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Common fixed point theorems for single and set-valued maps without continuity

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Abstract

The main purpose of this paper is to establish some common fixed point theorems for single and set-valued maps, under strict contractive conditions with no compacity and without using continuity. These theorems generalize, extend and improve the result due to Ahmed [1] and others. We also give a generalization of theorem 2.1 of [1].

1 Introduction

In recent years several fixed point theorems for single and set-valued maps for pairs of mappings are proved and have numerous applications and by now there exists an extensive considerable and rich literature in this domain. Many authors have discussed and studied extensively various results on coincidence, existence and uniqueness of fixed and common fixed points for contractive and expansive maps in different spaces and they have applied to diverse problems. Note that common fixed point theorems for single and set-valued maps are interesting and play a major role in many areas.

Our work here establishes common fixed point theorems for single and set-valued maps under strict contractive conditions. These theorems use minimal type commutativity with no continuity and compacity requirement. Our theorems extend some results especially the recent result given by M. A. Ahmed [1].

Key Words: δ -compatible and weakly compatible maps; D-mappings; Single and set-valued maps; Common fixed points.
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2 Preliminaries

Throughout this paper, $(\mathcal{X}, d)$ denotes a metric space and $B(\mathcal{X})$ is the set of all nonempty bounded subsets of $\mathcal{X}$. As in [3,4], we define the functions $\delta(A, B)$ and $D(A, B)$ as follows:

$$\begin{aligned} D(A, B) &= \inf\{d(a, b) : a \in A, b \in B\} \\ \delta(A, B) &= \sup\{d(a, b) : a \in A, b \in B\}, \end{aligned}$$

for all A, B in $B(\mathcal{X})$. If A contains a single point a, we write $\delta(A, B) = \delta(a, B)$. Also, if B contains a single point b, it yields $\delta(A, B) = d(a, b)$.

The definition of the function $\delta(A, B)$ yields the following:

$$\begin{aligned} \delta(A, B) &= \delta(B, A), \\ \delta(A, B) &\leq \delta(A, C) + \delta(C, B), \\ \delta(A, B) &= 0 \text{ iff } A = B = \{a\}, \\ \delta(A, A) &= diam A, \end{aligned}$$

for all A, B, C in $B(\mathcal{X})$.

Definition 2.1. **[3]** A sequence $\{A_n\}$ of subsets of $\mathcal{X}$ is said to be *convergent to a subset A of $\mathcal{X}$* if

(i) Given $a \in A$, there is a sequence $\{a_n\}$ in $\mathcal{X}$ such that $a_n \in A_n$ for $n \in \mathbb{N}^*$ and $\{a_n\}$ converges to a.

(ii) Given $\varepsilon > 0$, there exists a positive integer N such that $A_n \subseteq A_\varepsilon$ for $n > N$ where A_ε is the union of all open spheres with centers in A and radius ε.

Lemma 2.1. [3,4] *If $\{A_n\}$ and $\{B_n\}$ are sequences in $B(\mathcal{X})$ converging to A and B in $B(\mathcal{X})$ respectively, then the sequence $\{\delta(A_n, B_n)\}$ converges to $\delta(A, B)$.*

Lemma 2.2. [4] *Let $\{A_n\}$ be a sequence in $B(\mathcal{X})$ and y be a point in $\mathcal{X}$ such that $\delta(A_n, y) \to 0$ as $n \to \infty$. Then, the sequence $\{A_n\}$ converges to the set $\{y\}$ in $B(\mathcal{X})$.*

Definition 2.2. **[4]** A set-valued mapping $\mathcal{F}$ of $\mathcal{X}$ into $B(\mathcal{X})$ is said to be *continuous at $x \in \mathcal{X}$* if the sequence $\{\mathcal{F}x_n\}$ in $B(\mathcal{X})$ converges to $\mathcal{F}x$ whenever $\{x_n\}$ is a sequence in $\mathcal{X}$ converging to x in $\mathcal{X}$. $\mathcal{F}$ is said to be *continuous on $\mathcal{X}$* if it is continuous at every point in $\mathcal{X}$.

Lemma 2.3. [4] *Let $\{A_n\}$ be a sequence of nonempty subsets of $\mathcal{X}$ and z in $\mathcal{X}$ such that*

$$\lim_{n\to\infty} a_n = z,$$

z independent of the particular choices of each $a_n \in A_n$. If a self-map $\mathcal{I}$ of $\mathcal{X}$ is continuous, then $\{\mathcal{I}z\}$ is the limit of the sequence $\{\mathcal{I}A_n\}$.

