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SOME ELEMENTARY APPLICATIONS OF TOPOLOGICAL DEGREE

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Dedicated to Professor Mirela Ştefănescu on the occasion of her 60th birthday

Abstract

In this paper we consider the degree theory for a mapping f from an oriented compact connected manifold X to a oriented, compact, connected manifold Y of the same dimension and some elementary problems of fixed points.

1. Topological degree on manifolds.

There are many approaches to the introduction of the notion of degree and the background involved in each of them may differ considerably.We briefly introduce the notion of topological degree on manifolds and for more details we refer the reader to Nirenberg,L., [N] and Berger,M. & Gostiaux,B., [BG].

Let X be an $n-$dimensional $C^{q'}$ manifolds and q, r integers such that $0 \leq q \leq q' - 1$ and $0 \leq r \leq n$. Let ω be a C^q differential form of degree r, or $r - form$, on X. The space of $r - forms$ of class C^q on X will be denoted by $\Omega_q^r(X)$. In this section everything is of class C^∞, and $\Omega^r = \Omega_\infty^r(X)$.

Definition. Let X and Y be two oriented compact connected manifolds of same dimension d, and $f \in C^\infty(X; Y)$ a map. There exists an integer, called the degree of f and denoted by $\deg(f)$, such that:

(i) if $\omega \in \Omega^d(Y)$ we have

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$$\int_X f^*\omega = \deg(f)\int_Y \omega.$$

(ii) if y is a regular value for f we have

$$\deg(f) = \sum_{x\in f^{-1}(y)} sgn(J_x(f)).$$

In particular, if $f^{-1}(y) = \Phi$ the degree is zero.

We have the following properties of the degree:

P1 (Normalization): If Id_X is the identity map on X, $Id_X(x) = x$ $\forall\ x \in X$, then $\deg(Id_X) = 1$.

P2 (Existence of solution): If $\deg(f) \neq 0$, then $f : X \to Y$ is surjective.

Indeed, if f is not surjective from Sard's Lemma for manifolds there is $y \in Y$ a regular value such that $f^{-1}(y) = \Phi$ and from definition $\deg(f) = 0$, contradiction.

P3 (Homotopy invariance):

Definition. Let X and Y be manifolds. Two maps $f, g \in C^\infty(X;Y)$ are said to be *homotopic* if there exists a homotopy between f and g, that is, a map $F : [0,1] \times X \to Y$ such that:

(i) For every $t \in [0,1]$ the map $F_t : x \mapsto F(t,x)$ is in $C^\infty(X;Y)$;

(ii) The map $TF : [0,1] \times TX \to TY$ defined by $(TF)(t,x) = T_xF_t$ is continuous;

(iii) $F_0 = f$ and $F_1 = g$.

Proposition. *Let X and Y be oriented, compact, connected manifolds of same dimension. If $f,g \in C^\infty(X;Y)$ are homotopic we have* $\deg(f) = \deg(g)$.

Indeed, take $\omega \in \Omega^d(Y)$ such that $\int_Y \omega \neq 0$ and we have:

$F_t \in C^\infty(X;Y) \Rightarrow \deg(F_t)\int_Y \omega = \int_X F_t^*\omega \Rightarrow \deg(F_t) = \frac{\int_X F_t^*\omega}{\int_Y}\omega$.

The map $t \in [0,1] \mapsto \int_X F_t^*\omega \in \mathbb{R}$ is continuous, implying that the map $t \in [0,1] \mapsto \deg(F_t) \in \mathbb{Z}$ is continuous therefore $\deg(F_t)$ is constant $\forall\ t \in [0,1]$ $\Rightarrow \deg(F_0) = \deg(F_1) \Rightarrow \deg(f) = \deg(g)$.

P4 (Multiplication property): Let X, Y and Z be oriented, compact, connected manifolds of same dimension d, and $f \in C^\infty(X;Y)$, $g \in C^\infty(Y;Z)$ maps.We have $\deg(g \circ f) = \deg(g) \cdot \deg(f)$.

Indeed, let $\omega \in \Omega^d(Z)$ and we have:
$\int_X (g \circ f)^* \omega = \deg(g \circ f) \int_Z \omega$
$\int_X (g \circ f)^* \omega = \int_X f^* \circ g^* \omega = \int_X f^*(g^* \omega) = \deg(f) \int_Y g^* \omega = \deg(f) \cdot \deg(g) \int_Z \omega$

Therefore, $\deg(g \circ f) = \deg(g) \cdot \deg(f)$.

A basic fact is that this degree is independent of the choice of differential form ω or the choice of the regular value y. If we switch the orientation of both X and Y the degree does not change.

Degree extends to continuous maps f from X to Y because of the fundamental fact that if $f, g \in C^1(X; Y)$, and are close in the C^0 topology, then they have the same degree. Degree theory is often defined directly for continuous maps via the action of the map on nth degree homology.

