DOI: 10.2478/auom-2020-0015
An. Şt. Univ. Ovidius Constanţa
sciendo
Vol. 28(1),2020, 219–236

The commutative quotient structure of m-idempotent hyperrings

Azam Adineh Zadeh, Morteza Norouzi, and Irina Cristea

Abstract

The α^*-relation is a fundamental relation on hyperrings, being the smallest strongly regular relation on hyperrings such that the quotient structure R/α^* is a commutative ring. In this paper we introduce on hyperrings the relation ξ_m^*, which is smaller than α^*, and show that, on a particular class of m-idempotent hyperrings R, it is the smallest strongly regular relation such that the quotient ring R/ξ_m^* is commutative. Some properties of this new relation and its differences from the α^*-relation are illustrated and discussed. Finally, we show that ξ_m^* is a new representation for α^* on this particular class of m-idempotent hyperrings.

1 Introduction

Regular equivalences play a fundamental role in algebra. They can be defined, for example, on graphs, classical structures (as groups or rings), or hyperstructures (hypergroups, hyperrings, hypermodules). Using the terminology in [7], two elements are regularly equivalent if they are equally related to equivalent other elements from the support set. In Group theory these relations are called congruences and defined as equivalences that are compatible with the group operation. Given a congruence on a group, the set of the equivalence classes determined by the congruence forms a group structure, i.e. a quotient group.

Key Words: Hyperring, m-idempotent hyperring, strongly regular relation.
2010 Mathematics Subject Classification: Primary 20N20.
Received: 23.04.2019
Accepted: 31.05.2019

On the other hand, since in a hypergroup the result of the interaction (i.e. the result of the hyperproduct) between two elements is a set, it has sense to speak about regular and strongly regular equivalences. More exactly, an equivalence relation ρ on a hypergroup $(H, \circ)$ is called *regular*, if $a\rho b$ and $c\rho d$ implies that, for any $x \in a \circ c$, there exists $y \in b \circ d$ such that $x\rho y$, and for any $u \in b \circ d$ there exists $v \in a \circ c$ such that $u\rho v$. Besides, ρ is called *strongly regular* if, for all $x \in a \circ c$ and for all $y \in b \circ d$, we have $x\rho y$. It is well known the property saying that the quotient of a hypergroup modulo a regular (strongly regular) equivalence is a hypergroup (group). Extending these definitions to hyperrings, we say that an equivalence is (strongly) regular on a hyperring if it is (strongly) regular with respect both addition and multiplication hyperoperations. The smallest (with respect to inclusion) strongly regular relations, having a certain property, defined on a hyperstructure are called *fundamental relations*, because they are as a bridge, as a fundamental tool, between the hyperstructures (on which they are defined) and the related classical structures (with the same behaviour, i.e. they connect hypergroups with groups, hyperrings with rings and so on). On hypergroups, Koskas [17] defined the β-relation such that the associated quotient structure is a group. Another fundamental relation on hypergroups is the γ-relation defined by Freni [13] such that the quotient is a commutative group.

Obviously, since on a hyperring two (hyper)operations are defined, there are several fundamental relations that can be considered. The first one, the γ-relation (it is denoted in the same way as Freni's relation on hypergroups, but it has a different form) was introduced by Vougiouklis [21] on general hyperrings (where both the addition and multiplications are hyperoperations) such that the related quotient structure is a ring. Besides, a commutative ring can be obtained as a quotient structure of a hyperring modulo the α^*-relation [12]. Clearly, when we want to link hyperstructures with structures by this method of factorization via a strongly regular relation, we need to consider particular structures, as nilpotent groups [1], engel groups [2], solvable groups [15], Boolean rings [10], commutative rings with identity [3], commutative modules [5].

Norouzi and Cristea [19] have recently started the study of a new relation defined on general hyperrings, denoted by ε_m, which is a relation smaller than the γ-relation, and which is not transitive in general. They have proved that the transitive closure ε_m^* is a fundamental relation on a particular subclass of m-idempotent hyperrings [20]. Moreover, on this type of m-idempotent hyperrings, the relations ε_m^* and γ^* are equal. Motivated by all these aspects, in this note we introduce and study the relation ξ_m on general hyperrings and prove that its transitive closure ξ_m^* is a strongly regular relation on hyperrings satisfying a certain property, while in general the related quotient ring is not commutative. It is commutative only if we define ξ^*_m on m-idempotent hyperrings. Moreover, relations between all these regular equivalences ε_m, γ, α, ξ_m defined on hyperrings are investigated, concluding that there is a subclass of weak commutative m-idempotent hyperrrings where all these equivalences coincide.

2 Preliminaries

In this section, we fix the notation and give some basic definitions and results concerning hyperrings and strongly regular relations, which will be used throughout the paper. For more details about hyperstructures theory, specially hyperrings, we refer the readers to [8, 11, 21, 22] and their references.

Let H be a nonempty subset and set "$\circ$" : $H \times H \longrightarrow \mathcal{P}^*(H)$ where $\mathcal{P}^*(H)$ is the set of all nonempty subsets of H. Then the mapping "$\circ$" is called a hyperoperation on H and $(H, \circ)$ is said to be a *hypergroupoid*, where for $x \in H$ and $A, B \in \mathcal{P}^*(H)$, we have $A \circ B = \bigcup_{a \in A, b \in B} a \circ b$ and $A \circ x = A \circ \{x\}$. A hypergroupoid $(H, \circ)$ is called a *semihypergroup* if for all x, y, z of H, we have $(x \circ y) \circ z = x \circ (y \circ z)$. We say that a semihypergroup $(H, \circ)$ is a *hypergroup* if for all $x \in H$, $x \circ H = H \circ x = H$.