Recently, Li-Shan [6] introduced the following definition:

Definition 2.3. **[6]** The mappings $\mathcal{F} : \mathcal{X} \to B(\mathcal{X})$ and $f : \mathcal{X} \to \mathcal{X}$ are *δ-compatible* if

$$\lim_{n\to\infty} \delta(\mathcal{F}fx_n, f\mathcal{F}x_n) = 0$$

whenever $\{x_n\}$ is a sequence in $\mathcal{X}$ such that $f\mathcal{F}x_n \in B(\mathcal{X})$, $\mathcal{F}x_n \to \{t\}$ and $fx_n \to t$ for some $t \in \mathcal{X}$.

Jungck and Rhoades [5] gave a generalization of the above definition as follows:

Definition 2.4. **[5]** The mappings $\mathcal{F} : \mathcal{X} \to B(\mathcal{X})$ and $f : \mathcal{X} \to \mathcal{X}$ are *weakly compatible* if they commute at coincidence points, that is

$$\{t \in \mathcal{X}/\mathcal{F}t = \{ft\}\} \subseteq \{t \in \mathcal{X}/\mathcal{F}ft = f\mathcal{F}t\}.$$

It can be seen that δ-compatible maps are weakly compatible but the converse is not true. Examples supporting this fact can be found in [5].

Recently, M. A. Ahmed has proved in his paper [1] the following theorem.

Theorem 2.1. [1] *Let $\mathcal{I}$,$\mathcal{J}$ be functions of a compact metric space $(\mathcal{X}, d)$ into itself and $\mathcal{F}$,$\mathcal{G} : \mathcal{X} \to B(\mathcal{X})$ two set-valued functions with*
(1) $\cup\mathcal{F}(\mathcal{X}) \subseteq \mathcal{J}(\mathcal{X})$ *and* $\cup\mathcal{G}(\mathcal{X}) \subseteq \mathcal{I}(\mathcal{X})$. *Suppose that*
(2) *the inequality*

$$\begin{aligned}\delta(\mathcal{F}x, \mathcal{G}y) \quad &< \quad \alpha \max\{d(\mathcal{I}x, \mathcal{J}y), \delta(\mathcal{I}x, \mathcal{F}x), \delta(\mathcal{J}y, \mathcal{G}y)\} \\ &\quad\quad +(1-\alpha)\left[aD(\mathcal{I}x, \mathcal{G}y) + bD(\mathcal{J}y, \mathcal{F}x)\right],\end{aligned}$$

for all x,$y \in \mathcal{X}$ where
(3) $0 \le \alpha < 1, a \ge 0, b \ge 0, a \le \dfrac{1}{2}, b < \dfrac{1}{2}, \alpha\,|a-b| < 1-(a+b)$, *holds whenever the righ hand side of* (2) *is positive.*

If the pairs $\{\mathcal{F},\mathcal{I}\}$ and $\{\mathcal{G},\mathcal{J}\}$ are weakly compatible, and if the functions $\mathcal{F}$ and $\mathcal{I}$ are continuous, then there is a unique point u in $\mathcal{X}$ such that

$$\mathcal{F}u = \mathcal{G}u = \{u\} = \{\mathcal{I}u\} = \{\mathcal{J}u\}.$$

Our objective here is to generalize, improve and extend Theorem 2.1 above by dropping the hypothesis of compacity and without assuming the continuity with $a, b \in [0,1)$ on condition that their sum is strictly lower than 1 also without using the condition $\alpha\,|a-b| < 1-(a+b)$ but we use a new concept of mappings called D-mappings. That is, we shall prove that the assumptions of compacity and continuity made on $\mathcal{F}$ and $\mathcal{I}$ in the theorem of [1] are superfluous and can be removed.

Definition 2.5. **[2]** The mappings $\mathcal{F} : \mathcal{X} \to B(\mathcal{X})$ and $\mathcal{I} : \mathcal{X} \to \mathcal{X}$ are said to be *D-mappings* if there exists a sequence $\{x_n\}$ in $\mathcal{X}$ such that, $\lim_{n\to\infty} \mathcal{I}x_n = t$ and $\lim_{n\to\infty} \mathcal{F}x_n = \{t\}$ for some $t \in \mathcal{X}$.

