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On the structure of nilpotent endomorphisms and applications

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Abstract

The nilpotent endomorphisms over a finite free module over a domain with principal ideal are characterized. One may apply these results to the study of the maximal Cohen-Macaulay modules over the ring $R := A[[x]]/(x^n), n \geq 2$, where A is a DVR.

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1 Introduction

Our aim is to find some kind of a "normal form" for the nilpotent endomorphisms of a finite free module E over a principal ideal domain (briefly PID), similar with the Jordan form for the nilpotent endomorphisms of the linear spaces. We closely follow the procedure used to get the Jordan form for the nilpotent endomorphisms of linear spaces (see [G]). We shall see in Section 3 that this "normal form" looks very nice for those nilpotent endomorphisms which have the index of nilpotency equal to 2, but it becomes very complicate for bigger index. Next we apply these results in the study of the MCM modules over the ring $R := A[[x]]/(x^n)$, where $A = K[[y]]$, K being a field, but all the results can be applied for the MCM modules over the ring $A[[x]]/(x^n)$, where A is a DVR. R is a finite A–algebra and any maximal Cohen-Macaulay (briefly MCM) module over R is free of finite rank over A. Giving a MCM R–module M is equivalent to give a morphism of A–algebras, $f_M : R \to \mathrm{End}_A(M)$ which is uniquely determined by $u = f_M(x) \in \mathrm{End}_A(M)$. Since $x^n = 0$, we must

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have $u^n = 0$. Therefore we are interested to characterize the nilpotent endomorphisms of finite free A–modules.
Let us recall that, for a hypersurface ring $S/(f)$, where $(S, \mathfrak{m})$ is a local regular ring and f is a non–zero element in $\mathfrak{m}$, any MCM $S/(f)$–module has a minimal free resolution of periodicity 2 which is completely given by a matrix factorization (φ, ψ), φ, ψ being square matrices over S such that $\varphi\psi = \psi\varphi = f \cdot I_m$, for a certain positive integer m (see [E]).
Let us consider the hypersurface ring $R := A[[x]]/(x^n)$ where $A = K[[y]]$ or, more generally, a DVR. We show in the last section that any MCM R– module is described by a matrix factorization of the form

$$(\varphi = x \operatorname{Id}_m - T, \psi = x^{n-1} \operatorname{Id}_m + x^{n-2}T + \ldots + T^{n-1}),$$

for some $m \times m$–matrix T with the entries in A which gives the action of x on M, thus $T^n = 0$. Therefore, in order to find the matrix factorizations of the MCM R–modules, we need to study the structure of the nilpotent matrices over A.

2 Some general facts

For the beginning, let A be a PID, let $E = A^m$ be the finite free A–module of rank m, and let $u \in \operatorname{End}_A(E)$ be nilpotent. Let $n \geq 2$ such that $u^n = 0$ and $u^{n-1} \neq 0$.
For $0 \leq k \leq n$, let $E_k := \ker(u^k)$.

Claim 2.1. *For any* $0 \leq k \leq n-1, E_k \subsetneq E_{k+1}$.

Proof. Assume that there exists $0 \leq k \leq n-1$ such that $E_k = E_{k+1}$, and let $x \in E$. Then

$$0 = u^n(x) = u^{k+1} \circ u^{n-(k+1)}(x),$$

which implies that $u^{n-(k+1)}(x) \in E_{k+1} = E_k$. Thus,

$$0 = u^k(u^{n-(k+1)}(x)) = u^{n-1}(x), \ x \in E,$$

contradiction! □

It follows that

$$E_{k+1}/E_k \neq 0, \ 0 \leq k \leq n-1.$$

Claim 2.2. *For any* $0 \leq k \leq n-1, E_{k+1}/E_k$ *is free over* A.

Proof. Let $x + E_k \in E_{k+1}/E_k$ such that there exists $a \neq 0,\ a \in A$, with $a(x + E_k) = 0$, that is $ax \in E_k$. Then $au^k(x) = 0$. But E is free, so $u^k(x) = 0$, which implies $x \in E_k$, that is $x + E_k = 0$. This means that the torsion submodule of E_{k+1}/E_k is null. Since A is a domain with principal ideals, it results that E_{k+1}/E_k is free over A. □

Claim 2.3. *For any* $0 \leq k \leq n-1, u(E_{k+1}) \subset E_k$.