A commutative hypergroup $(H, \circ)$ is called *canonical* if (1) there exists $0 \in H$, such that $0 \circ x = \{x\}$, for every $x \in H$; (2) for all $x \in H$ there exists a unique $x^{-1} \in H$, such that $0 \in x \circ x^{-1}$; (3) $x \in y \circ z$ implies $y \in x \circ z^{-1}$.

Definition 2.1. [11] An algebraic system $(R, +, \cdot)$ is said to be a
(1) *(general) hyperring*, if $(R, +)$ is a hypergroup, $(R, \cdot)$ is a semihypergroup, and " $\cdot$ " is distributive with respect to " $+$ ". (In this case, if $(R, +)$ is a semihypergroup, then $(R, +, \cdot)$ is called a semihyperring.)
(2) *Krasner hyperring* ([18]), if $(R, +)$ is a canonical hypergroup and $(R, \cdot)$ is a semigroup such that 0 is a zero element (called also *absorbing element*), i.e. for all $x \in R$, we have $x \cdot 0 = 0$.
(3) *multiplicative hyperring*, if $(R, +)$ is a commutative group, $(R, \cdot)$ is a semihypergroup, and $x \cdot (-y) = (-x) \cdot y = -(x \cdot y)$, $x \cdot (y + z) \subseteq x \cdot y + x \cdot z$ and $(y + z) \cdot x \subseteq y \cdot x + z \cdot x$, for all $x, y, z \in R$.

A hyperring $(R, +, \cdot)$ is called *additive* (*multiplication*), if $x \cdot y$ $(x + y)$ is a singleton, for all $x, y \in R$ [22]. Moreover, a nonempty subset I of a hyperring $(R, +, \cdot)$ is a *hyperideal*, if $x, y \in I$ implies $x + y \subseteq I$ and for $r \in R$ we have $r \cdot x \cup x \cdot r \subseteq I$.

An equivalence relation ρ on a hypergroup $(H, \circ)$ is called *regular*, if $a \rho b$ and $c \rho d$, then $(a \circ c)\ \overline{\rho}\ (b \circ d)$ for $a, b, c, d \in R$, that is,

$$\forall x \in a \circ c,\ \exists y \in b \circ d\ :\ x \rho y \quad \text{and} \quad \forall u \in b \circ d,\ \exists v \in a \circ c\ :\ u \rho v.$$

Besides, ρ is called *strongly regular* if, for all $x \in a \circ c$ and for all $y \in b \circ d$, we have $x\rho y$, denoted by $(a \circ c)\ \overline{\overline{\rho}}\ (b \circ d)$. Now, let ρ be an equivalence relation on the (semi)hypergroup $(H, \circ)$ and consider the hyperoperation $\rho(x) \otimes \rho(y) = \{\rho(z) \mid z \in \rho(x) \circ \rho(y)\}$ on the quotient $H/\rho = \{\rho(x) \mid x \in H\}$. Then, by [11], ρ is regular (strongly regular) on H if and only if $(H/\rho, \otimes)$ is a (semi)hypergroup ((semi)group).

Now briefly summarize the main properties of the regular relations on a general hyperring. Let $(R, +, \cdot)$ be a hyperring. We say that a relation ρ is *(strongly) regular* on R, if it is (strongly) regular with respect to both hyperoperations " + " and " $\cdot$ ".

Already in 1990, Vougiouklis ([21]) introduced a strongly regular relation on (semi)hyperrings, denoted by γ^*, proving that it is a fundamental relation. We recall here its definition. Let $(R, +, \cdot)$ be a (semi)hyperring and $x, y \in R$. Then

$$x\gamma y \Longleftrightarrow \exists n \in \mathbb{N},\ \exists k_1, \ldots, k_n \in \mathbb{N},\ \exists z_{i1}, \ldots, z_{ik_i} \in R\ (i = 1, \ldots, n): \\ \{x, y\} \subseteq \sum_{i=1}^{n} (\prod_{j=1}^{k_i} z_{ij}),$$

in other words, two elements x, y are related if and only if there exists a finite hyperaddition of finite hyperproducts of elements in R containing both x and y. Let γ^* be the transitive closure of γ, that is $x\gamma^* y$ if and only if $\exists z_1, \ldots, z_{n+1} \in R$ with $z_1 = x$ and $z_{n+1} = y$ such that $z_1 \gamma z_2 \gamma \ldots z_n \gamma z_{n+1}$. It was shown that γ^* is the smallest strongly regular relation on hyperrings such that the associated quotient $(R/\gamma^*, \oplus, \odot)$ is a classical ring, where $\gamma^*(x) \oplus \gamma^*(y) = \gamma^*(z)$, for all $z \in \gamma^*(x) + \gamma^*(y)$ and $\gamma^*(a) \odot \gamma^*(b) = \gamma^*(d)$, for all $d \in \gamma^*(a) \cdot \gamma^*(b)$. Hence, $(R/\gamma^*, \oplus, \odot)$ is called the *fundamental ring* associated with R obtained by the γ^*-relation. Moreover, consider the following binary relations on $(R, +, \cdot)$:

$$x\beta_{\cdot} y \ \Leftrightarrow\ \exists z_1, \ldots, z_n \in R:\ \{x, y\} \subseteq \prod_{i=1}^{n} z_i,$$
$$x\beta_{+} y \ \Leftrightarrow\ \exists d_1, \ldots, d_n \in R:\ \{x, y\} \subseteq \sum_{i=1}^{n} d_i.$$

According to [11], both $\beta_{\cdot}^*$ and β_{+}^* are fundamental relations on $(R, +, \cdot)$ with respect to hypermultiplication and hyperaddition, respectively, and we have $\beta_{\cdot}^* \subseteq \gamma^*$ and $\beta_{+}^* \subseteq \gamma^*$.

