit

$$\eta := (1, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, 1),$$

then $\iota \circ conj(\eta)$ and ι_1 differ by the GROUP AUTOMORPHISM induced by $a \longmapsto b$ and $b \longmapsto a$. Hence for our automorphism we can equally well choose the group anti-involution ι_1

2. The situation is completely different for the Hecke algebra. The Hecke algebra $\mathcal{H}_{S_3}$ has a basis $\{1, a, b, a\cdot b, b\cdot a, a\cdot b\cdot a\}$ and the map sending each of the above basis elements to its inverse is an anti-involution. The anti-involution ι_1 induced by this is however not $\mathbb{Z}[q,q^{-1}]$-linear, since the element a in the index representation is represented as $q \in \mathbb{Z}[q,q^{-1}]^q$, and so ι_1 inverts it; i. e. ι_1 is NOT $\mathbb{Z}[q,q^{-1}]$-LINEAR. Our anti-involution ι, though, is $\mathbb{Z}[q,q^{-1}]$-linear.

2.4 The quasi-hereditary quotients of $\mathbb{Z}S_3$ and $\mathcal{H}_{S_3}$

The ring $\mathcal{O}_1$ from Equation 4 above is a QUASI-HEREDITARY order in the sense of Definition 5.1. In fact it has an idempotent

$$e := \left(\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}, 0\right) \text{ such that } \overline{J_1} = \mathcal{O}_1 \cdot e \cdot \mathcal{O}_1 = \begin{pmatrix}R & \pi\cdot R\\ R & \pi\cdot R\end{pmatrix}$$

is a pure [‡] R-free ideal with $\mathcal{O}_2 = \mathcal{O}_1/\overline{J_1} = R$ hereditary. We formulate this as

Lemma 2.6 *Let $R = \mathbb{Z}$ or $R = \mathbb{Z}[q,q^{-1}]$ and $[p] = p$ or $[p] = 1 + q + q^2 + \cdots + q^{p-1}$. Then $J_0 := [2]\cdot[3]\cdot R^q$ is a pure R-free ideal in $\mathbb{Z}S_3$ and $\mathcal{H}_{S_3}$ resp. such that $\mathbb{Z}S_3/J_0$ and $\mathcal{H}_{S_3}/J_0$ are R-free quasi-hereditary orders.*

We point out that in general a quotient modulo a pure ideal of a finitely generated free $\mathbb{Z}[q,q^{-1}]$-module need not be free; e. g. if $\mathfrak{m} =< p, f >$ is a maximal ideal, then we have an exact sequence

$$0 \longrightarrow R \longrightarrow R^2 \longrightarrow \mathfrak{m} \longrightarrow 0\,.$$

2.5 Localizations of $\mathcal{H}_{S_3}$ and $\mathbb{Z}_3S_3$

If we look at the structure of $\mathcal{H}_{S_3}$ in Diagram 3, we see that the only ideals involved are the minimal prime ideals $< [2] >=< 1 + q >$ and $< [3] >=< 1 + q + q^2 >$. For the group ring $\mathbb{Z}S_3$ the primes involved are $< 2 >$ and $< 3 >$; moreover $< [p] >$ in $\mathbb{Z}[q,q^{-1}]$ maps to $< p >$ modulo $< q - 1 >$.
If R is a commutative ring and $\wp$ a prime ideal, then we denote by $R_\wp$ the COMPLETION at $\wp$.
We are interested in maximal ideals $\wp$ lying above $< q-1 >$, since we still want to have an epimorphism $\mathbb{Z}[q,q^{-1}]_\wp \otimes_{\mathbb{Z}[q,q^{-1}]} \mathcal{H}_{S_3} \longmapsto \mathbb{Z}_pS_3$ modulo $< q-1 >$.

[‡]This means that the quotient is torsion-free.

Recall that the maximal ideals of $\mathbb{Z}[q]$ are of the form $< p, f >$, where f is a monic irreducible polynomial in $\mathbb{Z}[q]$, which is irreducible modulo p. So we are interested in the maximal ideals of the form $\mathfrak{m}_p :=< p, q-1 >$ for a rational prime p. We first observe that $q \notin \mathfrak{m}_p$, since then $1 \notin \mathfrak{m}_p$. This implies in particular that

$$\mathbb{Z}[q, q^{-1}]_{\mathfrak{m}_p} = \mathbb{Z}[q]_{\mathfrak{m}_p} \,.$$

Case 1: Let us look at the maximal ideal $\mathfrak{m}_2 =< q-1, 2 >$ and put $R_2 := \mathbb{Z}[q]_{\mathfrak{m}_2}$. We then have $1 = [3] - q \cdot ((q-1)+2)$ and so $[3]$ is a unit in $\mathbb{Z}[q]_{\mathfrak{m}_2}$. For the Hecke algebra we then have

$$R_2 \otimes_R \mathcal{H}_{S_3} = \begin{pmatrix} R_2 & R_2 \\ R_2 & R_2 \end{pmatrix} \oplus R^q \overset{[2]}{\text{———}} R^- \text{ with } R = \mathbb{Z}[q] \,.$$

It decomposes into two blocks, one of which is a separable order, the other the Hecke algebra $R_2 \cdot \mathcal{H}_2$ of the cyclic group of order 2.
Case 2: Let us look at the maximal ideal $\mathfrak{m}_3 =< q-1, 3 >$ and put $R_3 := \mathbb{Z}[q]_{\mathfrak{m}_3}$. We then have $1 = (q-1) + 3 - [2]$ and so $[2]$ is a unit in $\mathbb{Z}[q]_{\mathfrak{m}_3}$. Thus for the Hecke algebra we have

$$R_3 \otimes_R \mathcal{H}_3 = \begin{tikzcd} (R^q_{p_1}) \arrow[d, dash, "{[p_1]}"'] & \\ \begin{pmatrix} R_{p_1} & R_{p_1} \\ <[p_1]>_{p_1} & R_{p_1} \end{pmatrix} \arrow[r, dash, "{[p_1]}"] & (R^-_{p_1}) \end{tikzcd} \,, \tag{5}$$

where $[p_1] = [3]$.
Case 3: Let us look at a maximal ideal $\mathfrak{m} =< q-1, p > \neq \mathfrak{m}_2 \,, \mathfrak{m}_3 \,$; i. e. $p \neq 2 \,, 3$ and put $R_{\mathfrak{m}} := \mathbb{Z}[q]_{\mathfrak{m}}$. We then have $2 \cdot \alpha + p \cdot \beta = 1$ and $3 \cdot \gamma + p \cdot \delta = 1$ for rational integers $\alpha \,, \beta \,, \gamma \,, \delta \in \mathbb{N}$. Consequently

$$\beta \cdot p + \alpha \cdot ((q-1) - (1+q)) = 1 \text{ and } \gamma \cdot ((q+2) \cdot (q-1) - ([3])) + \delta \cdot p = 1 \,.$$

Thus $[2]$ and $[3]$ are units in $R_{\mathfrak{m}}$. Hence the Hecke algebra is separable

$$R_{\mathfrak{m}} \cdot \mathcal{H}_{S_3} = \begin{pmatrix} R_{\mathfrak{m}} & R_{\mathfrak{m}} \\ R_{\mathfrak{m}} & R_{\mathfrak{m}} \end{pmatrix} \oplus R^q_{\mathfrak{m}} \oplus R^-_{\mathfrak{m}} \,.$$

The localizations and completions of $\mathbb{Z}S_3$ follow easily from those of $\mathcal{H}_{S_3}$ by specializing q to 1.

