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# USE OF THE INTERIOR-POINT METHOD FOR CORRECTING AND SOLVING INCONSISTENT LINEAR INEQUALITY SYSTEMS

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*Dedicated to Professor Mirela Ştefănescu on the occasion of her 60th birthday*

**Abstract**

By the correction of an inconsistent linear system we mean avoiding its contradictory nature by means of relaxing the constraints. In $[Va2]$ it was shown that for inconsistent linear equation systems $Ax = b$, the correction of the whole augmented matrix $(A, b)$ using Euclidean norm criterion, is a problem equivalent to finding the least eigenvalues (and corresponding eigenvectors) of the matrix $(-b, A)^T(-b, A)$. In $[Va1]$ Vatolin proposed an algorithm based on linear programming, which finds minimal corrections of the constraint matrix and RHS vector. In $[PM]$ is analyzed correction problem for an inconsistent linear inequality system $Ax \leq b$, using two criteria $\|.\|_1$ and $\|.\|_\infty$. In this paper, we use interior-point techniques for solving the associated linear program.

## 1. Introduction

Consider the linear inequality system:

$$\begin{cases} \langle a_i, x\rangle \leq b_i, \ i \in M_0 \cup M_1 \\ x_j \geq 0, \ j = 1, ..., n, \end{cases} \tag{1}$$

where $a_i^T$, $i = 1, ..., m$, forms the $i^{th}$ row of the matrix $A$, $b_i$ is the $i^{th}$ component of $b$, $M_0$, $M_1$ are finite index sets and $\langle ., .\rangle$ stands for the standard inner

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product in $\mathbf{R}^n$.

With system (1) we associate the *corrected system*:

$$\begin{cases} \langle a_i, x\rangle \leq b_i,\ i \in M_0 \\ \langle a_i + h_i', x\rangle \leq b_i - h_{i,n+1},\ i \in M_1 \\ x_j \geq 0,\ j = 1, ..., n, \end{cases} \tag{2}$$

where $h_i' \in \mathbf{R}^n$ and $h_{i,n+1} \in \mathbf{R}$. Let $h_i \in \mathbf{R}^{n+1}$,

$$h_i = (h_i', h_{i,n+1}) = (h_{i1,...,}h_{i,n+1})$$

be the vector correcting the $i^{th}$ row of system (1), $i \in M_1$.
The rows with indices $i \in M_0$ are not corrected (are assumed to be fixed). We can fix also arbitrary columns of the augmented matrix $(A, b)$, with indices $j \in J_0 \subset \{1, ..., n+1\}$. Thus we set $h_{ij} = 0,\ i \in M_1,\ j \in J_0$.
Let $M_1 = \{i_1, ..., i_p\}$ and $J_1 = \{1, ..., n+1\} \diagdown J_0$ the complement of $J_{0,}$ i.e. the set of indices of columns to be corrected.

*The correction problem of system (1)* may be expressed as

$$min\,\{\Phi(H)/H \in S\} \tag{3}$$

where $H(h_{ij})_{p\times(n+1)}$,

$$S = \{H/h_{ij} = 0,\ i \in M_1,\ j \in J_0 \textit{and system (2) is consistent}\}.$$

$\Phi(H)$ is the *correction criterion* estimating the quality of correction.

## 2. The LP-based algorithm for solving correction problem (3)

The main difficulties in solving the problem (3) are that the left-hand sides of system (2) are bilinear in $h_i$ and $x$. The idea of Vatolin algorithm is to take $h_i$ of the form:

$$h_i = t_i c,\quad i \in M_1,\quad t_i \in \mathbf{R},$$

where $c \in \mathbf{R}^{n+1}$, $c = (c_1,...,c_{n+1})$ is defined bellow.

Thus, the problem (2) is also bilinear, but it can be converted into a linear one by:

a) changing variable $x \in \mathbf{R}^n$ for variable $h_0 \in \mathbf{R}^{n+1}$ so that

$$x = h_{0,n+1}^{-1}(h_{01}, ..., h_{0,n})^T,$$

where it is assumed that $0 \notin M_1$, $h_0 = (h_{01}, ..., h_{0,n}, h_{0,n+1}), h_{0,n+1} > 0$ and by

b) introducing additional constraint

$$\langle c, h_0 \rangle = -1.$$

Consequently, the algorithm reduces solving the correction problem (3) to solving a linear programming problem. If $\Phi(H)$ takes form:

$$\Phi(H) = \max_{i,j} |h_{ij}| , \tag{4}$$

then the vector $c \in \mathbf{R}^{n+1}$ is of the form

$$c_j = \begin{cases} 0, & j \in J_0 \\ -1, & j \in J_1. \end{cases}$$

We have to solve a linear program:

$$\begin{cases} \quad \mathit{min}\ \theta \\ \text{subject to} \\ \quad \langle d_i, h_0 \rangle \le 0,\ i \in M_0 \\ \quad \langle d_i, h_0 \rangle \le t_i,\ i \in M_1 \\ \quad 0 \le t_i \le \theta,\ i \in M_1 \\ \quad \sum\limits_{j \in J_1} h_{0,j} = 1 \\ \quad h_{0,j} \ge 0,\ j = 1, ..., n+1, \end{cases} \tag{5}$$

where $d_i = (a_i, -b_i) \in \mathbf{R}^{n+1}$, $i \in M_0 \cup M_1$. Using the criterion (4), the rows $i \in M_1$ and all columns $j \in J_1$ are effectively corrected.