Notice that in the associated fundamental ring R/γ^*, the commutativity as well as the existence of the unit element, is not allays verified, while in the classical rings theory the commutativity of the addition is assumed and it is related with the existence of the unit element in the multiplication law (see [11, page 245]). On the other side, in a hyperring $(R, +, \cdot)$, generally, the hyperaddition $+$ is not commutative and there is no unit element with respect to the hypermultiplication.

In order to overcome this difference and uniformize the property of the fundamental ring R/γ^* to the general theory of classical rings, Davvaz and Vougiouklis [12] defined on hyperrings the strongly regular relation α^* as follow:
If R is a hyperring, we set $\alpha_0 = \{(x, x) \mid x \in R\}$ and for every integer $n \geq 1$, the relation α_n is defined as:

$$\begin{aligned} x\alpha_n y \iff & \exists k_1, \ldots, k_n \in \mathbb{N},\ \exists \sigma \in \mathbb{S}_n \ \text{ and} \\ & \exists z_{i1}, \ldots, z_{ik_i} \in R,\ \exists \sigma_i \in \mathbb{S}_{k_i} \ \text{ for } \ 1 \leq i \leq n \ \text{ such that} \\ & x \in \sum_{i=1}^{n} (\prod_{j=1}^{k_i} z_{ij}) \ \text{ and } \ y \in \sum_{i=1}^{n} A_{\sigma(i)}, \end{aligned}$$

where $A_i = \prod_{j=1}^{k_i} z_{i\sigma_i(j)}$. Now, put $\alpha = \bigcup_{n \geq 0} \alpha_n$. Clearly, α is symmetric and reflexive. The transitive closure of α-relation is denoted by α^*. It was shown [11] that α^* is the smallest strongly regular relation on a hyperring R such that $(R/\alpha^*, \oplus, \odot)$ is a commutative ring. Hence, $(R/\alpha^*, \oplus, \odot)$ is called the *fundamental commutative ring* obtained from the α^*-relation.

3 ξ_m-relation on hyperrings

Following the idea and the methodology initiated by Vougiouklis [21] for the definition of the γ-relation and then used by Davvaz and Vougiouklis [12] for the study of the relation α, in [19] the authors defined on (semi)hyperrings a new relation, denoted by ε_m, smaller than the γ-relation, such that its transitive closure on particular class of hyperrings is the smallest strongly regular relation endowing the quotient set with a ring structure. We first recall here its definition.

Let $(R, +, \cdot)$ be a semihyperring and select a constant m, such that $2 \leq m \in \mathbb{N}$. Consider $\{(x, x) \mid x \in R\} \subseteq \varepsilon_m$ and for all $a, b \in R$ define

$$a\varepsilon_m b \iff \exists n \in \mathbb{N}, \exists (z_1, \ldots, z_n) \in R^n : \{a, b\} \subseteq \sum_{i=1}^{n} z_i^m,$$

where $z_i^m = \underbrace{z_i \cdot z_i \cdot \ldots \cdot z_i}_{m \ \ times}$.

It was proved that $\varepsilon_m \subsetneq \gamma$ and $\varepsilon_m^* \subsetneq \gamma^*$. Moreover, if $(R, +, \cdot)$ is a hyperring such that $(R, \cdot)$ is commutative satisfying the condition

$$B \subseteq \sum_{i=1}^{n} A_i^m \Longrightarrow \exists x_i \in A_i \ (1 \leq i \leq n) : B \subseteq \sum_{i=1}^{n} x_i^m, \tag{1}$$

for all $B, A_1, \ldots, A_n \subseteq R$, then the relation ε_m^* is the smallest strongly regular equivalence on R such that the quotient set R/ε_m^* is a ring (not necessary commutative). Therefore ε_m^* is a fundamental relation on such hyperrings. Besides, the quotient ring R/ε_m^* is not commutative in general.

Now it seems naturally to define a strongly regular relation, smaller than α^*-relation, in order to obtain commutative fundamental rings and this is the principal aim of this study. We define the relation ξ_m^* on general (semi)hyperrings as follow:

Consider a general (semi)hyperring $(R, +, \cdot)$, select a constant m such that $2 \leq m \in \mathbb{N}$ and set $\{(x, x) \mid x \in R\} \subseteq \xi_m$, where we define on R:

$$(x, y) \in \xi_m \Longleftrightarrow \exists n \in \mathbb{N}, \exists z_1, \ldots, z_n \in R, \exists \sigma \in \mathbb{S}_n : x \in \sum_{i=1}^{n} z_i^m \ , \ y \in \sum_{i=1}^{n} z_{\sigma(i)}^m.$$

Clearly, ξ_m is reflexive and symmetric. Also, it is easy to see that $\xi_m \subseteq \alpha$, but in general these two relations are not equal, as one can see in the following example.