2.6 The localized Hecke order $R_3 \otimes_R \mathcal{H}_3$ and the group ring $\mathbb{Z}_3 S_3$ as Brauer tree orders

We shall study the order $\Gamma_3 := R_3 \otimes_R \mathcal{H}_3$ from Equation 5 in more detail. It has two non-isomorphic indecomposable projective modules

$$P_1 = \{(u, \begin{pmatrix} u + [3] \cdot a & 0 \\ b & 0 \end{pmatrix}, 0) \, : \, u, a, b \in R_3\} \text{ and}$$

$$P_2 = \{(0, \begin{pmatrix} 0 & c \\ 0 & v + [3] \cdot d \end{pmatrix}, v) \, : \, c, d, v \in R_3\} \, .$$

Moreover, it has 4 special irreducible CM-lattices [§]:

$$M_1 := (R_3^q, 0, 0) \, , \, M_2 := (0, \begin{pmatrix} R_3 & 0 \\ R_3 & 0 \end{pmatrix}, 0) \, ,$$

$$M_3 := (0, \begin{pmatrix} 0 & [3] \cdot R_3 \\ 0 & R_3 \end{pmatrix}, 0) \text{ and } M_4 := (0, 0, R_3^-) \, .$$

Over the quotient field K of R_3 the algebra $K \cdot \Gamma_3$ has three rational components,

$$K \cdot \Gamma = K^q \times (K)_2 \times K^- \, .$$

We now define a graph, the Brauer tree of Γ_3 as follows:

- The vertices correspond to the rational components of $K \cdot \Gamma$, so let us label them $1^q = K^q$, $2 = (K)_2, 3^- = K^-$
- The projective modules P_1 and P_2 have each exactly two simple rational components. We draw an edge between two vertices, if a projective module has components in the rational components corresponding to these vertices.

So Γ has the Brauer tree (cf. Theorem 4.9)

$$1^q \xrightarrow{P_1} 2 \xrightarrow{P_2} 3^- \, .$$

This Brauer tree gives also important information about projective resolutions – "Green's walk around the Brauer-tree". We look at the projective resolution

[§]This is a left R_3-free Γ_3-module, which generates a simple module over the total ring of quotients.

of M_1:

$$\begin{tikzcd}
& P_1 \arrow[dr] & & P_2 \arrow[dr] & & P_2 \arrow[dr] & & P_1 \arrow[dr] & \\
M_1 \arrow[ur] & & M_2 \arrow[ur] & & M_4 \arrow[ur] & & M_3 \arrow[ur] & & M_1
\end{tikzcd} .$$

This is obtained by walking clockwise around the tree. We summarize this as

Lemma 2.7 *The Hecke-order $\mathbb{Z}[q]_{<q-1,3>} \otimes_{\mathbb{Z}[q,q^{-1}]} \mathcal{H}_{S_3}$ and the group ring $\mathbb{Z}_3 S_3$ are Brauer-tree-orders in the sense of Definition 4.2.*

2.7 A separable deformation of $\mathbb{Z}S_3$

Let us look at the algebra $\mathcal{H}_3^+ := \mathbb{Z}[q] < a, b >$ from Section 2.1 – note that now q is not invertible anymore, and the structural results from above do not hold any more (cf. Lemma 2.2).
We shall now modify the algebra $\mathcal{H}_{S_3}^+$ a bit to obtain almost a separable deformation of $\mathbb{Z}S_3$. This Hecke algebra was described in Equation 3 for $R = \mathbb{Z}[q, q^{-1}]$. We now look at the algebra $\mathcal{D}_3(q) := \mathcal{O}$ from Equation 3 for $R = \mathbb{Z}[q]$.
We consider the algebra $\mathbb{Z}_p \cdot \mathcal{D}_3(q)$ for the various rational primes p and specialize q to various elements in $\mathbb{Z}_p$.
Case 1: $p = 3$.

- Let $q = 1 + 3 \cdot x \in \mathbb{Z}_3$. Then $< 1 + q >= \mathbb{Z}_3$ and $< 1 + q + q^2 >= 3 \cdot \mathbb{Z}_3$. In this case $\mathbb{Z}_3 \cdot \mathcal{D}_3(q) = \mathbb{Z}_3 S_3$.

- If $q = 3 \cdot x \in \mathbb{Z}_3$, then both $q + 1$ and $1 + q + q^2$ are units and so we get the separable order

$$\mathbb{Z}_3 \cdot \mathcal{D}_3(q) = \mathbb{Z}_3 \times (\mathbb{Z}_3)_2 \times \mathbb{Z}_3 \,.$$

- If $q = 2 + 3 \cdot x \in \mathbb{Z}_3$, then $< 1 + q + q^2 >= \mathbb{Z}_3$ but $1 + q = 3 \cdot (1 + x)$. Here we have two different cases:

  – If $x = -1$ then $1 + q = 0$ and we have a DECREASE IN DIMENSION:

  $$\mathbb{Z}_3 \cdot \mathcal{D}_3(q) = \mathbb{Z}_3 \times (\mathbb{Z}_3)_2 \,.$$

  This order is also separable, but of rank 5.

– If $x \neq -1$, then x can be chosen in such a way that we have

$$< 1+q >=< 3 \cdot (1+x) >=< 3^n > \text{ for ANY } n \in \mathbb{N}.$$

In this case we obtain an inseparable order:

$$\mathbb{Z}_3 \cdot \mathcal{D}_3(q) = \{(a, a+3^n \cdot b) \,:\, a\,,\, b \in \mathbb{Z}_3\} \times (\mathbb{Z}_3)_2\,.$$

Case 2: $p = 2$.

- $q = 1 + 2 \cdot x$ in this case $1 + q + q^2$ is a unit and $q + 1 = 2(1 + x)$.
  - For $x = -1$ we obtain a DECREASE IN DIMENSION:

    $$\mathbb{Z}_2 \cdot \mathcal{D}_3(q) = (\mathbb{Z}_2)_2 \times \mathbb{Z}_2$$

    has dimension 5.
  - For $x \neq -1$ the ideal $< 2 \cdot (1+x) >$ can be any power $< 2^n >$, and we have:

    $$\mathbb{Z}_2 \cdot \mathcal{D}_3(q) = (\mathbb{Z}_2)_2 \times \{(a, a+2^n \cdot b) \,:\, a\,,\, b \in \mathbb{Z}_2\}\,,$$

    which is inseparable.
- For $q = 2 \cdot x$, we get the separable order

  $$\mathbb{Z}_2 \cdot \mathcal{D}_3(q) \simeq (\mathbb{Z}_2)_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2\,.$$

Case 3: $p \neq 2\,,\,3$. In this case it is easily seen that we obtain a separable order, since both $1 + q$ and $1 + q + q^2$ are units.
So we see that apparently we CANNOT GET A GLOBAL $\mathbb{Z}[q]$-ORDER, WHICH UPON TENSORING WITH $\mathbb{Z}_p$ GIVES A SEPARABLE DEFORMATION OF $\mathbb{Z}_pS_3$ in the sense of Definition 6.1.
We have to do this a PRIME AT A TIME. The proof of the next result follows from the previous calculations:

Lemma 2.8 • *For $p = 3$, the $\mathbb{Z}_3[q]$-order*

$$\{(u, v, \begin{pmatrix} u+[3]\cdot a & [3]\cdot b \\ c & v+[3]\cdot d \end{pmatrix}) \,:\, u, v, a, b, c, d \in \mathbb{Z}_3[q]\}$$

is a separable deformation of $\mathbb{Z}S_3$ in the sense of Definition 6.1.

- *For $p = 2$, the $\mathbb{Z}_2[q]$-order*

  $$(\mathbb{Z}_2[q])_2 \times \{(a\,,\, a+(q+1)\cdot b) \,:\, a\,,\, b \in \mathbb{Z}_2[q]\}$$

  is a separable deformation of $\mathbb{Z}S_3$ in the sense of Definition 6.1.

3 Hecke-orders of rank 2 BN-pairs

The HECKE-ORDER of a BN-pair of rank 2 is described as follows.

