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A generalized system of nonlinear relaxed cocoercieve variational inclusions with (A, η)-monotone mappings

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Abstract

In this paper, we introduce a generalized system of nonlinear relaxed cocoercive variational inclusions involving (A, η)-monotone mappings in the framework of Hilbert spaces. Based on the generalized resolvent operator technique associated with (A, η)-monotonicity, we consider the approximation solvability of solutions. Since (A, η)-monotonicity generalizes A-monotonicity and H-monotonicity, our results improve and extend the recent ones announced by many others.

1. Introduction

Variational inclusions problems are among the most interesting and intensively studied classes of mathematical problems and have wide applications in the fields of optimization and control, economics and transportation equilibrium and engineering sciences. Variational inclusions problems have been generalized and extended in different directions using the novel and innovative techniques. Various kinds of iterative algorithms to solve the variational inequalities and variational inclusions have been developed by many authors. There exists a vast literature [1-12] on the approximation solvability of nonlinear variational inequalities as well as nonlinear variational inclusions using projection type methods, resolvent operator type methods or averaging techniques. In most of the resolvent operator methods, the maximal monotonicity

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has played a key role, but more recently introduced notions of A-monotonicity [10] and H-monotonicity [3,4] have not only generalized the maximal monotonicity, but gave a new edge to resolvent operator methods. Recently Verma [12] generalized the recently introduced and studied notion of A-monotonicity to the case of (A,η)-monotonicity. Resolvent operator techniques have been in use for a while in literature, especially with the general framework involving set-valued maximal monotone mappings, but it got a new empowerment by the recent developments of A-monotonicity and H-monotonicity. Furthermore, these developments added a new dimension to the existing notion of the maximal monotonicity and its applications to several other fields such as convex programming and variational inclusions. Inspired and motivated by the recent research going on in this area, in this paper, we explore the approximation solvability of a generalized system of nonlinear variational inclusion problems based on (A,η)-resolvent operator technique in the framework Hilbert spaces.

2. Preliminaries

In this section we explore some basic properties derived from the notion of (A,η)-monotonicity. Let H denote a real Hilbert space with the norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$, respectively. Let $\eta : H \times H :\to H$ be a single-valued mapping. The mapping η is called τ-Lipschitz continuous if there is a constant $\tau > 0$ such that

$$\|\eta(u,v)\| \leq \tau\|u-v\|, \quad \forall u,v \in H.$$

Let $M : H \to 2^H$ be a multi-valued mapping from a Hilbert space H to 2^H, the power set of H. We recall following:

(i) The set $D(M)$ defined by

$$D(M) = \{u \in H : M(u) \neq \emptyset\},$$

is called the effective domain of M.

(ii) The set $R(M)$ defined by

$$R(M) = \bigcup_{u \in H} M(u),$$

is called the range of M.

(iii) The set $G(M)$ defined by

$$G(M) = \{(u,v) \in H \times H : u \in D(M), v \in M(u)\},$$

is the graph of M.

Definition 2.1. Let $\eta : H \times H \to H$ be a single-valued mapping and let $M : H \to 2^H$ be a multi-valued mapping on H.

(i) The map M is said to be (r, η)-strongly monotone if

$$\langle u^* - v^*, \eta(u, v)\rangle \geq r\|u - v\|, \quad \forall (u, u^*), (v, v^*) \in G(M).$$

(ii) η-pseudo-monotone if $\langle v^*, \eta(u, v)\rangle \geq 0$ implies

$$\langle u^*, \eta(u, v)\rangle \geq 0, \quad \forall (u, u^*), (v, v^*) \in G(M).$$

(iii) (m, η)-relaxed monotone if there exists a positive constant m such that

$$\langle u^* - v^*, \eta(u, v)\rangle \geq -m\|u - v\|^2, \quad \forall (u, u^*), (v, v^*) \in G(M).$$

Definition 2.2 [3,4]. Let $H : X \to X$ be a nonlinear mapping on a Hilbert space X and let $M : X \to 2^X$ be a multi-valued mapping on X. The map M is said to be H-monotone if $(H + \rho M)X = X$ for $\rho > 0$.