Example 2.1. Let $\mathcal{X} = [0,\infty)$. Define $\mathcal{F} : \mathcal{X} \to B(\mathcal{X})$ and $\mathcal{I} : \mathcal{X} \to \mathcal{X}$ by

$$\mathcal{F}x = [0, 2x] \text{ and } \mathcal{I}x = 3x, \forall x \in \mathcal{X}.$$

Consider the sequence $x_n = \dfrac{1}{4n}$ for all $n \in \mathbb{N}^*$. Obviously

$$\lim_{n\to\infty} \mathcal{F}x_n = \{0\} \text{ and } \lim_{n\to\infty} \mathcal{I}x_n = 0.$$

Then $\mathcal{F}$ and $\mathcal{I}$ are D-mappings.

3 Main results

We start with our first main result.

Theorem 3.1. *Let $(\mathcal{X}, d)$ be a metric space, let $\mathcal{F}$, $\mathcal{G} : \mathcal{X} \to B(\mathcal{X})$ and $\mathcal{I}$, $\mathcal{J} : \mathcal{X} \to \mathcal{X}$ be set and single-valued mappings, respectively satisfying the conditions:*

(1) $\mathcal{F}\mathcal{X} \subset \mathcal{J}\mathcal{X}$ *and* $\mathcal{G}\mathcal{X} \subset \mathcal{I}\mathcal{X}$,

(2)

$$\begin{aligned}\delta(\mathcal{F}x, \mathcal{G}y) \quad &< \quad \alpha \max\{d(\mathcal{I}x, \mathcal{J}y), \delta(\mathcal{I}x, \mathcal{F}x), \delta(\mathcal{J}y, \mathcal{G}y)\} \\ &\quad +(1-\alpha)\,[aD(\mathcal{I}x, \mathcal{G}y) + D(\mathcal{J}y, \mathcal{F}x)]\end{aligned}$$

for all $x, y \in \mathcal{X}$*, where* $0 \leq \alpha < 1, a \geq 0, b \geq 0, a+b < 1,$ *whenever the right hand side of* (2) *is positive. If either*

(3) $\mathcal{F}, \mathcal{I}$ *are weakly compatible D-mappings;* $\mathcal{G}$, $\mathcal{J}$ *are weakly compatible and* $\mathcal{F}\mathcal{X}$ *or* $\mathcal{J}\mathcal{X}$ *is closed or*

(3′) $\mathcal{G}$, $\mathcal{J}$ *are weakly compatible D-mappings;* $\mathcal{F}, \mathcal{I}$ *are weakly compatible and* $\mathcal{G}\mathcal{X}$ *or* $\mathcal{I}\mathcal{X}$ *is closed, then there is a unique common fixed point* t *in* $\mathcal{X}$ *such that*

$$\mathcal{F}t = \mathcal{G}t = \{t\} = \{\mathcal{I}t\} = \{\mathcal{J}t\}\,.$$

Proof. Suppose that $\mathcal{F}$ and $\mathcal{I}$ are D-mappings, then there exists a sequence $\{x_n\}$ in $\mathcal{X}$ such that, $\lim\limits_{n\to\infty} \mathcal{I}x_n = t$ and $\lim\limits_{n\to\infty} \mathcal{F}x_n = \{t\}$ for some $t \in \mathcal{X}$. Since $\mathcal{F}\mathcal{X}$ is closed and $\mathcal{F}\mathcal{X} \subseteq \mathcal{J}\mathcal{X}$, then, there exists a point u in $\mathcal{X}$ such that $\mathcal{J}u = t$. Then inequality (2) gives

$$\begin{aligned}\delta(\mathcal{F}x_n, \mathcal{G}u) \quad &< \quad \alpha \max\{d(\mathcal{I}x_n, \mathcal{J}u), \delta(\mathcal{I}x_n, \mathcal{F}x_n), \delta(\mathcal{J}u, \mathcal{G}u)\} \\ &\quad\quad +(1-\alpha)\left[aD(\mathcal{I}x_n, \mathcal{G}u) + bD(\mathcal{J}u, \mathcal{F}x_n)\right].\end{aligned}$$

Taking the limit as n tends to infinity and using Lemma 2.1, it comes

$$\begin{aligned}\delta(\mathcal{J}u, \mathcal{G}u) \quad &\leq \quad \alpha\delta(\mathcal{J}u, \mathcal{G}u) + (1-\alpha)aD(\mathcal{J}u, \mathcal{G}u) \\ &\leq \quad [\alpha + (1-\alpha)a]\,\delta(\mathcal{J}u, \mathcal{G}u).\end{aligned}$$