Definition 3.1 *Let m be a positive integer. Let $\mathcal{H}_{D_m}$ be the $\mathbb{Z}[q_a, q_b, q_a^{-1}, a_b^{-1}]$-algebra generated by two elements $\{a := a_m, b := b_m\}$ which satisfy the following relations:*

1. ***Quadratic relations:***

$$x^2 = (q_x - 1) \cdot x + q_x \cdot 1 \text{ for } x = a, b\,.$$

2. ***Homogeneous relations:***

$$(a \cdot b)^k = (b \cdot a)^k \text{ if } m = 2 \cdot k \text{ and}$$

$$(a \cdot b)^k \cdot a = (b \cdot a)^k \cdot b \text{ if } m = 2 \cdot k + 1 \text{ and in this case } q_a = q_b\,.$$

These Hecke orders are for $m = p^n$ with p an ODD PRIME described in [Ro; 97] – in this case, $q := q_a = q_b$. There an inductive description of the Hecke algebra $\mathcal{H}_{D_m}$ is given. This description, which is done as a subring of the separable algebra $\mathbb{Q}(q) \otimes_{\mathbb{Z}[q,q^{-1}]} \mathcal{H}_{D_m}$, shows at the amalgamations of the various "simple" constituents that the modular and the integral representation theory of the dihedral groups are intimately interrelated in the Hecke algebra; it also shows the relevance of the roots of unity for the Hecke algebras. The interaction between modular and integral representations in the Hecke algebras has been successfully used for the study of representations of Coxeter–groups (cf. [CR; 87]).
Contrary to the common definition, where the Hecke algebras are defined over $\mathbb{Q}[q_1, \cdots, q_n, q_1^{-1}, \cdots, q_n^{-1}]$ for POSITIVE REAL NUMBERS $\{q_i\}$, or over the LAURENT POLYNOMIALS OVER $\mathbb{Q}$, we treat the $\{q_i\}$ as variables over $\mathbb{Z}$. Thus for the above dihedral groups D_m for p odd we consider it as ORDER over the 2–dimensional ring $\mathbb{Z}^q := \mathbb{Z}[q, q^{-1}]$ of Laurent polynomials over $\mathbb{Z}$.
Let us recall the importance of the polynomial ring $\mathbb{Z}[q]$ which lies in the TWO DIFFERENT TYPES OF MINIMAL PRIME IDEALS ¶, which are principal, $\mathbb{Z}[q]$ being factorial.
The ARITHMETIC PRIME IDEALS are of the form $< f(q) >$ for a $\mathbb{Q}$-irreducible polynomial $f(q) \in \mathbb{Z}[q]$ whose coefficients have one as greatest common divisor. As quotient field of the residue ring EVERY algebraic number field occurs, so that in some sense the ring $\mathbb{Z}[q]$ incorporates all of algebraic number theory, and from these ideals we get the integral theory.

¶This means hight one primes in our case.

The GEOMETRIC PRIME IDEALS [‖] are of the form $< p >$ for a rational prime p, and the quotient ring is $\mathbb{F}_p[q]$, the polynomial ring, which in turn maps onto ALL finite fields, so from these ideals we get the modular theory.
The integral theory and the modular theory are interrelated as follows: Let $f(q)$ be a monic irreducible polynomial and let p be a rational prime number. Then we have a pull-back diagram

$$\begin{array}{ccc} \mathbb{Z}[q] & \longrightarrow & \mathbb{Z}[q]/ < f(q) > \\ \Big\downarrow & & \phi\Big\downarrow \\ \mathbb{F}_p[q] & \longrightarrow & \mathbb{F}_{p^n} \end{array} ,$$

where ϕ is reduction modulo a maximal ideal above p in $\mathbb{Z}[q]/ < f(q) >$.
It is well known that $\mathcal{H}_{D_m}$ is $\mathbb{Z}^q$–free with a semi-simple ring of quotients, isomorphic to the group algebra $\mathbb{Q}(q)D_m$ [Ro; 97].
In describing $\mathcal{H}_{D_m}$ as a subring of $\mathbb{Q}(q)D_m$, one has to choose better suited irreducible representations than those given by Curtis [CR; 87], in order to describe the projections of $\mathcal{H}_{D_m}$ into the simple "rational" components of $\mathbb{Q}(q)D_m$. The importance of the proper choice of the irreducible representation should be compared with the importance of a "good" basis for Artin algebras. These "simple" projections are very similar to those of the integral group ring of $\mathbb{Z}D_m$; as a matter of fact, they are two–dimensional Cohen–Macaulay–orders BUILT AS ANALOGA OF CLASSICAL HEREDITARY ORDERS.
Definition: We assume that $m = p^n$ for some odd prime p and put $R_m := \mathbb{Z}[\eta_m]$ with $\eta_m = \theta_m + \theta_m$ and θ_m a primitive p^n-th root of unity. We define $R^q_m := \mathbb{Z}^q \otimes_{\mathbb{Z}} \mathbb{Z}[\eta_m]$ with field of fractions K^q_m, $\gamma_m := (1-\theta_m)\cdot(1-\theta_m^{-1})$ and $\gamma^q_m := (q-\theta_m)\cdot(q-\theta_m^{-1})$ [**]. π_m stands for the R_m-ideal generated by γ_m and π^q_m denotes the R^q_m-ideal generated by γ^q_m.
We denote by Λ_m the $\mathbb{Z}^q$-order in $(K^q_m)_2$ generated by the FAITHFUL REPRESENTATION of $\mathcal{H}_{D_m}$.
The result is then:
Proposition I: Assume that m IS ODD. Then

$$\Lambda_m = \begin{pmatrix} R^q_m & \pi^q_m \\ R^q_m & R^q_m \end{pmatrix} .$$

The Hecke order $\mathcal{H}_{D_m} = \mathcal{H}_{D_{p^n}}$ will be described inductively as pull-back of the order corresponding to the faithful representation Λ_{p^n} and the Hecke order $\mathcal{H}_{D_{p^{n-1}}}$. The amalgamating ring is quite involved; it involves the integral representation of D_m and the modular representations of D_m.

[‖]The name was suggested to me by Claus M. Ringel.
[**]It is VERY important not to confuse γ_m and γ^q_m!

To describe the amalgamating rings, we have to describe the "Hecke order" $\mathcal{H}_n$ of a cyclic group of order $m = p^n$ which is generated over $\mathbb{Z}[q, q^{-1}]$ by c_n subject to the $p^n = m$-TH POWER RELATION $R_m(X)$, where

$$R_m(X) := (X - q) \cdot (\sum_{i=1}^{m-1} X^i)\,.$$

The "Hecke order" $\mathcal{H}_n$ for $n > 1$ is then inductively described by pull-backs. To this end we put

$$\omega_n(c_n) := (\sum_{i=0}^{p^{n-1}-1} c_n^i) \cdot (c_n - q) = (R_{p^{n-1}}(c_n)) \text{ and } tr_n = (\sum_{i=0}^{p-1} c_n^{i \cdot p^{n-1}})\,.$$

Proposition II: The ideal generated by $\omega_n^q(c_n)$ is

$$< (1 - \theta_m)^{p^{n-1}-1} > \cdot < \theta_m - q >$$

and the Hecke order $\mathcal{H}_n$ is for $n > 1$ inductively described by the following commutative diagram with exact rows and columns.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & tr_n \cdot \mathcal{H}_n \arrow[r, "\simeq"] \arrow[d] & \overline{tr_n} \cdot \mathcal{H}_{n-1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \omega_n(c_n) \cdot \mathcal{H}_n \arrow[r] \arrow[d] & \mathcal{H}_n \arrow[r] \arrow[d] & \mathcal{H}_{n-1} \arrow[r] \arrow[d] & 0\,, \\
0 \arrow[r] & \omega_n(\theta_m) \cdot \mathbb{Z}^q[\theta_m] \arrow[r] \arrow[d] & \mathbb{Z}^q[\theta_m] \arrow[r] \arrow[d] & \Sigma_n \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Σ_n is again described as a pull-back.

$$\begin{array}{ccc}
\Sigma_n & \longrightarrow & \mathbb{F}_p[X]/X^{p^{n-1}-1}[q, q^{-1}] \\
\big\downarrow & & \big\downarrow \\
\mathbb{Z}[\theta_m] & \longrightarrow & \mathbb{F}_p[X]/ < X^{p^{n-1}-1} >
\end{array}\,.$$