Definition 2.3 [10]. Let $A : H \to H$ be a nonlinear mapping on a Hilbert space H and let $M : H \to 2^H$ be a multivalued mapping on H. The map M is said to be A-monotone if
(i) M is m-relaxed monotone.
(ii) $A + \rho M$ is maximal monotone for $\rho > 0$.

Remark 2.1. A-monotonicity generalizes the notion of H-monotonicity introduced by Fang and Huang [2,3].

Definition 2.4 [8]. A mapping $M : H \to 2^H$ is said to be maximal (m, η)-relaxed monotone if
(i) M is (m, η)-relaxed monotone,
(ii) for $(u, u^*) \in H \times H$ and

$$\langle u^* - v^*, \eta(u, v)\rangle \geq -m\|u - v\|^2, \quad (v, v^*) \in \text{graph}(M),$$

we have $u^* \in M(u)$.

Definition 2.5 [8]. Let $A : H \to H$ and $\eta : H \times H \to H$ be two single-valued mappings. The map $M : H \to 2^H$ is said to be (A, η)-monotone if
(i) M is (m, η)-relaxed monotone,
(ii) $R(A + \rho M) = H$ for $\rho > 0$.
Note that alternatively, the map $M : H \to 2^H$ is said to be (A, η)-monotone if
(i) M is (m, η)-relaxed monotone,
(ii) $A + \rho M$ is η-pseudomonotone for $\rho > 0$.

Remark 2.2. (A, η)-monotonicity generalizes the notion of A-monotonicity introduced by Verma [10].

Definition 2.6. Let $A : H \to H$ be an (r, η)-strong monotone mapping and let $M : H \to H$ be an (A, η)-monotone mapping. Then the generalized resolvent operator $J_{M,\rho}^{A,\eta} : H \to H$ is defined by

$$J_{M,\rho}^{A,\eta}(u) = (A + \rho M)^{-1}(u), \quad \forall u \in H,$$

where $\rho > 0$ is a constant.

Definition 2.7. The map $N : H \times H$ is said to be relaxed (β, γ)-cocoercive with respect to A in the first argument if there exists two positive constants α, β such that

$$\langle T(x, u) - T(y, u), Ax - Ay\rangle \geq (-\beta)\|T(x, u) - T(y, u)\|^2 + \gamma\|x - y\|^2,$$

for all $(x, y, u) \in H \times H \times H$.

Proposition 2.1 [3]. Let $H : X \to X$ be a strictly monotone mapping and let $M : X \to 2^X$ be an H-monotone mapping. Then the operator $(H + \rho M)^{-1}$ is single-valued.

Proposition 2.2 [9,10]. Let $A : H \to H$ be an r-strongly monotone mapping and let $M : H \to 2^H$ be an A-monotone mapping. Then the operator $(A + \rho M)^{-1}$ is single-valued.

Proposition 2.3 [12]. Let $\eta : H \times \to H$ be a single-valued mapping, $A : H \to H$ be (r, η)-strongly monotone mapping and $M : H \to 2^H$ be an (A, η)-monotone mapping. Then the mapping $(A + \rho M)^{-1}$ is single-valued.

3. Results on algorithmic convergence analysis

Let $N : H \times H \to H$, $\eta : H \times H \to H$ $g : H \to H$ be three nonlinear mappings. Let $M : H \to 2^H$ be an (A, η)-monotone mapping. Then the nonlinear system of variational inclusion (NSVI) problem: determine elements $u, v \in H$ such that

$$0 \in Ag(u) - Ag(v) + \rho_1[N(v, u) + Mg(u)], \tag{3.1}$$

$$0 \in Ag(v) - Ag(u) + \rho_2[N(u, v) + Mg(v)]. \tag{3.2}$$

Next, we consider some special cases of NSVI problem (3.1)-(3.2).

(I) If $g = I$ in NSVI (3.1)-(3.2), then NSVI problem (3.1)-(3.2) reduces to the following NSVI problem: find $u, v \in H$ such that

$$0 \in Au - Av + \rho_1[N(v, u) + Mu], \tag{3.3}$$

$$0 \in Av - Au + \rho_2[N(u, v) + Mv]. \tag{3.4}$$

(II) If $g = I$, $\rho_1 = \rho_2$ and $u = v$ in NSVI (3.1)-(3.2), we have the following NVI problem: find an element $u \in H$ such that

$$0 \in N(u, u) + Mu, \tag{3.5}$$

In order to prove our main results, we need the following lemmas.