Proof. This is obvious. □

Claim 2.4. *The morphism*

$$\bar{u}_k : \frac{E_{k+1}}{E_k} \to \frac{E_k}{E_{k-1}},\ \bar{u}_k(x + E_k) = u(x) + E_{k-1},\ x \in E_{k+1},$$

induced by u, *is injective,* $\forall\ 1 \leq k \leq n-1$. *In particular, this implies that*

$$r_k := \operatorname{rank}_A(\frac{E_k}{E_{k-1}}) \geq r_{k+1} = \operatorname{rank}_A(\frac{E_{k+1}}{E_k}),\ 1 \leq k \leq n-1.$$

We also note that $m = r_1 + r_2 + \ldots + r_n$.

Proof. The composition $E_{k+1} \to E_k \to E_k/E_{k-1}$ of $u : E_{k+1} \to E_k$ with the canonical surjection $E_k \to E_k/E_{k-1}$ has the kernel E_k. □

3 The structure of nilpotent endomorphisms over a PID

3.1 The case n=2

Theorem 3.1. *Let A be a PID and E be a finite free A–module of rank m. Let $u \in \operatorname{End}_A(E)$ such that $u^2 = 0$ and $u \neq 0$. There exists a basis B of E such that the matrix of u in the basis B is of the form*

$$M_B(u) = \left(\begin{array}{c|c} 0 & \Lambda \\ \hline 0 & 0 \end{array}\right),$$

where the first left corner is of size $r_2 \times r_1, r_1 \geq r_2, r_1 + r_2 = m$, $\Lambda = \operatorname{diag}(a_1, \ldots, a_{r_2})$, where $a_1, \ldots, a_{r_2} \in A - \{0\}$ such that $a_1 \mid a_2 \mid \ldots \mid a_{r_2}$, the left down corner is of size $r_1 \times r_1$, and the right down corner is of size $r_1 \times r_2$.

Proof. With the notations of the previous section, we have $(0) = E_0 \subset E_1 \subset E_2 = E$, $u(E) \subset E_1$, by Claim 2.3, and

$$\bar{u}_1 : E/E_1 \to E_1,\ \bar{u}_1(x + E_1) = u(x),\ x \in E,$$

is injective. Moreover $E/E_1 \simeq \bar{u}_1(E/E_1) \subset E_1$ is free by Claim 2.2. We also have $r_1 = \mathrm{rank}_A(E_1) \geq r_2 = \mathrm{rank}_A(E/E_1)$ and $m = r_1 + r_2$.

Let

$$F_1 := \bar{u}_1(E/E_1) \subset E_1.$$

There exists $\{x_1, \ldots, x_{r_1}\}$ a basis of E_1 and $a_1 \mid a_1 \mid \ldots \mid a_{r_2} \in A - \{0\}$ such that $\{a_1x_1, \ldots a_{r_2}x_{r_2}\}$ is a basis of F_1. For each $1 \leq i \leq r_2$, we choose $z_i \in E$ such that $\bar{u}_1(z_i + E_1) = a_ix_i$, that is, $u(z_i) = a_ix_i$. Then we claim that $B = \{x_1, \ldots, x_{r_1}, z_1, \ldots, z_{r_2}\}$ is a basis of E.
B is linearly independent: Let

$$\sum_{i=1}^{r_1} \alpha_i x_i + \sum_{j=1}^{r_2} \beta_j z_j = 0.$$

We apply u and obtain

$$\sum_{j=1}^{r_2} \beta_j u(z_j) = 0,$$

that is

$$\sum_{j=1}^{r_2} \beta_j a_j x_j = 0,$$

which implies $\beta_j = 0$ for all j. Next, we get

$$\sum_{i=1}^{r_1} \alpha_i x_i = 0,$$

which implies $a_i = 0$, for all i.
B generates E over A: Let $y \in E$. Then $y + E_1 \in E/E_1$. We apply $\bar{u}_1$ and get $u(y) \in F_1$. It results that

$$u(y) = \sum_{i=1}^{r_2} \beta_i a_i x_i = \sum_{i=1}^{r_2} \beta_i u(z_i).$$

Next, we get

$$u(y - \sum_{i=1}^{r_2} \beta_i z_i) = 0,$$

which implies that

$$y - \sum_{i=1}^{r_2} \beta_i z_i = \sum_{i=1}^{r_1} \alpha_i x_i,$$

for some $\alpha_i \in A$.
Now, it is obvious that the matrix of u in the basis B of E is given as in the statement of the theorem. □

3.2 The case n=3

We shall see in this section that the structure of the nilpotent endomorphisms which have the nilpotency index equal to 3 is more complicate. The general case can be manipulated as the case $n = 3$. We prefer to give all the proofs in this case since the general case involves only similar calculations but which are complicate as writing.