2. Fixed points

In this section all maps are at least of class C^0.

Proposition 1.
Let $X = Y = S^d = \{x \in \mathbb{R}^{d+1} |\ \|x\| = 1\}$. The degree of antipodal map $g : S^d \to S^d$, $g(x) = -x\ \forall\ x \in S^d$, is deg$(g) = (-1)^{d+1}$.

Proof: If ω is the canonical volume form on S^d we have $g^* \omega = (-1)^{d+1} \omega$.
$\Rightarrow \int_{S^d} g^* \omega = (-1)^{d+1} \int_{S^d} \omega \Rightarrow \deg(g) = (-1)^{d+1}$.

Proposition 2.
Let $f : S^d \to S^d$ be a continuous map such that $\deg(f) \neq (-1)^{d+1}$. *Then f has at least a fixed point.*

Proof: Suppose that $f(x) \neq x\ \forall\ x \in S^d \Rightarrow -f(x) \neq -x,\ \forall\ x \in S^d$.
Let $g : S^d \to S^d$, $g(x) = -x,\ \forall\ x \in S^d$, the antipodal map.
Thus
$g(x) = -x \neq -f(x) \forall x \in S^d$ and $f(x), g(x) \in S^d$.
We obtain
$\|(1-t)f(x) + tg(x)\| \neq 0,\ \ \forall\ x \in S^d, \forall t \in [0;1]$.
Then it results that $F : [0;1] \times S^d \to S^d$ is a homotopy between f and g, where
$F[t,x] = \frac{(1-t)f(x)+tg(x)}{\|(1-t)f(x)+tg(x)\|}$ and $F[0,x] = f(x), F[1,x] = g(x)$.

It follows from homotopy invariance that $\deg(f) = \deg(g) = (-1)^{d+1}$, contradiction.

Proposition 3.
If d is even and if $f : S^d \to S^d$ is homotopic with Id_{S^d}, then f has a fixed point.

Proof: Assuming that f has no fixed points we shall derive a contradiction. Indeed by the proof of Proposition 2, we have that $\deg(f) = (-1)^{d+1} = -1$, since d is even. On the other hand,by multiplication property, $\deg(Id_{S^d}) = \deg(Id_{S^d} \circ Id_{S^d}) = \deg(Id_{S^d}) \cdot \deg(Id_{S^d})$ and therefore $\deg(Id_{S^d}) = +1$. But by homotopy invariance, $\deg(f) = \deg(Id_{S^d})$, contradiction.

Proposition 4.
Any continuous mapping $f : S^{2k} \to S^{2k}, k \in \mathbb{N}$ either has a fixed point or sends some point into its antipode.If $\deg(f) \neq -1$ then f always has a fixed point. If $\deg(f) \neq 1$ then there is a point on the sphere mapped into its antipode.

Proof: Let $g : S^{2k} \to S^{2k}$, $g(x) = -x$, $\forall\, x \in S^{2k}$ be the antipodal map.Let us first assume that $\deg(f) \neq -1$, but the mapping has no fixed points.We have $f(x) \neq -g(x)$, $\forall x \in S^{2k}$, and by the proof of Proposition 2, $\deg(f) = \deg(g) = (-1)^{2k+1} = -1$, contradiction.

Let $\deg(f) \neq 1$ and $f(x) \neq -x$ $\forall x \in S^{2k}$. We have $f(x) \neq -Id_{S^{2k}}(x)$ $\forall x \in S^{2k}$, and by the proof of *proposition2,* $\deg(f) = \deg(Id_{S^{2k}}) = 1$, contradiction.Suppose that we have no prior information on the degree of the mapping f. If it has no fixed points, then we can conclude, as before, that $\deg(f) = -1$. Assuming that no point x is mapped by f into its antipode,we can again conclude, as before, that $\deg(f) = +1$. The proposition is proved.

Proposition 5.
(i) If $f, g : S^{2k} \to S^{2k}$, then at least one of the three mappings f, g, and $g \circ f$ has a fixed point. In particular, the composition $f \circ f$ of any mapping f with itself has a fixed point.

(ii) Any mapping $f : S^{2k} \to S^{2k}$ either has a fixed point or has a pair of points that exchange their positions.

Proof: If $\deg(f) \neq -1$ or $\deg(g) \neq -1$ by Proposition 4, it results f or g has a fixed point.

If $\deg(f) = -1$ and $\deg(g) = -1$, then $\deg(g \circ f) = \deg(g) \cdot \deg(f) = +1$, thus $\deg(g \circ f) \neq -1$ and $g \circ f$ has a fixed point.In particular, $\deg(f \circ f) = (\deg(f))^2 \neq -1$. Thus, $\exists\, x \in S^{2k}$ such that $f(f(x)) = x$. Let $y = f(x)$ and we have $f(y) = x$. If $x = y$ this is a fixed point,else f has a pair of points that exchange their positions.

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