Lemma 3.1 [8,12]. *Let H be a real Hilbert space and let $\eta : H \times H \to H$ be a τ-Lipschitz continuous nonlinear mapping. Let $A : H \to H$ be a (r, η)-strongly monotone and let $M : H \to 2^H$ be (A, η)-monotone. Then the generalized resolvent operator $J^{A,\eta}_{M,\rho} : H \to H$ is $\tau/(r - \rho m)$, that is,*

$$\|J^{A,\eta}_{M,\rho}(x) - J^{A,\eta}_{M,\rho}(y)\| \leq \frac{\tau}{r - \rho m}\|x - y\|, \quad \forall x, y \in H.$$

Lemma 3.2. *Let H be a real Hilbert space, let $A : H \to H$ be (r, η)-strongly monotone, and let $M : H \to 2^H$ be (A, η)-monotone. Let $\eta : H \times H \to H$ be a τ-Lipschitz continuous nonlinear mapping. Then (u, v) is the solution of NSVI (3.1)-(3.2) if and only if it satisfies*

$$g(u) = J^{A,\eta}_{M,\rho_1}[Ag(v) - \rho_1 N(v, u)], \tag{3.6}$$

$$g(v) = J^{A,\eta}_{M,\rho_2}[Ag(u) - \rho_2 N(u, v)]. \tag{3.7}$$

Proof. The fact directly follows from the Definition 2.6.

Next, we consider the following algorithms.

Algorithm 3.1. For any $u_0, v_0 \in H$, compute the sequences $\{u_n\}$ and $\{v_n\}$ by the iterative process:

$$\begin{cases} u_{n+1} = u_n - g(u_n) + J^{A,\eta}_{M,\rho_1}[Ag(v_n) - \rho_1 N(v_n, u_n)], \\ g(v_n) = J^{A,\eta}_{M,\rho_2}[Ag(u_n) - \rho_2 N(u_n, v_n)]. \end{cases} \tag{3.8}$$

(I) If $g = I$ in Algorithm 3.1, then we have the following algorithm:

Algorithm 3.2. For any $u_0, v_0 \in H$, compute the sequence $\{u_n\}$ and $\{v_n\}$ by the iterative process:

$$\begin{cases} u_{n+1} = J^{A,\eta}_{M,\rho_1}[Ag(v_n) - \rho_1 N(v_n, u_n)], \\ g(v_n) = J^{A,\eta}_{M,\rho_2}[Ag(u_n) - \rho_2 N(u_n, v_n)]. \end{cases} \tag{3.9}$$

Remark 3.1. Algorithm 3.2 gives the approximate solution to the NSVI (3.3)-(3.4).

(II) If $g = I$, $\rho_1 = \rho_2$ and $u_n = v_n$ in Algorithm 3.1, then we have the following algorithm:

Algorithm 3.3. For any $u_0 \in H$, compute the sequence $\{u_n\}$ by the iterative processes:

$$u_{n+1} = J_{M,\rho}^{A,\eta}[Au_n - \rho N(u_n, u_n)]. \tag{3.10}$$

Remark 3.2. Algorithm 3.3 gives the approximate solution to the NVI (3.5).

Now, we are in the position to prove our main results.

Theorem 3.1. *Let H be a real Hilbert space, let $A : H \times H \to H$ be (r, η)-strongly monotone and s-Lipschitz continuous and let $M : H \to 2^H$ be (A, η)-monotone. Let $\eta : H \times H \to H$ be a τ-Lipschitz continuous nonlinear mapping and let $N : H \times H \to H$ be relaxed (α, β)-cocoercive (with respect to Ag) and μ-Lipschitz continuous in the first variable. Let N be ν-Lipschitz continuous in the second variable and $g : H \to H$ be relaxed (γ, δ)-cocoercive and σ-Lipschitz. Let (u^*, v^*) be the solution of NSVI problem (3.1)-(3.2), $\{u_n\}$ and $\{v_n\}$ be sequences generated by Algorithm 3.1. Suppose the following condition are satisfied:*