Theorem 3.2. *Let A be a PID and let E be a finite free A–module of rank m. Let $u \in \mathrm{End}_A(E)$ such that $u^3 = 0$ and $u^2 \neq 0$. There exists a basis B of E such that the matrix of u in the basis B is of the form*

$$M_B(u) = \left(\begin{array}{c|c|c} 0_{r_1\times r_1} & \Lambda_{r_1\times r_2} & \Delta_{r_1\times r_3} \\ \hline 0_{r_2\times r_1} & 0_{r_2\times r_2} & \Gamma_{r_2\times r_3}\Lambda'_{r_3\times r_3} \\ \hline 0_{r_3\times r_1} & 0_{r_3\times r_2} & 0_{r_3\times r_3} \end{array}\right),$$

where $r_1 \geq r_2 \geq r_3,\ r_1 + r_2 + r_3 = m$, $\Lambda = \left(\begin{array}{c} \mathrm{diag}(a_1, \ldots, a_{r_2}) \\ \hline 0 \end{array}\right)$ has the last $r_1 - r_2$ rows 0, $\Lambda' = \mathrm{diag}(b_1, \ldots, b_{r_3})$, $a_1 \mid a_2 \mid \ldots \mid a_{r_2}$, $b_1 \mid b_2 \mid \ldots \mid b_{r_3} \in A - \{0\}$, and the matrix Γ is left invertible.

Proof. We preserve the notations and we have $(0) = E_0 \subset E_1 \subset E_2 \subset E_3 = E$, $\bar{u}_1 : E_2/E_1 \to E_1$, $\bar{u}_2 : E/E_2 \to E_2/E_1$, and let $F_1 = \mathrm{Im}\,\bar{u}_1 \subset E_1$, $F_2 = \mathrm{Im}\,\bar{u}_2 \subset E_2/E_1$.
Let

$$\{x_{11}, \ldots, x_{1r_1}\}$$

be a basis of E_1 and $a_1 \mid a_2 \mid \ldots \mid a_{r_2} \in A - \{0\}$ such that

$$\{a_1x_{11}, \ldots, a_{r_2}x_{1r_2}\}$$

is a basis of F_1. For $1 \leq j \leq r_2$, we choose $x'_{1j} \in E_2$ such that

$$\bar{u}_1(x'_{1j} + E_1) = a_jx_{1j},\ 1 \leq j \leq r_2,$$

that is

$$u(x'_{1j}) = a_jx_{1j}\ 1 \leq j \leq r_2.$$

Then

$$\{x'_{11} + E_1, \ldots, x'_{1r_2} + E_1\}$$

is a basis of E_2/E_1.
Now, let

$$\{x_{21} + E_1, \ldots, x_{2r_2} + E_1\}$$

be a basis of E_2/E_1 and $b_1 \mid b_2 \mid \ldots \mid b_{r_3} \in A - \{0\}$ such that

$$\{b_1(x_{21} + E_1), \ldots, b_{r_3}(x_{2r_3} + E_1)\}$$

is a basis of F_2.
For $1 \le j \le r_3$, we choose $x'_{2j} \in E$ such that

$$\bar{u_2}(x'_{2j} + E_2) = b_j(x_{2j} + E_1).$$

Then

$$\{x'_{21} + E_2, \ldots, x'_{2r_3} + E_2\}$$

is a basis of E/E_2. Moreover,

$$u(x'_{2j}) + E_1 = b_j x_{2j} + E_1,\ 1 \le j \le r_3.$$

We claim that

$$B := \{x_{11}, \ldots, x_{1r_1}\} \cup \{x'_{11}, \ldots, x'_{1r_2}\} \cup \{x'_{21}, \ldots, x'_{2r_3}\}$$

is a basis of E.
B is linearly independent: Let

$$\delta = \sum_{j=1}^{r_1} \alpha_j x_{1j} + \sum_{j=1}^{r_2} \beta_j x'_{1j} + \sum_{j=1}^{r_3} \gamma_j x'_{2j} = 0.$$