Example 3.1. *Define on $R = \{a, b, c, d, e, f, g\}$ a hyperaddition by the following Cayley table and define*

$+$	a	b	c	d	e	f	g
a	$\{a,b\}$	$\{a,b\}$	c	d	e	f	g
b	$\{a,b\}$	$\{a,b\}$	c	d	e	f	g
c	c	c	$\{a,b\}$	f	g	d	e
d	d	d	g	$\{a,b\}$	f	e	c
e	e	e	f	g	$\{a,b\}$	c	d
f	f	f	e	c	e	g	$\{a,b\}$
g	g	g	d	e	c	$\{a,b\}$	f

and a hypermultiplication by taking $x \cdot y = \{a, b\}$ for every $x, y \in R$. Then $(R, +, \cdot)$ is a hyperring ([4]). One finds that $\alpha(a) = \{a, b, f, g\}$ and $\alpha(c) = \{c, d, e\}$, while $\xi_m(a) = \{a, b\}$ and $\xi_m(x) = \{x\}$ for all $x \in R \setminus \{a, b\}$. Thus, $\xi_m \neq \alpha$ (in particular $\xi_m \subsetneq \alpha$). Moreover, since $\xi_m = \xi_m^*$ and $\alpha = \alpha^*$, it follows that $\xi_m^* \neq \alpha^*$.

Next, we define a new subclass of hyperrings $(R, +, \cdot)$, having commutative the multiplicative part $(R, \cdot)$ and satisfying the condition: for any nonempty subsets $B, C, A_1, \ldots, A_n$ of R and a permutation $\sigma \in \mathbb{S}_n$, if $B \subseteq \sum_{i=1}^n A_i^m$ and $C \subseteq \sum_{i=1}^n A_{\sigma(i)}^m$, then there exist $x_i \in A_i$, for $1 \le i \le n$, such that

$$B \subseteq \sum_{i=1}^{n} x_i^m \quad \text{and} \quad C \subseteq \sum_{i=1}^{n} x_{\sigma(i)}^m. \tag{2}$$

Note that relation (2) is valid if and only if, for all $A_1, \ldots, A_n \subseteq R$, there exist $x_i \in A_i$, $1 \le i \le n$, such that $\sum_{i=1}^n A_i^m \subseteq \sum_{i=1}^n x_i^m$ and $\sum_{i=1}^n A_{\sigma(i)}^m \subseteq \sum_{i=1}^n x_{\sigma(i)}^m$. Moreover, similar to what happens for the relation ε_m^* (see [19, Proposition 3.2]), if we consider the set $X = \bigcup\{\sum_{i=1}^n A_i^m \mid A_1, \ldots, A_n \subseteq R\}$, then we get $\xi_m^* \subseteq X \times X$, while the converse implication is not generally valid. It holds under supplementary conditions, stated in the following result.

Proposition 3.2. *Let $(R, +, \cdot)$ be a hyperring satisfying relation (2) and such that there exists $0 \in R$ with the property that $x + 0 = \{x\} = 0 + x$ and $x \cdot 0 = \{0\}$, for all $x \in R$. If $A_1, \ldots, A_n$ are hyperideals of R, then X is an equivalence class of ξ_m^* and ξ_m is transitive.*

Proof. Since $\varepsilon_m \subseteq \xi_m$, the proof is completed by [19, Proposition 3.2]. $\square$

We know that α^* is a strongly regular relation on hyperrings [11]. In the following we show that ξ_m^* is, under some special conditions, a strongly regular relation, too.

Theorem 3.3. *The relation ξ_m^* is a strongly regular relation on a hyperring $(R, +, \cdot)$ satisfying relation (2).*

Proof. Let $a' \xi_m^* a$ and $b' \xi_m^* b$ for $a, b, a', b' \in R$. Then, for $x_1^{s+1}, y_1^{t+1} \in R$, with $x_1 = a'$, $x_{s+1} = a$, $y_1 = b'$ and $y_{t+1} = b$ we have $x_1 \xi_m x_2 \xi_m \ldots \xi_m x_s \xi_m x_{s+1}$ and $y_1 \xi_m y_2 \xi_m \ldots \xi_m y_t \xi_m y_{t+1}$. So, for every $i \in \{1, \ldots, s\}$ there exist $n_i \in \mathbb{N}$, $z_{i1}, \ldots, z_{in_i} \in R$ and $\sigma_i \in S_{n_i}$ such that

$$x_i \in \sum_{l=1}^{n_i} z_{il}^m \quad \text{and} \quad x_{i+1} \in \sum_{l=1}^{n_i} z_{i\sigma_i(l)}^m.$$

Besides, for all $j \in \{1, \ldots, t\}$ there exist $n_j' \in \mathbb{N}$, $d_{j1}, \ldots, d_{jn_j'} \in R$ and $\tau_j \in S_{n_j'}$, where

$$y_j \in \sum_{k=1}^{n_j'} d_{jk}^m \quad \text{and} \quad y_{j+1} \in \sum_{k=1}^{n_j'} d_{j\tau_j(k)}^m.$$