$$\frac{\tau^2\theta_2\theta_1}{(r - \rho_1 m)[(1 - \theta_3)(r - \rho_2 m) - \tau\rho_2\nu]} + \frac{\tau\rho_1\nu}{r - \rho_1 m} < 1 - \theta_3,$$

where $\theta_1 = \sqrt{\sigma^2 s^2 - 2\rho_1\beta + 2\rho_1\alpha\mu^2 + \rho_1^2\mu^2}$, $\theta_2 = \sqrt{\sigma^2 s^2 - 2\rho_2\beta + 2\rho_2\alpha\mu^2 + \rho_2^2\mu^2}$, and $\theta_3 = \sqrt{1 + 2\sigma^2\gamma - 2\delta + \sigma^2}$. Then the sequences $\{u_n\}$ and $\{v_n\}$ converges strongly to u^* and v^*, respectively.

Proof. Let $(u^*, v^*) \in H$ is the solution of NSVI problem (3.1)-(3.2), we have

$$\begin{cases} u^* = u^* - g(u^*) + J_{M,\rho_1}^{A,\eta}[Ag(v^*) - \rho_1 N(v^*, u^*)], \\ g(v^*) = J_{M,\rho_2}^{A,\eta}[Ag(u^*) - \rho_2 N(u^*, v^*)]. \end{cases}$$

It follows that

$$
\begin{aligned}
&\|u_{n+1} - u^*\| \\
&= \|u_n - g(u_n) + J_{M,\rho_1}^{A,\eta}[Ag(v_n) - \rho_1 N(v_n, u_n)] - u^*\| \\
&= \|u_n - g(u_n) + J_{M,\rho_1}^{A,\eta}[Ag(v_n) - \rho_1 N(v_n, u_n)] - u^* + g(u^*) \\
&\quad - J_{M,\rho_1}^{A,\eta}[Ag(v^*) - \rho_1 N(v^*, u^*)]\| \\
&\leq \|u_n - u^* - [g(u_n) - g(u^*)]\| \\
&\quad + \|J_{M,\rho_1}^{A,\eta}[Ag(v_n) - \rho_1 N(v_n, u_n)] - J_{M,\rho_1}^{A,\eta}[Ag(v^*) - \rho_1 N(v^*, u^*)]\| \\
&\leq \|u_n - u^* - [g(u_n) - g(u^*)]\| \\
&\quad + \frac{\tau}{r - \rho_1 m}\|Ag(v_n) - Ag(v^*) - \rho_1[N(v_n, u_n) - N(v^*, u_n)] \\
&\quad - \rho_1[N(v^*, u_n) - N(v^*, u^*)]\|.
\end{aligned}
\tag{3.11}
$$

It follows from relaxed (α, β)-cocoercive monotonicity and μ-Lipschitz continuity of N in the first variable, A is s-Lipschitz continuous and g is σ-Lipschitz continuous that

$$
\begin{aligned}
&\|Ag(v_n) - Ag(v^*) - \rho(N(v_n, u_n) - N(v^*, u_n))\|^2 \\
&= \|Ag(v_n) - Ag(v^*)\|^2 - 2\rho_1\langle N(v_n, u_n) - N(v^*, u_n), Ag(v_n) - Ag(v^*)\rangle \\
&\quad + \rho_1^2\|N(v_n, u_n) - N(v^*, u_n)\|^2 \\
&\leq \theta_1^2\|v_n - v^*\|^2,
\end{aligned}
\tag{3.12}
$$

where $\theta_1 = \sqrt{\sigma^2 s^2 - 2\rho_1\beta + 2\rho_1\alpha\mu^2 + \rho_1^2\mu^2}$. Observe that the ν-Lipschitz continuity of N in the second argument yields that