Then $0 = \delta + E_2 = \sum_{j=1}^{r_3} \gamma_j (x'_{2j} + E_2)$, which implies that all γ_j are zero. Next we consider $\delta + E_1$ and we get that all β_j are zero and, finally, all α_j are zero.
B generates E: Let $z \in E$. Then $\bar{u}_2(z + E_2) = u(z) + E_1 \in F_2$. It results that there are some $\alpha_j \in A,\ j = \overline{1, r_3}$, such that

$$u(z) + E_1 = \sum_{j=1}^{r_3} \alpha_j b_j (x_{2j} + E_1).$$

It follows that

$$u(z) + E_1 = \sum_{j=1}^{r_3} \alpha_j u(x'_{2j}) + E_1,$$

hence

$$u(z - \sum_{j=1}^{r_3} \alpha_j x'_{2j}) \in E_1,$$

which implies

$$z-\sum_{j=1}^{r_3}\alpha_j x'_{2j}\in E_2,$$

and, next,

$$(z-\sum_{j=1}^{r_3}\alpha_j x'_{2j})+E_1=\sum_{j=1}^{r_2}\beta_j x'_{1j}+E_1,$$

for some $\beta_j \in A$. From this last relation we get the conclusion.

For the matrix of u in the basis $B,\ M_B(u)$, we have

$$u(x_{1j})=0,\ j=\overline{1,r_1},$$

thus the first r_1 columns of $M_B(u)$ have all the entries 0, next,

$$u(x'_{1j})=a_j x_{1j}\ j=\overline{1,r_2},$$

which means that the first entry in the column r_1+1 is a_1, and all the others are 0, the second entry in the column r_1+2 is a_2, and all the others are 0, and so on, until we fill the columns up to r_1+r_2. For the last r_3 columns, observe first that

$$\{x'_{11}+E_1,\dots,x'_{1r_2}+E_1\}$$

and

$$\{x_{21}+E_1,\dots,x_{2r_2}+E_1\}$$

are bases of E_2/E_1. Then there exists an invertible matrix (γ_{tj}), with entries in A, such that

$$x_{2j}+E_1=\sum_{t=1}^{r_2}\gamma_{tj}(x'_{1t}+E_1),\ j=\overline{1,r_2}.$$

Then

$$u(x'_{2j})+E_1=b_j\sum_{t=1}^{r_2}\gamma_{tj}(x'_{1t}+E_1)=\sum_{t=1}^{r_2}\gamma_{tj}b_j x'_{1t}+E_1,\ j=\overline{1,r_3}.$$

It follows that

$$u(x'_{2j})=\sum_{t=1}^{r_2}\gamma_{tj}b_j x'_{1t}+w_j,$$

for some $w_j\in E_1, 1\le j\le r_3$. Let Δ be the $r_1\times r_3$–matrix whose columns are the coordinates of the vectors w_j in the basis $\{x_{11},\dots,x_{1r_1}\}$ of E_1. In conclusion, we may express the matrix of u in blocks as in the statement of the theorem. □

3.3 The general case

Let us consider now the general case, that is u nilpotent of arbitrary index $n \geq 2$. We recall that the morphisms

$$\bar{u}_k : \frac{E_{k+1}}{E_k} \to \frac{E_k}{E_{k-1}},\ \bar{u}_k(x + E_k) = u(x) + E_{k-1},\ x \in E_{k+1},$$

induced by u, are injective, $\forall\ 1 \leq k \leq n-1$. We denote $F_k = \operatorname{Im}(\bar{u}_k) \cong \frac{E_{k+1}}{E_k}$, for any k. Let

$$\{x_{11}, \ldots, x_{1r_1}\}$$

be a basis of E_1 and $a_{11} \mid a_{12} \mid \ldots \mid a_{1r_2} \in A - \{0\}$ such that

$$\{a_{11}x_{11}, \ldots, a_{1r_2}x_{1r_2}\}$$

is a basis of F_1. For $1 \leq j \leq r_2$, we choose $x'_{1j} \in E_2$ such that

