An observation on the determinant of a Sylvester-Kac type matrix

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Abstract

Based on a less-known result, we prove a recent conjecture concerning the determinant of a certain Sylvester-Kac type matrix related to some Lie Algebras. The determinant of an extension of that matrix is presented.

1 Introduction

Matrices and Lie algebras have an interesting long relation and share many problems. In a recent paper, Z. Hu and P.B. Zhang consider in [11] the polynomial

$$\det(z_0 I + z_1 A_1 + \cdots + z_s A_s),$$

where $A_1, \ldots, A_s$ are square matrices of the same order the I the identity matrix. Then they calculate the determinant of the finite dimensional irreducible representations of $sl(2, F)$, and show that is either zero or a product of some irreducible quadratic polynomials. In addition, it is proved that a finite dimensional Lie algebra is solvable if and only if the characteristic polynomial is completely reducible. For their purposes, they consider a specialised tridiagonal matrix with zero main diagonal, $(1, 2, \ldots, n)$ superdiagonal, and $(n, n-1, \ldots, 1)$ subdiagonal. Then they establish a conjecture, proved in two very particular cases.

The aim of this short note is to prove that conjecture based on a less-known result by W. Chu in [3]. We also provide a new general formula containing other particular known determinants. This formula can be used to extend [11], and useful both in Lie algebras and matrix theory.

2 The conjecture

Quite recently, in order to find formulas for the determinants of some Lie algebras, Z. Hu and P.B. Zhang proposed in [11] the following conjecture.

Conjecture 1. *The determinant of the matrix* $(n + 1) \times (n + 1)$

$$J_n(z_0, z_1) = \begin{pmatrix} z_0 + nz_1 & 1 & & & & & \\ n & z_0 + (n-2)z_1 & 2 & & & & \\ & n-1 & z_0 + (n-4)z_1 & \ddots & & & \\ & & \ddots & \ddots & n-1 & & \\ & & & 2 & z_0 - (n-2)z_1 & n \\ & & & & 1 & z_0 - nz_1 \end{pmatrix}$$

is

$$\prod_{k=0}^{n} \left(z_0 - (n - 2k)\sqrt{z_1^2 + 1} \right) .$$

Notice that Conjecture 1 is equivalent to state that the eigenvalues of $J_n(0, z_1)$ are

$$\pm (n - 2k)\sqrt{z_1^2 + 1}, \quad \text{for } k = 0, 1, \dots, \lfloor n/2 \rfloor.$$

The matrix $J_n(z_0, z_1)$ can be easily identified as an extension of the so-called Sylvester-Kac matrix. In fact, setting $z_1 = 0$ we find the characteristic matrix of the Sylvester-Kac matrix, also known as Clement matrix,

$$\begin{pmatrix} 0 & 1 & & & \\ n & 0 & 2 & & \\ & n-1 & 0 & \ddots & \\ & & \ddots & \ddots & n \\ & & & 1 & 0 \end{pmatrix} .$$

The characteristic polynomial of this matrix (that is, $\det J_n(x, 0)$) was first conjecture in [20], by the 19th century British mathematician James Joseph Sylvester celebrated, among other facets, as the founder of the American Journal of Mathematics, in 1878.

A fully comprehensive list of results on the different proofs for Sylvester's conjecture and the eigenpairs of non-trivial extensions of the Sylvester-Kac matrix can be found in [1–10, 12–19, 21, 22].

The aim of this short note is to prove Conjecture 1 based on a result by W. Chu in [3]. We also provide a general result containing other particular known determinants.

3   An extension to the Sylvester-Kac matrix

In 2010, cleverly based on two generalized Fibonacci sequences, W. Chu proved the following theorem.

Theorem 3.1 ([3]). *The determinant of the matrix* $(n+1) \times (n+1)$

$$
M_n(x,y,u,v) = \begin{pmatrix}
x & u \\
nv & x+y & 2u \\
 & (n-1)v & x+2y & \ddots \\
 & & \ddots & \ddots & n-1 \\
 & & & 2v & x-(n-1)y & nu \\
 & & & & v & x+ny
\end{pmatrix}
$$

is

$$
\prod_{k=0}^{n} \left(x + \frac{ny}{2} + \frac{n-2k}{2}\sqrt{y^2+4uv} \right) .
$$

Of course, the formula for the determinant in Theorem 3.1 can be rewritten as

$$
\prod_{k=0}^{\lfloor n/2 \rfloor} \left(\left(x + \frac{ny}{2}\right)^2 - \frac{(n-2k)^2}{4}(y^2+4uv) \right) .
$$

Now setting $x = z_0 + nz_1$, $y = -2z_1$, and $u = v = 1$, we prove immediately Conjecture 1.

Moreover, in the spirit of [1,9,10], using Theorem 3.1, we can also conclude the following theorem.

Theorem 3.2. *The eigenvalues of*

$$
M_n^{\pm}(a,b,r) = \begin{pmatrix}
nar & b \\
na & ((n-1)a \pm b)r & 2b \\
 & (n-1)a & ((n-2)a \pm 2b)r & 3b \\
 & & (n-2)a & \ddots & \ddots \\
 & & & \ddots & \ddots & nb \\
 & & & & a & \pm nbr
\end{pmatrix}
$$

are

$$
\frac{1}{2}\left(nr(a \pm b) + (n-2k)\sqrt{4ab + r^2(a \mp b)^2} \right) ,
$$

for $k = 0, 1, \ldots, n$.

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