Then, for $i = j = 1$ we can define $\mu \in \mathbb{S}_{n_1+n_1'}$ such that $\mu(l) = \sigma_1(l)$ for all $1 \leq l \leq n_1$ and $\mu(k) = k$ for all $1 \leq k \leq n_1'$, and put $z_{11} = b_1, \ldots, z_{1n_1} = b_{n_1}$, $d_{11} = b_{n_1+1}, \ldots, d_{1n_1} = b_{n_1+n_1'}$. Now, by

$$x_i + y_1 \subseteq \sum_{l=1}^{n_i} z_{il}^m + \sum_{k=1}^{n_1'} d_{1k}^m \quad \text{and} \quad x_{i+1} + y_1 \subseteq \sum_{l=1}^{n_i} z_{i\sigma_i(l)}^m + \sum_{k=1}^{n_1'} d_{1k}^m,$$

we have

$$x_1 + y_1 \subseteq \sum_{c=1}^{n_1+n_1'} b_c^m \quad \text{and} \quad x_2 + y_1 \subseteq \sum_{c=1}^{n_1+n_1'} b_{\mu(c)}^m,$$

which implies that $q_1 \xi_m q_2$, for all $q_1 \in x_1 + y_1$ and $q_2 \in x_2 + y_1$. Therefore, similarly we can choose some elements $q_1, \ldots, q_{s+t} \in R$ with $q_i \in x_i + y_1$, for $i = 1, 2, \ldots, s$, and $q_{s+j} \in x_{s+1} + y_{j+1}$, for $j = 1, \ldots, t$, such that

$$a' + b' \ni q_1 \xi_m q_2 \xi_m \ldots \xi_m q_s \xi_m q_{s+1} \xi_m \ldots \xi_m q_{s+t-1} \xi_m q_{s+t} \in a + b.$$

Hence, for all $q' \in a' + b'$ and $q \in a + b$, we have $q' \xi_m^* q$. It means that $a + b \,\overline{\overline{\xi_m^*}}\, a' + b'$. Then ξ_m^* is strongly regular on $(R, +)$. Similarly, one proves that ξ_m^* is a strongly regular relation on $(R, \cdot)$, too. $\square$

It is well-known that the quotient structure, constructed by a strongly regular relation on a hyperring, is a ring, property conserved also for the ξ_m^*-relation, as it is shown in the next result.

Theorem 3.4. *If R is a hyperring satisfying relation (2), then the quotient $R/\xi_m^* = \{\xi_m^*(x) \mid x \in R\}$ is a ring.*

Proof. For all $\xi_m^*(a), \xi_m^*(b) \in R/\xi_m^*$ consider $\xi_m^*(a) \oplus \xi_m^*(b) = \{\xi_m^*(c) \mid c \in \xi_m^*(a) + \xi_m^*(b)\}$ and $\xi_m^*(a) \odot \xi_m^*(b) = \{\xi_m^*(z) \mid z \in \xi_m^*(a) \cdot \xi_m^*(b)\}$. Let $p, q \in \xi_m^*(a) + \xi_m^*(b)$. Then there exist $a', a'' \in \xi_m^*(a)$ and $b', b'' \in \xi_m^*(b)$ such that $p \in a' + b'$ and $q \in a'' + b''$. By strongly regularity of ξ_m^*, we have $a' + b' \,\overline{\overline{\xi_m^*}}\, a'' + b''$. Hence, $p \xi_m^* q$ and so $\xi_m^*(p) = \xi_m^*(q)$. Therefore $\xi_m^*(a) \oplus \xi_m^*(b) = \{\xi_m^*(c)\}$, for all $c \in \xi_m^*(a) + \xi_m^*(b)$. Similarly, $\xi_m^*(a) \odot \xi_m^*(b) = \{\xi_m^*(z)\}$, for all $z \in \xi_m^*(a) \cdot \xi_m^*(b)$. Therefore, $\oplus$ and $\odot$ are trivial hyperoperations, i.e. they are both operations, and thus $(R/\xi_m^*, \oplus, \odot)$ is a ring. $\square$

Moreover, the quotient R/α^* is always a commutative ring [11], while R/ξ_m^* is not commutative in general, as one can see in the following example.

Example 3.5. *Consider the hyperring $R = \{a, b, c, d, e, f, g\}$ defined in Example 3.1. We have*

$$R/\xi_m^* = \big\{\xi_m^*(a) = \{a, b\}, \xi_m^*(c), \xi_m^*(d), \xi_m^*(e), \xi_m^*(f), \xi_m^*(g)\big\}$$

where $\xi_m^*(x) = \{x\}$ for all $x \notin \{a, b\}$. By Theorem 3.4, we have $\xi_m^*(d) \oplus \xi_m^*(e) = \xi_m^*(f)$, since $\xi_m^*(d) + \xi_m^*(e) = d + e = \{f\}$. On the other hand, $\xi_m^*(e) \oplus \xi_m^*(d) = \xi_m^*(g)$, since $\xi_m^*(e) + \xi_m^*(d) = e + d = \{g\}$. But, $\xi_m^*(g) \neq \xi_m^*(f)$. Hence, R/ξ_m^* is not commutative. Note that R is neither m-idempotent for all $2 \leq m \in \mathbb{N}$, since $x^m = \{a, b\}$ for all $x \in R$.

Actually, if R is not an m-idempotent hyperring, then R/ξ_m^* is not a commutative ring in general.

Theorem 3.6. *If $(R, +, \cdot)$ is an m-idempotent hyperring satisfying relation (2), then $(R/\xi_m^*, \oplus, \odot)$ is a commutative ring.*