$$\bar{u}_1(x'_{1j} + E_1) = a_{1j}x_{1j}, \forall\ 1 \leq j \leq r_2,$$

that is

$$u(x'_{1j}) = a_{1j}x_{1j}.$$

Then

$$\{x'_{11} + E_1, \ldots, x'_{1r_2} + E_1\}$$

is a basis of E_2/E_1. For $k \geq 2$, let

$$\{x_{k1} + E_{k-1}, \ldots, x_{kr_k} + E_{k-1}\}$$

be a basis of $\frac{E_k}{E_{k-1}}$ and

$$a_{k1} \mid a_{k2} \mid \ldots \mid a_{kr_{k+1}} \in A - \{0\}$$

such that

$$\{a_{k1}x_{k1} + E_{k-1}, \ldots, a_{kr_{k+1}}x_{kr_{k+1}} + E_{k-1}\}$$

is a basis of F_k. We choose $x'_{k1}, \ldots, x'_{kr_{k+1}} \in E_{k+1}$ such that

$$\bar{u}_k(x'_{kj} + E_k) = a_{kj}x_{kj} + E_{k-1},\ j = \overline{1, r_{k+1}}.$$

Then

$$\{x'_{kj} + E_k \mid j = \overline{1, r_{k+1}}\}$$

is a basis of $\frac{E_{k+1}}{E_k}$ since $\bar{u}_k$ is injective. Then one can prove as in the case $n = 3$ that the set of elements

$$B := \{x_{11}, \ldots, x_{1r_1}, x'_{11}, \ldots, x'_{1r_2}, x'_{21}, \ldots, x'_{2r_3}, \ldots, x'_{n-1,1}, \ldots, x'_{n-1,r_n}\}$$

is a basis of E. Performing the appropriate changes of coordinates in each factor space, the matrix of u in this basis looks as in the following:

Theorem 3.3. *Let A be a principal ideals domain and let E be a finite free A–module of rank m. Let $u \in \mathrm{End}_A(E)$ such that $u^n = 0$ and $u^{n-1} \neq 0,\ n \geq 2$. There exists a basis B of E such that the matrix of u in the basis B is of the form:*

$$M_B(u) = \left(\begin{array}{c|c|c|c|c|c} 0 & \Lambda_1 & \Delta_{11} & \Delta_{12} & \dots & \Delta_{1,n-2} \\ \hline 0 & 0 & \Gamma_1\Lambda_2 & \Delta_{22} & \dots & \Delta_{2,n-2} \\ \hline 0 & 0 & 0 & \Gamma_2\Lambda_3 & \dots & \Delta_{3,n-2} \\ \hline \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \Gamma_{n-2}\Lambda_{n-1} \\ \hline 0 & 0 & 0 & 0 & \dots & 0 \end{array}\right),$$

where $\Lambda_1 = \left(\frac{\mathrm{diag}(a_{11}, \dots, a_{1r_2})}{0}\right)$ is of size $r_1 \times r_2$ and has the last $r_1 - r_2$ rows 0, $\Lambda_k = \mathrm{diag}(a_{k1}, \dots, a_{kr_{k+1}})$, has the size $r_{k+1} \times r_{k+1},\ k \geq 2$, Γ_k is left invertible and of size $r_{k+1} \times r_{k+2}$, for any k, and Δ_{ij} is of size $r_i \times r_{j+2}$, for any i, j. Moreover, for any k, the elements $a_{k1} \mid a_{k2} \mid \dots \mid a_{kr_{k+1}} \in A - \{0\}$ are the invariants of the A–free modules $\bar{u}_k(\frac{E_{k+1}}{E_k}) \subset \frac{E_k}{E_{k-1}},\ k = \overline{1, n-1}$.

4 Applications

Let $A := K[[y]],\ S := K[[x, y]]$, and $R_n := A[[x]]/(x^n), n \geq 2$.

Proposition 4.1. *(i) Let T be a $m \times m$–matrix with entries in A such that $T^n = 0$. Then the pair of matrices*

$$(x \,\mathrm{Id}_m - T, x^{n-1} \,\mathrm{Id}_m + x^{n-2}T + \ldots + T^{n-1})$$

is a matrix factorization of x^n over S which defines a MCM R_n–module.