It is obvious that $[\alpha + (1-\alpha)a] < 1$, then the above contradiction demands that $\mathcal{G}u = \{\mathcal{J}u\}$. Since $\mathcal{G}$ and $\mathcal{J}$ are weakly compatible, $\mathcal{G}u = \{\mathcal{J}u\}$ implies that $\mathcal{G}\mathcal{J}u = \mathcal{J}\mathcal{G}u$ and hence

$$\mathcal{G}\mathcal{G}u = \mathcal{G}\mathcal{J}u = \mathcal{J}\mathcal{G}u = \{\mathcal{J}\mathcal{J}u\}\,.$$

Again by (2), we have

$$\begin{aligned}\delta(\mathcal{F}x_n, \mathcal{G}\mathcal{G}u) \quad &< \quad \alpha \max\{d(\mathcal{I}x_n, \mathcal{J}\mathcal{G}u), \delta(\mathcal{I}x_n, \mathcal{F}x_n), \delta(\mathcal{J}\mathcal{G}u, \mathcal{G}\mathcal{G}u)\} \\ &\quad\quad +(1-\alpha)\left[aD(\mathcal{I}x_n, \mathcal{G}\mathcal{G}u) + bD(\mathcal{J}\mathcal{G}u, \mathcal{F}x_n)\right].\end{aligned}$$

At infinity and by Lemma 2.1, we obtain

$$\begin{aligned}\delta(\mathcal{J}u, \mathcal{G}\mathcal{G}u) \quad &\leq \quad \alpha d(\mathcal{J}u, \mathcal{G}\mathcal{G}u) + (1-\alpha)(a+b)D(\mathcal{J}u, \mathcal{G}\mathcal{G}u) \\ &\leq \quad [\alpha + (1-\alpha)(a+b)]\,\delta(\mathcal{J}u, \mathcal{G}\mathcal{G}u)\end{aligned}$$

and, since $[\alpha + (1-\alpha)(a+b)] < 1$, then we have $\mathcal{G}\mathcal{G}u = \{\mathcal{J}u\}$. Hence $\{\mathcal{J}u\} = \mathcal{G}\mathcal{G}u = \mathcal{J}\mathcal{G}u$, i.e. $\mathcal{G}u = \mathcal{G}\mathcal{G}u = \mathcal{J}\mathcal{G}u$ and $\mathcal{G}u$ is a common fixed point of $\mathcal{G}$

and $\mathcal{J}$. Since $\mathcal{G}\mathcal{X} \subseteq \mathcal{I}\mathcal{X}$, then there is a point $v \in \mathcal{X}$ such that $\{\mathcal{I}v\} = \mathcal{G}u$. Moreover the use of (2) gives

$$\begin{aligned}
\delta(\mathcal{F}v, \mathcal{G}u) &< \alpha \max\{d(\mathcal{I}v, \mathcal{J}u), \delta(\mathcal{I}v, \mathcal{F}v), \delta(\mathcal{J}u, \mathcal{G}u)\} \\
&\quad +(1-\alpha)\left[aD(\mathcal{I}v, \mathcal{G}u) + bD(\mathcal{J}u, \mathcal{F}v)\right] \\
&= \alpha\delta(\mathcal{G}u, \mathcal{F}v) + (1-\alpha)bD(\mathcal{G}u, \mathcal{F}v) \\
&\leq \left[\alpha + (1-\alpha)b\right]\delta(\mathcal{G}u, \mathcal{F}v).
\end{aligned}$$

It is easy to see that $[\alpha + (1-\alpha)b] < 1$, implying that $\mathcal{F}v = \mathcal{G}u = \{\mathcal{I}v\}$. Since $\mathcal{F}v = \{\mathcal{I}v\}$, by the weak compatibility of $\mathcal{F}$ and $\mathcal{I}$, we get $\mathcal{F}\mathcal{I}v = \mathcal{I}\mathcal{F}v$ and hence

$$\mathcal{F}\mathcal{F}v = \mathcal{F}\mathcal{I}v = \mathcal{I}\mathcal{F}v = \{\mathcal{I}\mathcal{I}v\}.$$