This shows that Σ_n involves both the faithful integral representation on $\mathbb{Z}[\theta_m]$ of the cyclic group of order p^n and the "universal modular representation" over $\mathbb{F}_p[X]$ of the cyclic group of order p^{n-1}, namely $\mathbb{F}_p[q,q^{-1}]C_{p^{n-1}}/soc(\mathbb{F}_p[q,q^{-1}]C_{p^{n-1}})$. Moreover, the ring $\mathbb{Z}[\theta_m]$ is obtained as the quotient

$$\mathbb{Z}^q[\theta_m]/ < \theta_m - q > \simeq \mathbb{Z}[\theta_m]\,.$$

Here one sees the SINGULAR BEHAVIOR at the roots of unity, if one specifies q to θ_m.
Moreover, for arbitrary n we have a pull-back diagram for the Hecke-order

$$\begin{array}{ccc} \mathcal{H}_n & \xrightarrow{\alpha_1} & \mathbb{Z}^q \\ {\scriptstyle\beta_1}\big\downarrow & & {\scriptstyle\beta_2}\big\downarrow \\ \Delta_n & \xrightarrow{\alpha_2} & Hom_{\mathbb{Z}}(I(C_n),\mathbb{Z}) \end{array} \quad ,$$

where $I(C_n)$ is the augmentation ideal of $\mathbb{Z}C_n$, the map β_1 is reduction modulo $\sum_{i=0}^{p^n-1} c_n^i$, the map β_2 is reduction modulo $\sum_{i=0}^{p^n-1} q^i$, the maps α_1 and α_2 are reductions modulo $c_n - q$.
This result is then used to define the "Hecke orders" of abelian groups as tensor-products of its cyclic p-power factor groups.
The connection of the Hecke orders $\mathcal{H}_n$ of the cyclic groups of order $m := p^n$ and the Hecke orders $\mathcal{H}_{D_{p^n}} = \mathbb{Z}^q < a, b_m >$ of the dihedral groups of order $2 \cdot p^n$ lies in the fact that the $\mathbb{Z}^q$–order generated by the element $d_n = q$-$\cdot\, a \cdot b_m \in \mathcal{H}_{D_{p^n}}$ is ALMOST such a Hecke order of the corresponding cyclic group $\mathcal{H}_n$. As a matter of fact, the element d_n satisfies the relation:

$$R_m^d(X) = \frac{(X^m - 1)}{X - 1} \cdot (a - b_m)\,,$$

which for $q = 1$ has the the same solution as $R_m(X)$ at $q = 1$.
For the description of the Hecke orders of dihedral groups we need the structure of the Hecke order of the cyclic group of order 2.
Claim: We have the exact sequence:

$$0 \longrightarrow (a - b_m) \cdot \mathcal{H}_{D_m} \longrightarrow \mathcal{H}_{D_m} \longrightarrow \mathcal{H}_2 \longrightarrow 0\,.$$

Recall that $\mathcal{H}_2$ is the Hecke order of the cyclic group of order 2. Moreover, we have a pull-back diagram

$$\begin{array}{ccc} \mathcal{H}_{D_m} & \xrightarrow{\alpha_1} & \mathcal{H}_2 \\ {\scriptstyle\beta_1}\big\downarrow & & {\scriptstyle\beta_2}\big\downarrow \\ \Gamma_n^d & \xrightarrow{\alpha_2} & Hom_{\mathbb{Z}}(I(C_n),R) \times Hom_{\mathbb{Z}}(I(C_n),R) \end{array} \quad ,$$

where $I(C_n)$ is the augmentation ideal over $\mathbb{Z}$ of the cyclic group of order p^n and Γ_n^d is the projection of $\mathcal{H}_{D_m}$ onto the sum of the two-dimensional representations.
If one puts for $n > 1$

$$\omega_n^d(d_n) := (\sum_{i=0}^{p^{n-1}-1} d_n^i) \cdot (a - b_m) = R_{p^{n-1}}^d(d_n)) \text{ and } tr_n = (\sum_{i=0}^{p-1} d_n^{i \cdot p^{n-1}}) \,,$$

then we have:

Proposition III: The ideal generated by $\omega_n^d(d_n)$ is given by

$$< \omega_n^d(d_n) >= \pi_m^\nu \cdot < a - b_m >= \pi_m^\nu \cdot \begin{pmatrix} \pi_m^q \cdot R_m^q & \pi_m^q \cdot R_m^q \\ R_m^q & \pi_m^q \cdot R_m^q \end{pmatrix}$$

and the Hecke order$\mathcal{H}_{D_m}$ is inductively described by the following commutative diagram with exact rows and columns.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & tr_n \cdot \mathcal{H}_{D_m} \arrow[r, "\simeq"] \arrow[d] & \overline{tr_n} \cdot \mathcal{H}_{D_{p^{n-1}}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \omega_n(d_n) \cdot \mathcal{H}_{D_m} \arrow[r] \arrow[d] & \mathcal{H}_{D_m} \arrow[r] \arrow[d] & \mathcal{H}_{D_{p^{n-1}}} \arrow[r] \arrow[d] & 0 \,, \\
0 \arrow[r] & \omega_n(d_n) \cdot \Lambda_n^q \arrow[r] \arrow[d] & \Lambda_n^q \arrow[r] \arrow[d] & \Sigma_n^d \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

where Σ_n^d is the pull-back

$$\begin{tikzcd}
\Sigma_n^d \arrow[r] \arrow[d] & \mathbb{Z}[\theta_m] \times \mathbb{Z}[\theta_m] \arrow[d] \\
\Delta_n^d \arrow[r] & \mathbb{F}[X]/ < X^{2 \cdot \nu} > \times \mathbb{F}[X]/ < X^{2 \cdot \nu} >
\end{tikzcd} \;.$$

Here

$$\Delta_n^d = \begin{pmatrix} R_m^q/\pi_m^\nu & \pi_m^q/\pi_m^q \cdot \pi_m^\nu \\ R_m^q/\pi_m^\nu & R_m^q/\pi_m^\nu \end{pmatrix}$$

Since $\Sigma_n^d \simeq \mathcal{H}_{p^{n-1}}/\overline{tr_n} \cdot \mathcal{H}_{p^{n-1}}$ we can use our knowledge of Σ_n^d to describe $\mathcal{H}_{p^{n-1}}/\overline{tr_n} \cdot \mathcal{H}_{p^{n-1}}$. We have the pull-back diagram from the Claim. The element $\overline{tr_n}$ acts on $\mathcal{H}_2$ as $\sum_{i=0}^{p-1}(q^{-i}, q^i)$ and on Γ_{n-1}^d as multiplication with p. We can thus identify $\Gamma_{n-1}^d/p \cdot \Gamma_{n-1}^d$ with Δ_n^d.

4 Green orders

Definition 4.1 *Let $\mathcal{O}$ be a regular noetherian integral domain of dimension 2 with field of fractions $\mathcal{K}$. Let π be a minimal prime ideal in $\mathcal{O}$ with residue ring $\mathcal{O}/\pi := R_\pi$ of dimension 1 and let K be the quotient field of R_π. Then π is part of a regular sequence. Sometimes we just write R for R_π.*

Definition 4.2 *A* Cohen Macaulay $\mathcal{O}$-order $\mathcal{L}$ *in a separable $\mathcal{K}$-algebra A is an $\mathcal{O}$-algebra $\mathcal{L}$, which is $\mathcal{O}$-projective of finite rank, such that $\mathcal{K}\otimes_{\mathcal{O}} \mathcal{L} := A$. By ${}_{CM}\mathcal{L}$ we denote the left* Cohen Macaulay modules *for $\mathcal{L}$; i.e the left $\mathcal{L}$-modules, which are $\mathcal{O}$-projective of finite rank. $ind({}_{CM}\mathcal{L})$ denotes the indecomposable objects in ${}_{CM}\mathcal{L}$. Similar notation is used for right modules. A full*[††] *two-sided ideal P of $\mathcal{L}$ is said to be a* co-classical ideal, *if $P \cap \mathcal{O} = \pi$ is a minimal prime ideal in $\mathcal{O}$ such that $\Lambda = \mathcal{L}/P$ is a classical order over R_π in a separable K-algebra.*

Definition 4.3 *Let Ω_i for $i = 1, 2$ be Cohen Macaulay $\mathcal{O}$-orders with full principal co-classical ideals $\omega(i) := \omega_0(i)\cdot\Omega_i = \Omega_i\cdot\omega_0(i)$ and let Δ be a classical $R = R_\pi$-order in a separable algebra, where $\pi = \mathcal{O} \cap \omega(i)$ is independent of i. If we are given homomorphisms $\alpha_i\colon \mathcal{O}_i \longrightarrow \Delta$ with kernel ω_i, then we can form the pull-back*

$$\begin{tikzcd} \Omega_1 \underset{(\alpha_1,\alpha_2)}{\underline{\qquad}} \Omega_2 \arrow[r] \arrow[d] & \Omega_1 \arrow[d, "\alpha_1"] \\ \Omega_2 \arrow[r, "\alpha_2"] & \Delta \end{tikzcd} .$$

These pull-backs are the first ingredients to the Green-orders, the other ingredients are the triangular orders.

[††] I. e. $\mathcal{K}\otimes_{\mathcal{O}} P = \mathcal{K}\otimes_{\mathcal{O}} \mathcal{L}$.