Proof. By Theorem 3.4, $(R/\xi_m^*, \oplus, \odot)$ is a ring. We show that $\xi_m^*(x_1) \oplus \xi_m^*(x_2) = \xi_m^*(x_2) \oplus \xi_m^*(x_1)$, for every $\xi_m^*(x_1), \xi_m^*(x_2) \in R/\xi_m^*$. Let $a \in x_1 + x_2$ and $b \in x_2 + x_1$. Then, we have $\xi_m^*(x_1) \oplus \xi_m^*(x_2) = \xi_m^*(a)$ and $\xi_m^*(x_2) \oplus \xi_m^*(x_1) = \xi_m^*(b)$. Now, consider $\sigma \in \mathbb{S}_n$ such that $\sigma(1) = 2$ and $\sigma(2) = 1$. Since R is m-idempotent, we have

$$a \in x_1 + x_2 \subseteq x_1^m + x_2^m \quad \text{and} \quad b \in x_{\sigma(1)} + x_{\sigma(2)} \subseteq x_{\sigma(1)}^m + x_{\sigma(2)}^m$$

which implies that $a \xi_m b$ and so $\xi_m^*(a) = \xi_m^*(b)$. Hence, $\xi_m^*(x_1) \oplus \xi_m^*(x_2) = \xi_m^*(x_2) \oplus \xi_m^*(x_1)$ and so $\oplus$ is commutative on R/ξ_m^*. Moreover, since $(R, \cdot)$ is commutative, it clearly follows that $\odot$ is commutative on R/ξ_m^*. Therefore, $(R/\xi_m^*, \oplus, \odot)$ is a commutative ring. $\square$

Theorem 3.7. *If $(R, +, \cdot)$ is an m-idempotent hyperring satisfying relation (2), then ξ_m^* is the smallest strongly regular equivalence relation on R such that the quotient R/ξ_m^* is a commutative ring.*

Proof. For a strongly regular equivalence relation θ such that R/θ is a commutative ring, consider the canonical projection $\phi : R \longrightarrow R/\theta$ by $\phi(x) = \theta(x)$ for all $x \in R$. Let $x \xi_m y$ for $x, y \in R$. Then there exist $n \in \mathbb{N}$, $\sigma \in S_n$ and $z_1, z_2, \ldots, z_n \in R$ such that $x \in \sum_{i=1}^{n} z_i^m$ and $y \in \sum_{i=1}^{n} z_{\sigma(i)}^m$. So $\phi(x) = \oplus_{i=1}^{n} \theta(z_i)^m$ and $\phi(y) = \oplus_{i=1}^{n} \theta(z_{\sigma(i)})^m$. which implies that $\theta(x) = \theta(y)$ since the quotient R/θ is a commutative ring. Hence, $\xi_m \subseteq \theta$ and so $\xi_m^* \subseteq \theta^* = \theta$. Thus, ξ_m^* is the smallest strongly regular equivalence on an m-idempotent hyperring R satisfying relation (2), such that R/ξ_m^* is a commutative ring. $\square$

The fundamental relation α^* is the smallest strongly regular relation on hyperrings such that the related quotient is a commutative ring [11]. By Theorem 3.7, we can conclude that the relation ξ_m^* is a fundamental relation on m-idempotent hyperrings satisfying relation (2) such that the corresponding quotient ring is commutative.

According with the definition of the γ-relation on (semi)hypergroups introduced by Freni ([13]), the relations γ_+ and $\gamma_.$ can be also introduced ([11]) on a hyperring $(R, +, \cdot)$ with respect to " $+$ " and " $\cdot$ " as follows:

$$x\gamma_+ y \Longleftrightarrow \exists\ t \in \mathbb{N},\ \exists\ y_1, y_2, \ldots, y_t \in R,\ \tau \in S_n;\ x \in \sum_{i=1}^{t} y_i \ \text{ and } \ y \in \sum_{i=1}^{t} y_{\tau(i)},$$

$$x\gamma_. y \Longleftrightarrow \exists\ n \in \mathbb{N},\ \exists\ z_1, z_2, \ldots, z_n \in R,\ \sigma \in S_n;\ x \in \prod_{i=1}^{n} z_i \ \text{ and } \ y \in \prod_{i=1}^{n} z_{\sigma(i)}.$$

The transitive closures of γ_+ and $\gamma_.$ are denoted by γ_+^* and $\gamma_.^*$, respectively. Clearly, $\gamma_+ \cup \gamma_. \subseteq \alpha$ and $\gamma_+^* \cup \gamma_.^* \subseteq \alpha^*$. In the following we compare the γ_+-relation with ξ_m and ξ_m^* (while in the next section we study the connection between $\gamma_.$ and ξ_m).

Theorem 3.8. *For all m-idempotent hyperrings, $\gamma_+ \subseteq \xi_m$.*

Proof. In an m-idempotent hyperring R we have $x \in x^m$, for all $x \in R$. Hence, for $x, y \in R$ it holds

$$x\gamma_+ y \Longleftrightarrow \exists t \in \mathbb{N},\ \exists z_1, z_2, \ldots, z_t \in R,\ \exists \sigma \in \mathbb{S}_n :\ x \in \sum_{i=1}^{t} z_i \text{ and } y \in \sum_{i=1}^{t} z_{\sigma(i)},$$

$$\Longrightarrow x \in \sum_{i=1}^{t} z_i^m \text{ and } y \in \sum_{i=1}^{t} z_{\sigma(i)}^m.$$

Then $x\xi_m y$ and so $\gamma_+ \subseteq \xi_m$. □

In the following example, we can see that, if R is not an m-idempotent hyperring, then $\gamma_+ \nsubseteq \xi_m$.

Example 3.9. *Consider the hyperring $R = \{a, b, c, d, e, f, g\}$ defined in Example 3.1. Then we have*

$$\gamma_+(a) = \gamma_+(b) = \{a, b, f, g\} \quad \textit{and} \quad \gamma_+(c) = \{c, d, e\}.$$

Hence, $\gamma_+ = \alpha$ and so $\xi_m \subseteq \gamma_+$.