If $\Phi(H)$ takes form:

$$\Phi(H) = \sum_{i,j} |h_{ij}| ,$$

then the number of linear programming problems which will be solved is $|J_1|$. At each linear programming problem, only a column of augmented matrix $(A, b)$ is corrected (see [Po], [PM]).

Let $K$ be the set of feasible solutions $(\theta, t, h_0)$ of problem (5), where vector $t$ is composed of components $t_i$, $i \in M_1$. If $K = \phi$ then S=$\phi$. Else, for each optimal solution $(\theta, t, h_0)$ of the problem (5), there are obtained the optimal value $\sigma = \theta$, the optimal correction matrix $H = H(t)$ with $(i, j)$component

$$h_{ij} = \begin{cases} 0, & j \in J_0 \\ -t_i, & j \in J_1 \end{cases} ,\ i \in M_1$$

and the solution $x$ of the corrected system

$$x = h_{0,n+1}^{-1} (h_{01}, ..., h_{0,n})^T.$$

## 3. Interior-point method for solving linear programming problem (5)

The linear program (5) admits an equivalent program in the standard form, obtainable by adding slack variables: $s_i,\ i \in M_0,\ v_i,\ z_i,\ i \in M_1$ :

$$\begin{cases} \qquad\qquad min\ \theta \\ \text{subject to} \\ \qquad\qquad \langle d_i, h_0 \rangle + s_i = 0,\ i \in M_0 \\ \qquad\qquad \langle d_i, h_0 \rangle - t_i + v_i = 0,\ i \in M_1 \\ \qquad\qquad -\theta + t_i + z_i = 0,\ i \in M_1 \\ \qquad\qquad \langle c, h_0 \rangle = 1 \\ \qquad\qquad \theta, h_0, t, s, v, z \geq 0, \end{cases} \tag{6}$$

where the vectors $s, v$ and $z$ are composed of components $s_i,\ i \in M_0$ and $v_{i,}$ $z_i,\ i \in M_1$. Note that the strict inequality $h_{0,n+1} > 0$ in (6) was replaced by $h_{0,n+1} \geq 0$.

We introduce the notations: $f = (1, 0, ..., 0)^T$, $g = (0, ..., 1)^T$. Also, $y = (\theta, h_0, t, s, v, z)^T$ denotes vector composed of $\theta \in \mathbf{R}$ and vectors $h_0, t, s, v$ and $z$. The coefficient matrix in the linear problem (6) is:

$$G = \begin{pmatrix} 0 & d_i & 0 & I_{m-p} & 0 & 0 \\ 0 & d_i & -I_p & 0 & I_p & 0 \\ -e & 0 & I_p & 0 & 0 & I_p \\ 0 & c^T & 0 & 0 & 0 & 0 \end{pmatrix},$$

where $e$ stands for the all-one vector, $e = (1, 1, ..., 1)^{T\cdot}$ Matrix $G$ has a full row rank. Using these notations, the problem (6) becomes:

$$(P) \qquad \begin{cases} \qquad \min f^T y \\ \quad subject\ to \\ \qquad Gy = g \\ \qquad y \geq 0. \end{cases}$$

We define the feasible set $\mathcal{P}$ to be the set of vectors $y$ satisfying the constraints, i.e.

$$\mathcal{P} = \{y/\ Gy = g \text{ and } y \geq 0\},$$

and the associated set $\mathcal{P}^+$ to be the subset of $\mathcal{P}$ satisfying strict nonnegativity constraints

$$\mathcal{P}^+ = \{y/\ Gy = g \text{ and } y > 0\}.$$

*Interior-point methods* are iterative methods that compute a sequence of iterates belonging to $\mathcal{P}^+$ and converging to an optimal solution.
This is completely different from the simplex method which explores the vertices of the polyhedron $\mathcal{P}$ and an exact optimal solution is obtained after a finite number of steps (see fig. 0.1).

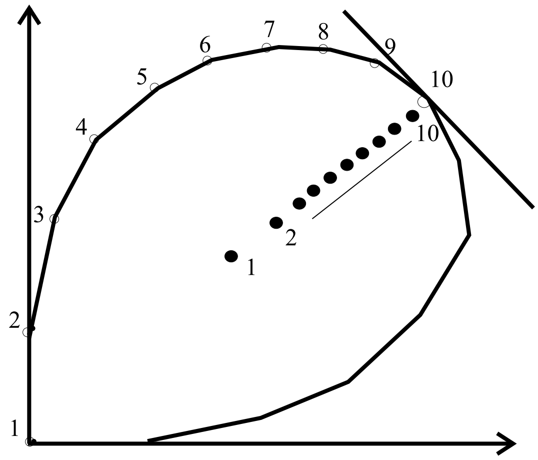


Figure 1:

Interior-point iterates tend to an optimal solution but never attain it (since the optimal solutions do not belong to $\mathcal{P}^+$ but to $\mathcal{P}\backslash\mathcal{P}^+$). Yet an approximate solution (with e.g. $10^{-8}$ relative accuracy) is sufficient for our purpose. In addition, these methods are practically efficient and can be used to solve large-scale problems. For such of problems, the chances that the system is self-contradictory (inconsistent) are high.