Definition 4.4 *The order*

$$\mathbb{H} = \mathcal{H}_{\Omega,\omega,n} := \begin{pmatrix} \Omega & \Omega & \Omega & \dots & \Omega & \Omega \\ \omega & \Omega & \Omega & \cdots & \Omega & \Omega \\ \omega & \omega & \Omega & \cdots & \Omega & \Omega \\ & & & \ddots & & \\ \dots & \dots & \dots & & \dots & \dots \\ \omega & \omega & \omega & \cdots & \Omega & \Omega \\ \omega & \omega & \omega & \cdots & \omega & \Omega \end{pmatrix}_n \tag{6}$$

is then the TRIANGULAR *(Cohen Macaulay) $\mathcal{O}$-*ORDER *to the data* (Ω, ω, n) , *where Ω and $< \omega_0 >$ are as Ω_1 and ω_1 in Definition 4.3.*
The left ideal ρ generated by

$$\rho_0 := \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ \omega_0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}_n \tag{7}$$

is two-sided; i. e. $\mathbb{H} \cdot \rho_0 = \rho_0 \cdot \mathbb{H}$, and so ρ is isomorphic to $\mathbb{H}$ as left and as right module; not as bimodule though.
Moreover, conjugation with ρ_0 induces an automorphism σ of $\mathbb{H}$, which cyclicly permutes the indecomposable projective direct summands of $\mathbb{H}$.
The quotient $\mathbb{H}/\rho$

$$\mathbb{H}/\rho = \prod_1^n \Delta$$

is a product of the classical order $\Delta := \Omega/\omega$.
We shall use the above notation for $\mathbb{H}$ and ρ also for one- dimensional (classical) orders.

Assume that a tree $T = (V, E)$ with vertex set $V = \{v_1, \cdots v_\nu\}$ and edges $E = \{e_1, \cdots, e_\mu\}$ is given, which is embedded in the plane. Each vertex v has local edges, numbered clockwise, $(\epsilon_1^v, \cdots, \epsilon_{\nu_v}^v)$. Given an edge from the vertex v to the vertex w, this edge meets v at the local edge ϵ_i^v and w at the local edge ϵ_j^w. We shall write for this edge $e_{\epsilon_i^v, \epsilon_j^w}$. (Note that this takes advantage of the fact that T is embedded in the plane.)
We now construct an order to T, given the following data:

Data 4.5 1. $\mathcal{O}$, π and $R = R_\pi$ are as in Definition 4.1.

2. Δ is an R-order in a separable K-algebra.

3. For a vertex v with valency ν_v the following data are given:

   (a) An order Ω_v with principal co-classical ideal ω_v as in Definition 4.3.

   (b) A surjective $\mathcal{O}$-algebra homomorphism $\alpha^v \colon \Omega_v \longrightarrow \Delta$, which has kernel ω_v.

   (c) The $\mathcal{O}$-order $\mathbb{H}_v = \mathcal{H}_{\Omega_v,\omega_v,\nu_v}$ as in Equation 6.

   (d) We number the quotients modulo "ρ" according to the numbering of the local edges of v:

$$\mathbb{H}_v/\rho_v \xrightarrow{(\alpha_1^v,\alpha_2^v,\cdots,\alpha_{\nu_v}^v)} \prod_{i=1}^{\nu_v} \Delta_i^v\,,$$

   where each of the orders are equal: $\Delta_i^v = \Delta$.

Definition 4.6 *The Green order $\mathcal{T}$ constructed from the above Tree and Data 4.5 is defined as a sub-order of $\mathcal{G} = \prod_{v\in V} \mathbb{H}_v$. The only difference between $\mathcal{T}$ and $\mathcal{G}$ lies in the diagonal entries: We replace the diagonal entry*

$$\mathbb{H}_v(i,i) \times \mathbb{H}_w(j,j) \ \textit{by}\ \Omega_v \underset{(\alpha_i^v,\alpha_j^w)}{\overline{\qquad}} \Omega_w\,,$$

according to Definition 4.3.

Let us point out that we can also define Green orders with respect to locally embedded graphs (cf. [Ka; 97]).
Obviously one can also define Green orders over a complete valuation ring, in a similar way as they are defined above. In this case the ring Δ is an artin algebra over R_π.
Let us pause to recall the essentials from block theory.

4.1 Blocks and defect groups

For details of block theory we refer to [CR; 87].
Let G be a finite group and let R be a complete Dedekind domain with maximal ideal $\mathfrak{p}$ and field of fractions K, a local number field, and residue field $\mathfrak{k}$ of characteristic $p > 0$.
In general, the group ring $RG := \{\sum_{g\in G} r_g \cdot g \ : \ r_g \in R\}$ will decompose into a direct sum of rings. The indecomposable ring direct summands of RG are called BLOCKS or more generally p-BLOCKS of RG. First of all we note that the Krull-Schmidt theorem holds, since R is complete. Then the reader should keep in mind that primitive central idempotents are uniquely determined. Now we state that such a block is uniquely determined by a primitive

central idempotent $e = e(B)$ with $B = RG \cdot e$, which is called the BLOCK IDEMPOTENT.
Since the representation theory of RG is determined by the representation theory of the blocks, these blocks are the main ingredients of p-adic representation theory. For a block B we denote by ${}_{CM}B$ the category of left B-lattices, i. e. left B-modules which are R-free of finite rank. The aim is the classification – if possible – of the indecomposable B-lattices.
An important tool of constructing RG-lattices is the process of INDUCTION. Let H be a subgroup of G and $M \in {}_{CM}RG$ be indecomposable, then there exists a – unique up to conjugation – p-subgroup V of G, called the VERTEX of M and an indecomposable RV-module S such that M is a direct summand of $S \uparrow_V^G := RG \otimes_{RH} S$; i. e. it is a direct summand of an induced module from V. The module S is unique up to conjugation and is called the SOURCE OF M.

Definition 4.7 *Let B be a block, then there exists a minimal – unique up to conjugation – p-group $D = D(B)$, the* DEFECT GROUP OF *B, such that every indecomposable B-module is a direct summand of a module induced from D; equivalently, the defect group of B is the vertex D of B as $R(G \times G^{op})$-module – this vertex can naturally be identified with a subgroup of G.*

Example 4.8

1. There is a unique block of RG containing the trivial representation; it is called the PRINCIPAL BLOCK B_0; its defect group is the Sylow p-subgroup of G.

2. If the Sylow p-subgroup is normal, then every block has the Sylow p-subgroup as vertex, since the defect group of every block is the intersection of two Sylow p-subgroups.

3. A block of DEFECT ZERO is a block whose defect group is trivial; i. e. every indecomposable lattice is induced from the trivial group and hence projective. It even turns out that $B \simeq Mat(n, S)$, where S is an unramified extension of R; i. e. $rad(S) = rad(R) \cdot S$. This means that B is an indecomposable separable R-order.

4. For example, $\mathbb{Z}_3 S_3$ is a single block, but $\mathbb{Z}_2 S_3 = Mat(2, \mathbb{Z}_2) \oplus \mathbb{Z}_2 C_2$ is the direct sum of a block of defect zero and the group ring of C_2, the cyclic group of order 2.

Theorem 4.9 *Let B be a block with cyclic defect of $\mathbb{Z}_p G$ for a finite group G. Then B is a Green order with tree, the* BRAUER TREE *of B.*

In dimension 2 we have the following examples:

Theorem 4.10 *Let $\mathcal{H}_{D_{p^n}}$ be the Hecke order (cf. Section 3) of the dihedral group of order $2 \cdot p^n$ for an odd prime p. Then it is a Green order, corresponding to the tree $\bullet$ —— $\bullet$ —— $\bullet$ (cf. [Ro; 97]).*

Moreover, the maximal and separable deformations of blocks with cyclic defect are also Green orders, whose tree is again the Brauer tree of the block (cf. [Ro; 98, I]).