(ii) Every MCM R_n–module has a matrix factorization of x^n over S of the form

$$(x \,\mathrm{Id}_m - T, x^{n-1} \,\mathrm{Id}_m + x^{n-2}T + \ldots + T^{n-1}),$$

for some square matrix T with the entries in A such that $T^n = 0$.

Proof. Any MCM R_n–module M is free over A of finite rank. Therefore, giving a MCM R_n– module M is equivalent with giving the action of x on the free A–module M, that is with giving an endomorphism $u \in \mathrm{End}_A(M)$ such that $u^n = 0$ which can be represented by its matrix T in some basis of M over A. Obviously, $T^n = 0$. If T is a $m \times m$–matrix with entries in A such that $T^n = 0$, then the pair of matrices

$$((x \,\mathrm{Id}_m - T), (x^{n-1} \,\mathrm{Id}_m + x^{n-2}T + \ldots + T^{n-1})),$$

is a matrix factorization of x^n over $K[[x,y]]$ which defines a MCM R_n–module M and the action of x on the finite free A–module M is given by the matrix T. Conversely, let us consider a MCM R_n–module M whose minimal free R–resolution is

$$\ldots \xrightarrow{\bar{\psi}} R^q \xrightarrow{\bar{\varphi}} R^q \xrightarrow{\bar{\psi}} R^q \xrightarrow{\bar{\varphi}} R^q \longrightarrow M \longrightarrow 0,$$

where (φ,ψ) is a matrix factorization of x^n over $K[[x,y]]$ which defines M. Let $m := \operatorname{rank}_A M$, let T be the nilpotent $m\times m$–matrix with entries in A which gives the action of x on the finite free A–module M, and let N be the MCM R_n–module given by the periodic resolution

$$\ldots \xrightarrow{\bar{\mu}} R^m \xrightarrow{\bar{\nu}} R^m \xrightarrow{\bar{\mu}} R^m \xrightarrow{\bar{\nu}} R^m \longrightarrow N \longrightarrow 0,$$

where

$$\nu = x\operatorname{Id}_m - T,\ \ \mu = x^{n-1}\operatorname{Id}_m + x^{n-2}T + \ldots + T^{n-1}.$$

Then N is an A–free module of rank m and the action of x over N is given by T. This means that the R–modules M and N are isomorphic, hence the module M has the matrix factorization (ν,μ). □

Remark 4.2. *The matrix (ν,μ) from (ii) can be not reduced, as we show in the following:*

Example 4.3. *Let us consider the MCM R_3 module given by the matrix factorization $\left(\varphi := \begin{pmatrix} x & -y \\ 0 & x^2 \end{pmatrix}, \psi := \begin{pmatrix} x^2 & y \\ 0 & x \end{pmatrix}\right)$. Then, as A–module, M has rank 3 and the action of x on M is given by the matrix $T := \begin{pmatrix} 0 & y & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, that is the matrix factorization (ν,μ) is given by*

$$\nu = \begin{pmatrix} x & -y & 0 \\ 0 & x & -1 \\ 0 & 0 & x \end{pmatrix}, \mu = \begin{pmatrix} x^2 & xy & y \\ 0 & x^2 & x \\ 0 & 0 & x^2 \end{pmatrix}.$$

As an immediate consequence of the above proposition we get the known form of the indecomposable MCM modules over $R = k[[x,y]]/(x^2)$ (see [BGS, Proposition 4.1], [Y, Example 6.5]).

Proposition 4.4. *Let M be an indecomposable MCM–module over $K[[x,y]]/(x^2)$. Then M has a matrix factorization of the following form:*

$$((x),(x)),\ \ \textit{or}\ \left(\begin{pmatrix} x & y^n \\ 0 & x \end{pmatrix}, \begin{pmatrix} x & -y^n \\ 0 & x \end{pmatrix}\right),$$

for some positive integer n.

Proof. Let M be a MCM–module over $K[[x,y]]/(x^2)$. If the $m\times m$–matrix T over A defines the action of x over M, then $T^2=0$, and $(\varphi,\psi)=((x\,\mathrm{Id}_m+T),(x\,\mathrm{Id}_m-T))$ is a matrix factorization over $K[[x,y]]$ of M. Next we apply Theorem 3.1. $\square$

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