Example 3.10. *Consider the commutative multiplicative hyperring $(\mathbb{Z}_3, \oplus, *)$ defined as:*

$\oplus$	$\bar{0}$	$\bar{1}$	$\bar{2}$
$\bar{0}$	$\bar{0}$	$\bar{1}$	$\bar{2}$
$\bar{1}$	$\bar{1}$	$\bar{2}$	$\bar{0}$
$\bar{2}$	$\bar{2}$	$\bar{0}$	$\bar{1}$

$*$	$\bar{0}$	$\bar{1}$	$\bar{2}$
$\bar{0}$	$\{\bar{0}\}$	$\{\bar{0}\}$	$\{\bar{0}\}$
$\bar{1}$	$\{\bar{0}\}$	$\mathbb{Z}_3$	$\mathbb{Z}_3$
$\bar{2}$	$\{\bar{0}\}$	$\mathbb{Z}_3$	$\mathbb{Z}_3$

It is an m-idempotent multiplicative hyperring for all $2 \leq m \in \mathbb{N}$. For every $x \in \mathbb{Z}_3$ we have $\gamma_+^*(x) = \{x\}$ and $\xi_m^*(x) = \mathbb{Z}_3$. So, $\gamma_+^* \subsetneq \xi_m^*$.

Theorem 3.11. *If R is an m-idempotent Krasner hyperring, then*

$$\beta_+ = \gamma_+ = \xi_m = \varepsilon_m.$$

Proof. By Theorem 3.8, we have $\gamma_+ \subseteq \xi_m$. Since R is an m-idempotent Krasner hyperring, it follows that $z = z^m$ for all $z \in R$. Hence, if $a\xi_m b$ for $a, b \in R$, then there exist $n \in \mathbb{N}$, $z_1, z_2, \ldots, z_n \in R$ and $\sigma \in \mathbb{S}_n$ such that

$$a \in \sum_{i=1}^{n} z_i^m = \sum_{i=1}^{n} z_i \quad \text{and} \quad b \in \sum_{i=1}^{n} z_{\sigma(i)}^m = \sum_{i=1}^{n} z_{\sigma(i)}.$$

This means that $a\gamma_+ b$. Then, $\xi_m \subseteq \gamma_+$ and thus $\gamma_+ = \xi_m$. Since, in a Krasner hyperring, the additive part is commutative, it holds $\beta_+ = \gamma_+ = \xi_m = \varepsilon_m$. □

Corollary 3.12. *For all m-idempotent Krasner hyperrings, we have $\gamma = \varepsilon_m = \xi_m = \alpha$.*

Proof. In all Krasner hyperrings, we have $\beta_+ = \gamma$ and $\gamma_+ = \alpha$. Hence, the proof is completed by Theorem 3.11. □

Remark 3.13. *Note that Corollary 3.12 introduces a characterization for well-known strongly regular relations on m-idempotent Krasner hyperrings by*

$$\beta_+ = \gamma_+ = \gamma = \varepsilon_m = \xi_m = \alpha.$$

The last part of this section is dedicated to the study of some properties of the Cartesian product of hyperrings and its related quotient by the ξ_m^*-relation. We start with a very preliminary result.

Lemma 3.14. *Let R_1 and R_2 be two hyperrings, $(a, b), (c, d) \in R_1 \times R_2$ and $\sigma \in \mathbb{S}_n$. Then $(a, b) \in \sum_{i=1}^n (x_i, y_i)^m$ and $(c, d) \in \sum_{i=1}^n (x_{\sigma(i)}, y_{\sigma(i)})^m$ if and only if $a \in \sum_{i=1}^n x_i^m$, $c \in \sum_{i=1}^n x_{\sigma(i)}^m$, $b \in \sum_{i=1}^n y_i^m$ and $d \in \sum_{i=1}^n y_{\sigma(i)}^m$ for some $x_i \in R_1$ and $y_i \in R_2$.*

Proof. We have

$$\begin{aligned}\sum_{i=1}^{n}(x_i,y_i)^m &= \underbrace{(x_1,y_1)(x_1,y_1)\ldots(x_1,y_1)}_{m\ \text{times}} + \underbrace{(x_2,y_2)(x_2,y_2)\ldots(x_2,y_2)}_{m\ \text{times}} \\ &\quad + \ldots + \underbrace{(x_n,y_n)(x_n,y_n)\ldots(x_n,y_n)}_{m\ \text{times}} \\ &= (x_1^m, y_1^m) + (x_2^m, y_2^m) + \ldots + (x_n^m, y_n^m) \\ &= \left(\sum_{i=1}^{n} x_i^m, \sum_{i=1}^{n} y_i^m\right),\end{aligned}$$

and similarly $\sum_{i=1}^{n}(x_{\sigma(i)}, y_{\sigma(i)})^m = \left(\sum_{i=1}^{n} x_{\sigma(i)}^m, \sum_{i=1}^{n} y_{\sigma(i)}^m\right)$. This completes the proof. □

Corollary 3.15. *Let R_1 and R_2 be two hyperrings, $\xi_m^*(R_1)$, $\xi_m^*(R_2)$ and $\xi_m^*(R_1 \times R_2)$ be the corresponding ξ_m^*-relations on R_1, R_2 and $R_1 \times R_2$, respectively. Then*

$$\big((a,b),(c,d)\big) \in \xi_m^*(R_1 \times R_2) \iff (a,c) \in \xi_m^*(R_1) \quad \textit{and} \quad (b,d) \in \xi_m^*(R_2).$$