Moreover, by (2), we can estimate

$$\begin{aligned}
\delta(\mathcal{F}\mathcal{F}v, \mathcal{G}u) &< \alpha \max\{d(\mathcal{I}\mathcal{F}v, \mathcal{J}u), \delta(\mathcal{I}\mathcal{F}v, \mathcal{F}\mathcal{F}v), \delta(\mathcal{J}u, \mathcal{G}u)\} \\
&\quad +(1-\alpha)\left[aD(\mathcal{I}\mathcal{F}v, \mathcal{G}u) + bD(\mathcal{J}u, \mathcal{F}\mathcal{F}v)\right] \\
&= \alpha \max\{d(\mathcal{I}\mathcal{F}v, \mathcal{J}u), 0, 0\} + (1-\alpha)(a+b)D(\mathcal{I}\mathcal{F}v, \mathcal{G}u) \\
&= \alpha d(\mathcal{F}\mathcal{F}v, \mathcal{G}u) + (1-\alpha)(a+b)D(\mathcal{F}\mathcal{F}v, \mathcal{G}u) \\
&\leq \left[\alpha + (1-\alpha)(a+b)\right]\delta(\mathcal{F}\mathcal{F}v, \mathcal{G}u) < \delta(\mathcal{F}\mathcal{F}v, \mathcal{G}u),
\end{aligned}$$

which is a contradiction, thus $\mathcal{F}\mathcal{F}v = \mathcal{G}u$, i.e., $\mathcal{F}\mathcal{G}u = \mathcal{G}u = \mathcal{I}\mathcal{G}u$ and $\mathcal{G}u$ is also a common fixed point of $\mathcal{F}$ and $\mathcal{I}$. Since $\mathcal{G}u = \{t\}$, then

$$\mathcal{F}t = \mathcal{G}t = \{t\} = \{\mathcal{I}t\} = \{\mathcal{J}t\}.$$

Similarly, one can obtain this conclusion by using (3′) instead of (3).

Finally, we prove that t is unique. Indeed, let t' be another common fixed point of the maps $\mathcal{I}$,$\mathcal{J}$,$\mathcal{F}$ and $\mathcal{G}$ such that $t' \neq t$. Then, by estimation (2), one may get

$$\begin{aligned}
d(t, t') &= \delta(\mathcal{F}t, \mathcal{G}t') < \alpha \max\{d(\mathcal{I}t, \mathcal{J}t'), \delta(\mathcal{I}t, \mathcal{F}t), \delta(\mathcal{J}t', \mathcal{G}t')\} \\
&\quad +(1-\alpha)\left[aD(\mathcal{I}t, \mathcal{G}t') + bD(\mathcal{J}t', \mathcal{F}t)\right] \\
&= \alpha d(t, t') + (1-\alpha)(a+b)D(t, t') \\
&\leq \left[\alpha + (1-\alpha)(a+b)\right] d(t, t') < d(t, t').
\end{aligned}$$

This contradiction implies that $t' = t$. Hence, t is the unique common fixed point of $\mathcal{I}$,$\mathcal{J}$,$\mathcal{F}$ and $\mathcal{G}$.

If we let in Theorem 3.1 $\mathcal{F} = \mathcal{G}$ and $\mathcal{I} = \mathcal{J}$, then we get the following result:

Corollary 3.1. *Let $\mathcal{I} : \mathcal{X} \to \mathcal{X}$ be a self-map of a metric space $(\mathcal{X}, d)$ and $\mathcal{F} : \mathcal{X} \to B(\mathcal{X})$ be a set-valued map. Assume that $\mathcal{F}$ and $\mathcal{I}$ satisfy the conditions*

(i) $\mathcal{F}\mathcal{X} \subseteq \mathcal{I}\mathcal{X}$,

(ii) the inequality

$$\begin{aligned}\delta(\mathcal{F}x, \mathcal{F}y) &< \alpha \max\{d(\mathcal{I}x, \mathcal{I}y), \delta(\mathcal{I}x, \mathcal{F}x), \delta(\mathcal{I}y, \mathcal{F}y)\} \\ &\quad +(1-\alpha)\left[aD(\mathcal{I}x, \mathcal{F}y) + bD(\mathcal{I}y, \mathcal{F}x)\right],\end{aligned}$$

for all x,$y \in \mathcal{X}$, where $0 \le \alpha < 1$,$a \ge 0$,$b \ge 0$,$a + b < 1$, whenever the right hand side of inequality (ii) is positive. If $\mathcal{F}$ and $\mathcal{I}$ are weakly compatible D-mappings and $\mathcal{F}\mathcal{X}$ or $\mathcal{I}\mathcal{X}$ is closed, then $\mathcal{F}$ and $\mathcal{I}$ have a unique common fixed point t in $\mathcal{X}$.