$$
\|N(v^*, u^*) - N(v^*, u_n)\| \leq \nu\|u_n - u^*\|. \tag{3.13}
$$

On the other hand, we have

$$
\begin{aligned}
&\|g(v_n) - g(v^*)\| \\
&= \|J_{M,\rho_2}^{A,\eta}[Ag(u_n) - \rho_2 N(u_n, v_n)] - J_{M,\rho_2}^{A,\eta}[Ag(u^*) - \rho_2 N(u^*, v^*)]\| \\
&\leq \frac{\tau}{r - \rho_2 m}\|Ag(u_n) - Ag(u^*) - \rho_2[N(u_n, v_n) - N(u^*, v^*)]\| \\
&\leq \frac{\tau}{r - \rho_2 m}\|Ag(u_n) - Ag(u^*) - \rho_2[N(u_n, v_n) - N(u^*, v_n)] \\
&\quad - \rho_2[N(u^*, v_n) - N(u^*, v^*)]\|.
\end{aligned}
\tag{3.14}
$$

It follows from relaxed (α, β)-cocoercive monotonicity and μ-Lipschitz continuity of N in the first variable, A_2 is s-Lipschitz continuous and g is σ-Lipschitz continuous that

$$\begin{aligned}
&\|Ag(u_n) - Ag(u^*) - \rho(N(u_n, v_n) - N(u^*, v_n))\|^2 \\
&= \|Ag(u_n) - Ag(u^*)\|^2 - 2\rho_2 \langle N(u_n, v_n) - N(u^*, v_n), Ag(u_n) - Ag(u^*)\rangle \\
&\quad + \rho_2^2 \|N(u_n, v_n) - N(u^*, v_n)\|^2 \\
&\leq \theta_2^2 \|u_n - u^*\|^2,
\end{aligned} \tag{3.15}$$

where $\theta_2 = \sqrt{\sigma^2 s^2 - 2\rho_2\beta + 2\rho_2\alpha\mu^2 + \rho_2^2\mu^2}$. Observe that the ν-Lipschitz continuity of N in the second argument yields that

$$\|N(u^*, v^*) - N(u^*, v_n)\| \leq \nu \|v_n - v^*\|. \tag{3.16}$$

Substituting (3.15) and (3.16) into (3.14), we have

$$\|g(v_n) - g(v^*)\| \leq \frac{\tau\theta_2}{r - \rho_2 m}\|u_n - u^*\| + \frac{\tau\rho_2\nu}{r - \rho_2 m}\|v_n - v^*\|. \tag{3.17}$$

Observe that

$$\|v_n - v^*\| \leq \|v_n - v^* - [g(v_n) - g(v^*)]\| + \|g(v_n) - g(v^*)\|. \tag{3.18}$$

Since the relaxed (γ, δ)-cocoercive monotonicity and σ-Lipschitz continuity of g that

$$\begin{aligned}
&\|v_n - v^* - g(v_n) - g(v^*)\|^2 \\
&= \|v_n - v^*\|^2 - 2\langle g(v_n) - g(v^*), v_n - v^*\rangle + \|g(v_n) - g(v^*)\|^2 \\
&\leq \|v_n - v^*\|^2 - 2[-\gamma\|g_2(v_n) - g_2(v^*)\|^2 + \delta\|v_n - v^*\|^2] + \|g_2(v_n) - g_2(v^*)\|^2 \\
&\leq \|v_n - v^*\|^2 + 2\sigma^2\gamma\|v_n - v^*\|^2 - 2\delta\|v_n - v^*\|^2 + \sigma^2\|v_n - v^*\|^2 \\
&= \theta_3^2\|v_n - v^*\|^2,
\end{aligned} \tag{3.19}$$

where $\theta_3 = \sqrt{1 + 2\sigma^2\gamma - 2\delta + \sigma^2}$. Substitute (3.17) and (3.19) into (3.18) yields that

$$\|v_n - v^*\| \leq \theta_3|v_n - v^*\| + \frac{\tau\theta_2}{r - \rho_2 m}\|u_n - u^*\| + \frac{\tau\rho_2\nu}{r - \rho_2 m}\|v_n - v^*\|,$$

which implies that

$$\|v_n - v^*\| \leq \frac{\tau\theta_2}{(1 - \theta_3)(r - \rho_2 m) - \tau\rho_2\nu}\|u_n - u^*\|. \tag{3.20}$$