5 Quasi-hereditary orders

Definition 5.1 *Let R be a complete Dedekind domain and let Λ be an R-order in a separable K-algebra A. A two-sided ideal J is said to be a* HEREDITY IDEAL, *provided*

1. *J is a pure submodule; i. e. Λ/J is an R-order in a separable K-algebra,*
2. *J is Λ-projective as left module,*
3. *J is of the form $J = \Lambda \cdot \epsilon \cdot \Lambda$ for an idempotent ϵ of Λ,*
4. *$\epsilon \cdot \Lambda \cdot \epsilon$ is a maximal R-order in $\epsilon \cdot A \cdot \epsilon$.*

The R-order Λ is called QUASI-HEREDITARY, *provided that there is a proper chain of two-sided ideals $J_0 = 0 \subset J_1 \subset \cdots \subset J_n = \Lambda$ such that for $1 \leq i \leq n$ the quotient J_i/J_{i-1} is a heredity ideal in the R-order Λ/J_{i-1}.*

Remark 5.2 This definition is DIFFERENT FROM THE ORIGINAL one given in [CPS; 88]. They even require that $e \cdot \Lambda \cdot e$ is a separable R-order. In case A is split, both definitions coincide. In general though they are different. We have chosen the above one for the following reason: Let S be a finite ramified extension of R. Then an S-order Λ is also an R-order. Now it may be that Λ is a separable S-order; but then it would not be a separable R-order. Hence in the Definition of [CPS; 88], the question of whether an S-order Λ is quasi-hereditary depends on whether it is considered as an S- or as an R-order. Our definition is independent of the ground ring over which Λ is considered.
One could also GENERALIZE THE DEFINITION – this would probably be more easily verified – and only require that $\epsilon \cdot \Lambda \cdot \epsilon$ is a hereditary order.

Claim 5.3 *Let $J = \Lambda \cdot \epsilon \cdot \Lambda$ be a heredity ideal. Then we may assume that $A_\epsilon := A \cdot \epsilon \cdot A$ is a simple algebra.*

Proof Assume on the contrary that A_ϵ is not simple, then we have a decomposition of the identity of A_e into central primitive idempotents, say $e_A = e_1 + e_2$.

Hence we have a decomposition $\epsilon = e_i + \epsilon_2$ with $\epsilon_i = \epsilon \cdot e_i$ for $i = 1, 2$. Since $\epsilon \cdot \Lambda \cdot \epsilon$ is maximal, we conclude that $\epsilon_i \in \Lambda$. Thus $J = \Lambda \cdot \epsilon_1 \cdot \Lambda \oplus \Lambda \cdot \epsilon_2$, since ϵ_1 and ϵ_2 lie in different rational components. If we put $J_1 = \Lambda \cdot \epsilon_1 \cdot \Lambda$, then the heredity chain will be extended $0 \subset J_1 \subset J$.
q.e.d.

Remark 5.4 We cannot assume though that in a heredity ideal in Λ the idempotent ϵ, which now lies totally in a simple component of A, is primitive. If ϵ is not primitive, then we have a proper decomposition $\epsilon = \epsilon_1 + \epsilon_2$ into orthogonal idempotent. We then have inclusions $\Lambda \cdot \epsilon_i \Lambda \subset \Lambda \cdot \epsilon \cdot \Lambda$, and both ideals span the same algebra. But whereas $\Lambda \cdot \epsilon \cdot \Lambda$ is a pure ideal, this is not so in general with $\Lambda \cdot \epsilon_1 \cdot \Lambda$.

We can give in some sense the structure of quasi-hereditary orders. Let $J := \Lambda \cdot \epsilon \cdot \Lambda$ be a heredity ideal. We then have the Pierce decomposition of $\Lambda_\epsilon := \Lambda \cdot e$, where e is the identity element in $A \cdot \epsilon \cdot A = Mat(m, D)$ for a skew-field D with maximal order Ω. Note that Λ_ϵ is uniquely determined. Let $\omega =< \omega_0 >$ be the radical of Ω. We may then assume that $\Lambda_\epsilon \subset Mat(m, \Omega)$.
We have

$$\Lambda_\epsilon = \begin{pmatrix} \epsilon \cdot \Lambda \cdot \epsilon & \epsilon \cdot \Lambda \cdot (\epsilon - 1) \\ (\epsilon - 1) \cdot \Lambda \cdot \epsilon & (\epsilon - 1) \cdot \Lambda \cdot (\epsilon - 1) \end{pmatrix} \text{ and } \tag{8}$$

$$J = \begin{pmatrix} \epsilon \cdot \Lambda \cdot \epsilon & \epsilon \cdot \Lambda \cdot (\epsilon - 1) \\ (\epsilon - 1) \cdot \Lambda \cdot \epsilon & (\epsilon - 1) \cdot \Lambda \cdot \epsilon \cdot \Lambda \cdot (\epsilon - 1) \end{pmatrix} \tag{9}$$

The first part

$$\Gamma := \epsilon \cdot \Lambda \cdot \epsilon = Mat(n, \Omega)$$

is the maximal order. We put $m = n \cdot \nu$
Moreover, ${}_\Gamma M := \epsilon \cdot \Lambda \cdot (\epsilon - 1)$ is a left lattice for Γ and $N_\Gamma := (\epsilon - 1) \cdot \Lambda \cdot \epsilon$ is a right lattice for Γ, and $(\epsilon - 1) \cdot \Lambda \cdot \epsilon \cdot \Lambda \cdot (\epsilon - 1) = N_\Gamma \otimes_\Gamma {}_\Gamma M$. Since J is a pure ideal in Λ, we must have a pull-back description of L as follows. Let Λ_η be the projection of Λ in $A \cdot (1 - e)$.

$$\begin{array}{ccc} \Lambda & \longrightarrow & \Lambda_\eta \\ \downarrow & & \alpha \downarrow \\ \Lambda_\epsilon & \xrightarrow{\beta} & \Lambda_\epsilon / J \end{array} . \tag{10}$$

But as we have seen above,

$$\Lambda_\epsilon / J = ((1 - \epsilon) \cdot \Lambda_\epsilon \cdot (1 - \epsilon))/((1 - \epsilon) \cdot \Lambda_\epsilon \cdot \epsilon \cdot \Lambda_\epsilon \cdot (1 - \epsilon)) .$$

It should be kept in mind that $(1 - \epsilon) \cdot \Lambda \cdot (1 - \epsilon) \subset Mat(\nu, \Omega)$.
Though Γ-lattices are fairly well understood, the situation may be quite involved as shows the next example.

Example 5.5 Let us recall the block-matrix multiplication:

$$\begin{pmatrix} \Lambda_1 & N \\ M & \Lambda_2 \end{pmatrix} \cdot \begin{pmatrix} \Lambda_1 & N \\ M & \Lambda_2 \end{pmatrix} = \begin{pmatrix} \Lambda_1 + N \cdot M & \Lambda_1 \cdot N + N \cdot \Lambda_1 \\ \Lambda_2 \cdot M + M \cdot \Lambda_2 & \Lambda_2 + \ \cdot N \end{pmatrix}.$$

where the multiplication is the matrix multiplication inside a matrix ring. In particular we must have that N is a (Λ_1, Λ_1)-bimodule and M is a (Λ_2, Λ_2)-bimodule. Moreover, $N \cdot M \subseteq \Lambda_1$ and $M \cdot N \subseteq \Lambda_2$; as a matter of fact, they are even ideals.
We consider the following special example

$$\Lambda_1 = \left\{ \begin{pmatrix} a_{11} & \pi \cdot a_{12} & \pi \cdot a_{13} \\ \pi \cdot a_{21} & \pi^2 \cdot a_{22} & \pi^2 \cdot a_{23} \\ \pi \cdot a_{21} + \pi^2 \cdot a_{31} & \pi \cdot a_{32} & a_{22} + \pi \cdot a_{33} \end{pmatrix} : a_{ij} \in R \right\} \text{ and}$$

$$J_1 = \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ \pi \cdot a_{21} & \pi \cdot a_{22} & \pi \cdot a_{23} \\ \pi \cdot a_{21} + \pi^2 \cdot a_{31} & \pi \cdot a_{22} + \pi^2 \cdot a_{32} & \pi \cdot a_{23} + \pi^2 \cdot a_{33} \end{pmatrix} : a_{ij} \in R \right\}.$$

where $N = (R, R)$ and $M = (R \xrightarrow{(\pi)} R)$. Though as R-module $M \simeq (R, R)$ we cannot replace M by this isomorphic copy. For the quotient we have with $\mathbf{R} = R/\pi$

$$\overline{\Lambda} := \begin{tikzcd} & \bullet_{\mathbf{R}} \arrow[dl, dash, "a"'] \arrow[dr, dash, "b"] & \\ \bullet_{\mathbf{R}} & & \bullet_{\mathbf{R}} \end{tikzcd} . \tag{11}$$

The order $\Lambda_0 = R \xrightarrow{\pi} R$ has the same quotient modulo(π^2, π^2). Hence the order Λ described as pull-back

$$\begin{tikzcd} \Lambda \arrow[r] \arrow[d] & \Lambda_1 \arrow[d] \\ \Lambda_0 \arrow[r] & \overline{\Lambda} \end{tikzcd}$$

is a quasi-hereditary order.