For the three maps, we have the following result:

Corollary 3.2. *Let $\mathcal{I} : \mathcal{X} \to \mathcal{X}$ be a self-map of a metric space $(\mathcal{X}, d)$ and $\mathcal{F}, \mathcal{G} : \mathcal{X} \to B(\mathcal{X})$ be two set-valued maps such that*

(i) $\mathcal{F}\mathcal{X} \subseteq \mathcal{I}\mathcal{X}$ and $\mathcal{G}\mathcal{X} \subseteq \mathcal{I}\mathcal{X}$,

(ii) the inequality

$$\begin{aligned}\delta(\mathcal{F}x, \mathcal{G}y) &< \alpha \max\{d(\mathcal{I}x, \mathcal{I}y), \delta(\mathcal{I}x, \mathcal{F}x), \delta(\mathcal{I}y, \mathcal{G}y)\} \\ &\quad +(1-\alpha)\left[aD(\mathcal{I}x, \mathcal{G}y) + bD(\mathcal{I}y, \mathcal{F}x)\right],\end{aligned}$$

holds for all x,$y \in \mathcal{X}$, where $0 \le \alpha < 1$,$a \ge 0$,$b \ge 0$,$a + b < 1$, whenever the right hand side of the above inequality is positive. Further, if either

(iii) $\mathcal{F}, \mathcal{I}$ are weakly compatible D-mappings; $\mathcal{G}$, $\mathcal{I}$ are weakly compatible and $\mathcal{F}\mathcal{X}$ or $\mathcal{I}\mathcal{X}$ is closed or

$(iii)'$ $\mathcal{G}, \mathcal{I}$ are weakly compatible D-mappings; $\mathcal{F}, \mathcal{I}$ are weakly compatible and $\mathcal{G}\mathcal{X}$ or $\mathcal{I}\mathcal{X}$ is closed, then $\mathcal{F}, \mathcal{G}$ and $\mathcal{I}$ have a unique common fixed point in $\mathcal{X}$.

Remark.

Truly, our result generalizes the result of M. A. Ahmed [1], since we have not assuming compacity nor continuity but only the so-called D-mappings and the minimal condition of the closedness.

Now, we give our second result which is a generalization of the above result

Theorem 3.2. *Let $\mathcal{I}$,$\mathcal{J} : \mathcal{X} \to \mathcal{X}$ be self-mappings and $\mathcal{F}_i : \mathcal{X} \to B(\mathcal{X})$,$i \in \mathbb{N}^*$ be set-valued maps such that*

(i) $\mathcal{F}_i\mathcal{X} \subseteq \mathcal{J}\mathcal{X}$ and $\mathcal{F}_{i+1}\mathcal{X} \subseteq \mathcal{I}\mathcal{X}$,

(ii) the inequality

$$\begin{aligned}\delta(\mathcal{F}_i x, \mathcal{F}_{i+1} y) &< \alpha \max\left\{d(\mathcal{I}x, \mathcal{J}y), \delta(\mathcal{I}x, \mathcal{F}_i x), \delta(\mathcal{J}y, \mathcal{F}_{i+1} y)\right\} \\ &\quad +(1-\alpha)\left[aD(\mathcal{I}x, \mathcal{F}_{i+1} y) + bD(\mathcal{J}y, \mathcal{F}_i x)\right]\end{aligned}$$

holds for all x,$y \in \mathcal{X}$,$\forall i \in \mathbb{N}^*$, where $0 \leq \alpha < 1$,$a \geq 0$,$b \geq 0$,$a+b < 1$, whenever the right hand side of (ii) is positive. Further, if either

(iii) $\mathcal{F}_i$,$\mathcal{I}$ are weakly compatible D-mappings; $\mathcal{F}_{i+1}$,$\mathcal{J}$ are weakly compatible and $\mathcal{F}_i\mathcal{X}$ or $\mathcal{J}\mathcal{X}$ is closed or

$(iii)'$ $\mathcal{F}_{i+1}$,$\mathcal{J}$ are weakly compatible D-mappings; $\mathcal{F}_i$,$\mathcal{I}$ are weakly compatible and $\mathcal{F}_{i+1}\mathcal{X}$ or $\mathcal{I}\mathcal{X}$ is closed.