Substitute (3.20) into (3.12) yields that

$$\begin{aligned}
&\|Ag(v_n) - Ag(v^*) - \rho(N(v_n, u_n) - N(v^*, u_n))\| \\
&\leq \frac{\tau\theta_2\theta_1}{(1 - \theta_3)(r - \rho_2 m) - \tau\rho_2\nu}\|u_n - u^*\|.
\end{aligned} \tag{3.21}$$

On the other hand, we can obtain similarly

$$\|u_n - u^* - g(u_n) - g(u^*)\| \le \theta_3 \|u_n - u^*\|. \tag{3.22}$$

Substituting (3.13), (3.21) and (3.22) into (3.11), we arrive at

$$\begin{aligned}
&\|u_{n+1} - u^*\| \\
&\le \theta_3 \|u_n - u^*\| + \frac{\tau^2 \theta_2 \theta_1}{(r - \rho_1 m)[(1-\theta_3)(r - \rho_2 m) - \tau \rho_2 \nu]} \|u_n - u^*\| \\
&\quad + \frac{\tau \rho_1 \nu}{r - \rho_1 m} \|u_n - u^*\| \\
&= (\theta_3 + \frac{\tau^2 \theta_2 \theta_1}{(r - \rho_1 m)[(1-\theta_3)(r - \rho_2 m) - \tau \rho_2 \nu]} + \frac{\tau \rho_1 \nu}{r - \rho_1 m}) \|u_n - u^*\|.
\end{aligned} \tag{3.23}$$

Observing condition $\theta_3 + \frac{\tau^2 \theta_2 \theta_1}{(r-\rho_1 m)[(1-\theta_3)(r-\rho_2 m) - \tau \rho_2 \nu]} + \frac{\tau \rho_1 \nu}{r - \rho_1 m} < 1$, we can prove the desired conclusion. This completes the proof.

From Theorem 2.1, we have the following results immediately.

Theorem 3.2. *Let H be a real Hilbert space, let $A : H \times H \to H$ be (r, η)-strongly monotone and s-Lipschitz continuous and let $M : H \to 2^H$ be (A, η)-monotone. Let $\eta : H \times H \to H$ be a τ-Lipschitz continuous nonlinear mapping and let $N : H \times H \to H$ be relaxed (α, β)-cocoercive (with respect to A) and μ-Lipschitz continuous in the first variable. Let N be ν-Lipschitz continuous in the second variable. Let u^*, v^* be the solution of NSVI problem (3.3)-(3.4), $\{u_n\}$ and $\{v_n\}$ be sequences generated by Algorithm 3.2. Suppose the following condition are satisfied:*

$$\frac{\tau^2 \theta_2 \theta_1}{(r - \rho_1 m)[(r - \rho_2 m) - \tau \rho_2 \nu]} + \frac{\tau \rho_1 \nu}{r - \rho_1 m} < 1,$$

where $\theta_1 = \sqrt{s^2 - 2\rho_1 \beta + 2\rho_1 \alpha \mu^2 + \rho_1^2 \mu^2}$ and $\theta_2 = \sqrt{s^2 - 2\rho_2 \beta + 2\rho_2 \alpha \mu^2 + \rho_2^2 \mu^2}$. Then the sequences $\{u_n\}$ and $\{v_n\}$ converges strongly to u^* and v^*, respectively.

Theorem 3.3. *Let H be a real Hilbert space, let $A : H \times H \to H$ be (r, η)-strongly monotone and s-Lipschitz continuous and let $M : H \to 2^H$ be (A, η)-monotone. Let $\eta : H \times H \to H$ be a τ-Lipschitz continuous nonlinear mapping and let $N : H \times H \to H$ be relaxed (α, β)-cocoercive (with respect to A) and μ-Lipschitz continuous in the first variable. Let N be ν-Lipschitz continuous in the second variable. Let u^* be the solution of NVI problem (3.5), $\{u_n\}$ be a sequence generated by Algorithm 3.3. Suppose the following condition are satisfied:*

$$(\theta + \rho \nu)\tau < (r - \rho m),$$

where $\theta = \sqrt{s^2 - 2\rho\beta + 2\rho\alpha\mu^2 + \rho^2\mu^2}$. *Then the sequence* $\{u_n\}$ *converges strongly to* u^*.

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