These quasi-hereditary orders occur in various situations in representation theory:

- A general reference for integral quasi-hereditary algebras is the basic paper of E. Cline, B. Parshall and L. Scott [CPS; 90].

- J. A. Green [Gr; 93] gave a combinatorial proof that the classical Schur orders (over $\mathbb{Z}$) are quasi-hereditary.
- R. Dipper and G. James [DiJa; 89] showed that the q-Schur algebras are quasi-hereditary.
- R. Dipper, G. James and A. Mathas defined Schur-algebras to the Ariki-Koike algebras and noted that they are quasi-hereditary.

6 Separable and maximal deformations

Definition 6.1 *Let R be a complete Dedekind domain with maximal ideal $\mathfrak{m}$ and let Λ be an R-order in a separable algebra over the field of fractions of R. A Cohen Macaulay $R[q]$-order $\mathbb{H}$ is called a* SEPARABLE DEFORMATION OF Λ *provided*

1. $\mathbb{H}/ < q-1 > \simeq \Lambda$,
2. *for $r \in R$ not congruent to* 1 *modulo* $\mathfrak{m}$ *the order* $\mathbb{H}/ < q-r >$ *is a separable R-order in a separable algebra.*

$\mathbb{H}$ *is called a* MAXIMAL DEFORMATION OF Λ *provided*

1. $\mathbb{H}/ < q-1 > \simeq \Lambda$,
2. *for $r \in R$ not congruent to* 1 *modulo* $\mathfrak{m}$ *the order* $\mathbb{H}/ < q-r >$ *is a maximal R-order in a separable algebra.*

Note 6.2 We point out that for a separable deformation we have not any influence on what happens when we specialize q to an element in $\mathfrak{m}$. Moreover, maximal deformations have the advantage that the definition is independent of the ground ring over which we consider the order Λ.

Example 6.3

1. The Hecke order completed at 3 of the symmetric group S_3 on three letters Lemma 2.8 in Subsection 2.7 is a separable deformation of the 3-adic group ring $\mathbb{Z}_3 S_3$.
2. The Hecke order of the dihedral group D_8 of order 16 (cf. 3) completed at 2 is a maximal deformation of the 2-adic group ring $\mathbb{Z}D_8$; it is not a separable deformation [Ro; 97].
3. More generally, p-adic integral blocks with cyclic defect (cf. Section 4.1) have maximal deformations [Ro; 98, I].
4. Modular blocks with cyclic defect have semi-simple deformations [Sch; 94] in a weaker sense.

7 Cellular orders

In [KoXi; 96] the authors S. Knig and C. Xi gave a definition of cellular algebras which is equivalent to the original one of J. J. Graham and G. I. Lehrer [GrLe; 96].

Definition 7.1 (Knig-Xi [KoXi; 96]) *Let Λ be an R-algebra, where R is a commutative Noetherian integral domain. Assume that there is an R-linear anti-involution ι on Λ. A two-sided ideal J of Λ is called a* CELL-IDEAL *provided*

- $\iota(J) = J$ *and*
- *there exists a left Λ-ideal $\Delta \subset J$ such that*
  - *Δ is finitely generated and free over R and*
  - *there is an isomorphism of bimodules*

$$\alpha\colon J \longrightarrow \Delta \otimes_R \iota(\Delta)\,, \tag{12}$$

making the following diagram commute

$$\begin{array}{ccc} J & \xrightarrow{\alpha} & \Delta \otimes_R \iota(\Delta) \\ \iota\big\downarrow & & \big\downarrow x\otimes y \longmapsto \iota(y)\otimes\iota(x) \\ J & \xrightarrow{\alpha} & \Delta \otimes_R \iota(\Delta) \end{array}$$

The algebra Λ is called CELLULAR *(with respect to the anti-involution ι) provided there exists a chain of two-sided Λ-ideals*

$$0 = J_0 \subset J_1 \subset J_2 \subset \cdots \subset J_n = A\,,$$

each of them fixed by ι, such that for each index $0 < i$ the quotients J_i/J_{i-1} is a cell ideal (with respect to ι) in Λ/J_{i-1}.

The importance of cellular algebras stems from the following examples.

1. Hecke algebras of type A or B or more generally Ariki-Koike algebras (cf. [GrLe; 96]).
2. Brauer algebras (cf. [GrLe; 96]),
3. Temperly-Lieb algebras (cf. [GrLe; 96]),
4. q-Schur algebras (cf. [DiJa; 89]),

5. Jones' annular algebras (cf. [GrLe; 96]).

An immediate consequence of the definition is that a CELLULAR R-ALGEBRA IS FINITELY GENERATED AND FREE OVER R AS MODULE. In [GrLe; 96] and [KoXi; 96] the main emphasis is on the case where R is a field; however, when looking at Coxeter groups and Hecke algebras, it is necessary to look at noetherian integral domains R of finite Krull dimension, in particular at $R = \mathbb{Z}[q, q^{-1}]$.
Moreover, in all of these cases the algebra $K \otimes_R \Lambda$ is separable, where K is the field of fractions of R. So we make the following GENERAL ASSUMPTIONS:

Assumption 7.2 (and Definition) *R with field of fractions K is a noetherian integral domain of finite Krull dimension. The R-algebra Λ is a* COHEN-MACAULAY R-ORDER – IN THE SEPARABLE K-ALGEBRA $A := K \otimes_R \Lambda$, *and* $_{CM}\Lambda$ *denotes the* CATEGORY OF LEFT COHEN-MACAULAY-MODULES *for* Λ.

- *A Cohen-Macaulay-module M over Λ is said to be* IRREDUCIBLE, *provided $V := K \otimes_R M$ is a simple A-module, corresponding to the simple algebra $A_V := (D_V)_{n_V}$.*

- *For an irreducible Cohen-Macaulay-module M over Λ, its* LEFT ORDER *$\Lambda_l(M)$ is defined as*

$$\Lambda_l(M) = \{x \in A_V \, : \, x \cdot M \subseteq M\} .$$

  In general, there is no reason for $\Lambda_l(M)$ to be Cohen-Macaulay.

- *If M is a left Cohen-Macaulay-module for Λ, then the* DUAL MODULE *$M^* := Hom(M, R)$ is a right Cohen-Macaulay-module for Λ.*

We shall keep these notations and assumptions throughout.

Note 7.3
- The Definition 7.1 depends on whether you look at Λ as an R-algebra or as an S-algebra for some finite extension ring; e. g. $(\mathbb{C})_n$ is a cellular $\mathbb{C}$-algebra, but it is not a cellular $\mathbb{R}$-algebra. Such a definition is not suited for group rings, since there, in general, matrix rings over several different commutative rings, even orders in skew-fields, are involved.

- With Definition 7.1 the group ring of the Coxetergroup D_7 [‡‡] over $\mathbb{Z}_7$, the ring of the 7-adic integers is not a cellular algebra; neither is its Hecke algebra.

[‡‡]This is the dihedral group of order $2 \cdot 7$.

- We thus have to change the definition a bit; the point is, that when taking in the definition $J \simeq \Delta \otimes_R \iota(\Delta)$, the tensor product over R is too restrictive.