Since 1984, when Karmarkar introduced this new class of methods in [Ka], many different interior-point methods have been developed. For solving linear program (P), we will use the *affine-scaling method* which has been previously proposed by Dikin, 17 years before Karmarkar. This is in fact a projective gradient method. Also, at each iteration, the $y$ variable is simply scaled by $y = Dw$, where $D$ is a positive diagonal matrix (this scaling operation is responsible for the denomination of the method).

Let us consider the current iterate $y_{k>0}$ and $D = Y_k$, where $Y_k$ is diagonal matrix made up with vector $y_k$. Choosing this special matrix, which maps the current iterate $y_k$ to $e$ ($Y_k^{-1} y_k = e$), we obtain the following problem:

$$(P_D) \qquad \begin{cases} \qquad \min(Y_k f)^T w \\ subject\ to \\ \qquad GY_k w = g \\ \qquad w \geq 0. \end{cases}$$

It is easy to show that problem $(P_D)$ is equivalent to $(P)$. We introduce the notations $G_k = GY_k$ and $f_k = Y_k f$.

The scaled current iterate $w_k$ is located far inside of the polyhedron (see fig. 0.2), at equal distance to each face. The iterate is centered because by this, a significant shift toward the optimal solution $y^*$ can be executed (without touching the faces of the polyhedron).

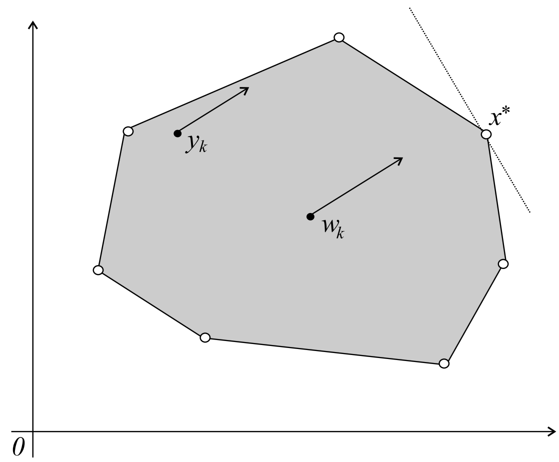


Figure 2:

The displacing direction for the scaled problem $\triangle w_k$ is defined as the projection of the scaled problem gradient, with sign changed $-f_k$, onto $ker(G_k)$. The projection matrix onto $ker(G_k)$ is:

$$P_{G_k} = I - G_k^T \left(G_k G_k^T\right)^{-1} G_k.$$

Then,

$$\triangle w_k = -P_{G_k} f_k = -Y_k[f - G^T(GY_k^2 G^T)^{-1} GY_k^2 f].$$

Using the notation

$$u_k = (GY_k^2G^T)^{-1}GY_k^2f, \tag{7}$$

the displacing direction $\triangle w_k$ becomes:

$$\triangle w_k = -Y_k\left(f - G^Tu_k\right).$$

Back in the original space we obtain

$$\triangle y_k = Y_k\triangle w_k = -Y_k^2(f - G^Tu_k),$$

where the vector $u_k$ is the solution of the linear system:

$$(GY_k^2G^T)u_k = GY_k^2f. \tag{8}$$

The next iterate $y_{k+1} = y_k + \triangle y_k$ is expected to be closer to the optimal solution than $y_k$.

Since the iterates must always satisfy the strict nonnegativity conditions, we will reduce the step with a factor $\alpha_k < 1$ in order to make it stay within the strictly feasible region $\mathcal{P}^+$ :

$$y_{k+1} = y_k + \alpha_k\triangle y_k.$$

The sequence of iterates is converging to the optimal solution (see [An]). While the simplex method may potentially make a number of moves that grows exponentially with the problem size, interior-point methods need a number of iterations that is polynomially bounded by the problem size to obtain a given accuracy.

The stopping criterion is usually a small predefined duality gap $\varepsilon$ :

$$\frac{\left\|f^Ty_k - g^Tu_k\right\|}{\|f^Ty_k\| + 1} < \varepsilon,$$

where $u_k$, defined in (7), are the dual variables.

The resolution of the system (8) takes up most of the computing time in affine-scaling algorithm (80-90% of the total CPU-time). It should be therefore very carefully implemented (with a Cholesky factorization, taking advantage of the fact that matrix $GY_k^2G^T$ is positive definite and with application of sparse matrix techniques).

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