Then there exists a unique common fixed point $t \in \mathcal{X}$ such that

$$\mathcal{F}_i t = \{\mathcal{I}t\} = \{\mathcal{J}t\} = \{t\}, \forall i \in \mathbb{N}^*.$$

We ended our paper by giving a generalization of Theorem 2.1 of [1].

Theorem 3.3. *Let $\mathcal{I}$,$\mathcal{J}$ be mappings of a metric space $(\mathcal{X}, d)$ into itself and $\mathcal{F}_i : \mathcal{X} \to B(\mathcal{X})$,$i \in \mathbb{N}^*$ be set-valued maps such that*

(i) $\cup\mathcal{F}_i\mathcal{X} \subseteq \mathcal{J}\mathcal{X}$ and $\cup\mathcal{F}_{i+1}\mathcal{X} \subseteq \mathcal{I}\mathcal{X}$,

(ii) the inequality

$$\begin{aligned}\delta(\mathcal{F}_i x, \mathcal{F}_{i+1} y) &\leq \alpha \max\left\{d(\mathcal{I}x, \mathcal{J}y), \delta(\mathcal{I}x, \mathcal{F}_i x), \delta(\mathcal{J}y, \mathcal{F}_{i+1} y)\right\} \\ &\quad +(1-\alpha)\left[aD(\mathcal{I}x, \mathcal{F}_{i+1} y) + bD(\mathcal{J}y, \mathcal{F}_i x)\right]\end{aligned}$$

holds for all x,$y \in \mathcal{X}$,$\forall i \in \mathbb{N}^*$, where $0 \leq \alpha < 1$,$a \geq 0$,$b \geq 0$,$a+b < 1, \alpha\left|a-b\right| < 1-(a+b)$. Suppose that one of $\mathcal{I}\mathcal{X}$ or $\mathcal{J}\mathcal{X}$ is complete. If both pairs $\{\mathcal{F}_i, \mathcal{I}\}$ and $\{\mathcal{F}_{i+1}, \mathcal{J}\}$ are weakly compatible, then there exists $z \in \mathcal{X}$ such that

$$\{z\} = \{Iz\} = \{Jz\} = \mathcal{F}_i z, \forall i \in \mathbb{N}^*.$$

Proof. Letting $i = 1$, we get the hypotheses of Theorem 2.1 of [1] for the maps $\mathcal{I}, \mathcal{J}, \mathcal{F}_1$ and $\mathcal{F}_2$ with the unique common fixed point t. Now, t is a unique common fixed point of $\mathcal{I}, \mathcal{J}, \mathcal{F}_1$ and of $\mathcal{I}, \mathcal{J}, \mathcal{F}_2$. Otherwise, if t' is a second distinct fixed point of $\mathcal{I}, \mathcal{J}$ and $\mathcal{F}_1$, then by inequality (ii), we get

$$\begin{aligned}d(t, t') &= \delta(\mathcal{F}_1 t, \mathcal{F}_2 t') \leq \alpha \max\left\{d(\mathcal{I}t, \mathcal{J}t'), \delta(\mathcal{I}t, \mathcal{F}_1 t), \delta(\mathcal{J}t', \mathcal{F}_2 t')\right\} \\ &\quad +(1-\alpha)\left[aD(\mathcal{I}t, \mathcal{F}_2 t') + bD(\mathcal{J}t', \mathcal{F}_1 t)\right] \\ &= (\alpha + (1-\alpha)(a+b))d(t, t'),\end{aligned}$$

since $(\alpha + (1-\alpha)(a+b)) < 1$, hence $t' = t$.

By the same method, we prove that t is the unique common fixed point of the mappings $\mathcal{I}, \mathcal{J}$ and $\mathcal{F}_2$.

Now, by letting $i = 2$, we get the hypotheses of Theorem 2.1 of Ahmed for the maps $\mathcal{I}, \mathcal{J}, \mathcal{F}_2$ and $\mathcal{F}_3$ and consequently they have a unique common fixed point t'. Analogously, t' is the unique common fixed point of $\mathcal{I}, \mathcal{J}, \mathcal{F}_2$ and of $\mathcal{I}, \mathcal{J}, \mathcal{F}_3$. Thus $t' = t$. Continuing in this way, we clearly see that t is the required point.

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