We shall thus modify the definition to also include the integral group rings of the Coxeter groups, where the rationals are not necessarily a splitting field. We also want to make sure that we have a local-global principle. The local-global principle can be guaranteed if we require that the left Λ-ideal Δ in Definition 7.1 is a Cohen-Macaulay-module over Λ. We shall concentrate here on R-orders in separable K-algebras. Note that group rings and generic Hecke algebras do satisfy these requirements. In all the examples, the left module Δ in Definition 7.1 is irreducible. If we keep in mind that every simple K-algebra $\mathcal{S} := Mat(n, D)$, where D is a separable skew-field over K – not necessarily central over K – has a representation as

$$\mathcal{S} \simeq L \otimes_D L^* \text{ where } L \text{ is simple}$$

the following SEEMS TO BE a natural definition of INTEGRAL CELLULAR ALGEBRAS:

Definition 7.4 *Let Λ be a Cohen-Macaulay R-order in the separable K-algebra $A := K \otimes_R \Lambda$. Assume that ι is an anti-involution. A two-sided ideal J of Λ is called a* CELL-IDEAL *provided*

- *$\iota(J) = J$ and*
- *there exists a left Λ-ideal $\Delta \subset J$ such that*
  - *Δ is a Cohen-Macaulay-module,*
  - *Δ is irreducible,*
  - *$E(\Delta) := End_\Lambda(\Delta) \subset \Lambda$ under the natural inclusion $End(K \otimes_R \Delta) \subset A$,*
  - *$\iota(E(\Delta)) = E(\Delta)$; these conditions guarantee that $\iota(\Delta)$ is an $(E(\Delta), \Lambda)$ bi-module. We furthermore require that*
  - *there is an isomorphism*

$$\alpha \colon J \longrightarrow \Delta \otimes_{E(\Delta)} \iota(\Delta)\,, \tag{13}$$

making the following diagram commute

$$\begin{array}{ccc} J & \xrightarrow{\alpha} & \Delta \otimes_{E(\Delta)} \iota(\Delta) \\ \iota \big\downarrow & & \big\downarrow x \otimes y \longmapsto \iota(y) \otimes \iota(x) \\ J & \xrightarrow{\alpha} & \Delta \otimes_{E(\Delta)} \iota(\Delta) \end{array}$$

The algebra Λ is called CELLULAR *(with respect to the anti-involution ι), provided there exists a chain of two-sided Λ-ideals*

$$0 = J_0 \subset J_1 \subset J_2 \subset \cdots \subset J_n = A\,,$$

each of them fixed by ι, such that for each index $0 < i$ the quotient J_i/J_{i-1} is a cell ideal (with respect to ι) in Λ/J_{i-1}.

We note that in case $E(\Delta) = R$ this coincides with Definition 7.1.

Remark 7.5 In the few examples I know, the ideal left Δ has an additional property, which I shall describe now: Δ is an irreducible Cohen-Macaulay module, and so its left order $\Lambda_l(M)$ is an order in a simple algebra. Let us put

$$\Lambda(M) := \Lambda_l(M) \cap \iota(\Lambda_l(M)\,.$$

Then J is a two-sided $\Lambda(M)$-ideal, and $\Lambda(M)$ is the largest such order. IN THE EXAMPLES THE FOLLOWING PHENOMENA DID OCCUR:

1. $\Lambda(M)$ is a Cohen-Macaulay R-order.
2. The anti-involution ι induces an anti-involution on $\Lambda(M)$ and fixes $E(\Delta)$.
3. Δ is a projective $\Lambda(M)$-module.
4. $\Lambda(M)$ is Morita equivalent to an order of the form

$$\mathcal{H} = \mathcal{H}(\nu_1, \ldots, \nu_{n-1}) :=$$

$$= \begin{pmatrix} \Omega & \Omega & \Omega & \cdots & \Omega & \Omega \\ \omega^{\nu_1}\cdot\Omega & \Omega & \Omega & \cdots & \Omega & \Omega \\ \omega^{\nu_2}\cdot\Omega & \omega^{\nu_1}\cdot\Omega & \Omega & \cdots & \Omega & \Omega \\ & & & \ddots & & \\ \cdots & \cdots & \cdots & & \cdots & \cdots \\ \omega^{\nu_{n-2}}\cdot\Omega & \omega^{\nu_{n-3}}\cdot\Omega & \omega^{\nu_{n-4}}\cdot\Omega & \cdots & \Omega & \Omega \\ \omega^{\nu_{n-1}}\cdot\Omega & \omega^{\nu_{n-2}}\cdot\Omega & \omega^{\nu_{n-3}}\cdot\Omega & \cdots & \omega^{\nu_1}\cdot\Omega & \Omega \end{pmatrix}_n \,. \qquad (14)$$

   Where $\nu_1 \leq \nu_2 \leq \cdots \leq \nu_{n-1}$ and Ω is a maximal Cohen-Macaulay-order in the skew-field $D - A_V = Mat(n, D)$ (cf. Assumption 7.2). Moreover, $\omega \cdot \Omega = \Omega \cdot \omega =: \underline{\omega}$ is a principal prime ideal.

Definition 7.6 (projectively cellular orders) *A cellular Cohen-Macaulay-order Λ is said to be a* PROJECTIVELY CELLULAR ORDER*, provided the cell ideals satisfy the conditions 1.), 2.) and 3.) from Remark 7.5. It is called a* TRIANGULAR PROJECTIVELY CELLULAR ORDER *if in addition it satisfies Condition 4.) from Remark 7.5.*

Proposition 7.7 *It was shown in [Ro; 98, II] that the integral group ring of the dihedral group D_m of order $2 \cdot m$ is a projectively cellular order.*

Remark 7.8 One could also replace in the definition of a cellular order the condition that Δ be irreducible by the condition that J is a full ideal in a simple algebra. Then the notion of a cellular order and a quasi-hereditary order are not too far apart. We can talk about quasi-hereditary cellular orders by requiring that a heredity ideal $J = \Lambda \cdot \epsilon \cdot \Lambda$ is at the same time cellular with $\Delta = \Lambda \cdot \epsilon$. For example, if D_{p^n} is the dihedral group of order $2 \cdot p^n$ with π odd, then the dual of the augmentation ideal of $\mathbb{Z}D_{p^n}$ is a quasi-hereditary cellular algebra (cf. [Ro; 98, II]).

References

[CPS; 88] Cline, E. – B. Parshal – L. Scott, Finite dimensional algebras and highest weight categories, J. Reine Angew. Math. 391 (1988), 85-99

[CPS; 90] Cline, E. – B. Parshall – L.Scott, Integral and graded quasi–hereditary algebras, I. J. Alg. 131 (1990), 126–160

[CR; 87] Curtis, C. W. – I. Reiner, Methods of Representation Theory, Vol. II, John Wiley & Sons, Interscience Publication, 1987

[Ca; 85] Carter, R. W. Finite Groups of Lie Type: Conjugacy Classes and Complex Characters, John Wiley & Sons, Interscience Publications, 1985

[Cu; 88] Curtis, C. W. Representations of Hecke algebras, Astérisque (1988), 13–60

[DiJa; 89] Dipper, R. – G.James, The q–Schur algebra. Proc. London Math.Soc. 59 (1989), 23–50

[DiJaMa; 98] Dipper, R. – G.James – A.Mathas, Cyclotomic q–Schur algebras. Math.Z. 229 (1998), 385–416

[GrLe; 96] Graham, J. J. – G. I. Lehrer, Cellular Algebras, Invent. Math. 123 (1996) 1–34

[Gr; 93] Green, J. A. Combinatorics and the Schur algebra. J.Pure Appl.Alg. 88 (1993), 89–106

[Ka; 97] Kauer, M. – K. W. Roggenkamp, Higher dimensional orders, graph-orders, and derived equivalences, J.Pure Appl.Alg. 155 (2001), 181–202

[Ko; 97] Knig, S. A criterion for quasi–hereditary, and an abstract straightening formula. Invent. Math. 127 (1997), 481–488

[KoXi; 96] Knig, S. – C.-C. Xi, On the Structure of Cellular Algebras, Algebras and modules, II (Geiranger, 1996), Amer.Math.Soc., Providence, RI (1998), 365–386

[PaSc; 88] Parshal, B. – L. L. Scott, Derived categories, quasi-hereditary algebras and algebraic groups, Proc. of the Ottawa-Moosonee Workshop in Algebras 1987, Math. Lect. Note Series, Carlton University and Univeristé d' Ottawa (1988)

[Ro; 84] Roggenkamp, K. W. Automorphisms and isomorphisms of integral group rings of finite groups, Proceedings of "Groups Korea" in Springer Lecture Notes in Math. 1098 (1984), 118-135

[Ro; 92] Roggenkamp, K. W. Blocks with cyclic defect and Green orders, Communications in Algebra, 20 (1992) , 1715–1734

[Ro; 97] Roggenkamp, K. W. Hecke orders of Dihedral groups, MS Stuttgart, 1997

[Ro; 98, I] Roggenkamp, K. W. Deformations of blocks of cyclic defect and Hecke algebras, MS Stuttgart, 1998

[Ro; 98, II] Roggenkamp, K. W. The cellular structure of integral group rings of dihedral groups, An. Şt. Univ. Ovidius Constanţa, vol. 6(2)(1998), 119–138

[Sch; 94] Schaps, M. A modular version of Maschkes theorem for groups with cyclic p-Sylow subgroups, J. Algebra 163 (1994), 623-635

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