Proof. By Lemma 3.14, we have

$$\begin{aligned}\big((a,b),(c,d)\big) \in \xi_m^*(R_1 \times R_2) &\iff \exists n \in \mathbb{N},\ \exists x_i \in R_1,\ \exists y_i \in R_2\ (1 \le i \le n),\ \exists \sigma \in \mathbb{S}_n; \\ &\quad (a,b) \in \sum_{i=1}^{n}(x_i,y_i)^m \quad \text{and} \quad (c,d) \in \sum_{i=1}^{n}\big(x_{\sigma(i)}, y_{\sigma(i)}\big)^m \\ &\iff a \in \sum_{i=1}^{n} x_i^m,\ c \in \sum_{i=1}^{n} x_{\sigma(i)}^m,\ b \in \sum_{i=1}^{n} y_i^m,\ d \in \sum_{i=1}^{n} y_{\sigma(i)}^m \\ &\iff (a,c) \in \xi_m^*(R_1) \quad \text{and} \quad (b,d) \in \xi_m^*(R_2).\end{aligned}$$

□

Theorem 3.16. *Let R_1 and R_2 be two hyperrings, $\xi_m^*(R_1)$, $\xi_m^*(R_2)$ and $\xi_m^*(R_1 \times R_2)$ be the corresponding ξ_m^*-relations on R_1, R_2 and $R_1 \times R_2$, respectively. Then*

$$(R_1 \times R_2)/\xi_m^*(R_1 \times R_2) \cong R_1/\xi_m^*(R_1) \times R_2/\xi_m^*(R_2).$$

Proof. Define the map f from $(R_1 \times R_2)/\xi_m^*(R_1 \times R_2)$ to $R_1/\xi_m^*(R_1) \times R_2/\xi_m^*(R_2)$ by $f(\xi_m^*(R_1 \times R_2)(a,b)) = (\xi_m^*(R_1)(a), \xi_m^*(R_2)(b))$, for $(a,b) \in R_1 \times R_2$. Clearly, f is onto. It is easy to verify that f is well-defined and one to one, by Corollary 3.15. Moreover, f is a homomorphism. □

Now, consider the ring R/ξ_m^* and the canonical projection $\varphi : R \longrightarrow R/\xi_m^*$. Define $K = \ker \varphi = \varphi^{-1}(0_{R/\xi_m^*})$.

Theorem 3.17. *Let $(R, +, \cdot)$ be a Krasner hyperring satisfying relation (2). Then K is a hyperideal of R.*

Proof. For all $x \in R$ we have $x \in x + 0$, since R is a Krasner hyperring. This implies that $\xi_m^*(0) = 0_{R/\xi_m^*}$ and so $0 \in K$. Also, if $x \in K$, since $0 \in x - x$ we have $0_{R/\xi_m^*} = \xi_m^*(0) = \xi_m^*(x) \oplus \xi_m^*(-x) = 0_{R/\xi_m^*} \oplus \xi_m^*(-x) = \xi_m^*(-x)$, that is, $-x \in K$. Now, let $x, y \in K$. For all $z \in x+y$, we have $\xi_m^*(z) = \xi_m^*(x) \oplus \xi_m^*(y) = 0_{R/\xi_m^*} \oplus 0_{R/\xi_m^*} = 0_{R/\xi_m^*}$. Hence, $z \in K$. Thus $x + y \subseteq K$. Moreover, for all $r \in R$ and $x \in K$ we have $\xi_m^*(r \cdot x) = \xi_m^*(r) \odot \xi_m^*(x) = \xi_m^*(r) \odot 0_{R/\xi_m^*} = 0_{R/\xi_m^*}$. Hence $r \cdot x \in K$. Therefore, K is a hyperideal of the Krasner hyperring R by [11, Lemma 3.2.3]. □

4 New form of the α-relation on m-idempotent hyperrings

As already mentioned before, the fundamental relation α^* is the smallest strongly regular relation on hyperrings such that the related quotient is a commutative ring [11]. Theorem 3.7 states that the relation ξ_m^* is also a fundamental relation on m-idempotent hyperrings satisfying relation (2). Besides, ξ_m^* is a new representation for the α^*-relation on m-idempotent hyperrings satisfying relation (2).

Theorem 4.1. *On m-idempotent hyperrings satisfying relation (2), there is $\xi_m = \alpha$.*

Proof. Let $(R, +, \cdot)$ be an m-idempotent hyperring satisfying relation (2). Clearly, $\xi_m \subseteq \alpha$. Let $x \alpha y$ for $x, y \in R$. Then, there exist $n, k_i \in \mathbb{N}$, $z_{i1}, \ldots, z_{ik_i} \in R$, $\sigma \in \mathbb{S}_n$ and $\sigma_i \in \mathbb{S}_{k_i}$, for $1 \leq i \leq n$, such that

$$x \in \sum_{i=1}^{n} \left(\prod_{j=1}^{k_i} z_{ij} \right) \quad \text{and} \quad y \in \sum_{i=1}^{n} \left(\prod_{j=1}^{k_{\sigma(i)}} z_{\sigma(i)\sigma_{\sigma(i)}(j)} \right).$$

Since R is m-idempotent, it follows that

$$x \in \sum_{i=1}^{n} \left(\prod_{j=1}^{k_i} z_{ij} \right) \subseteq \sum_{i=1}^{n} \left(\prod_{j=1}^{k_i} z_{ij}^m \right) \subseteq \sum_{i=1}^{n} \left(\prod_{j=1}^{k_i} z_{ij} \right)^m$$

and

$$y \in \sum_{i=1}^{n} \left(\prod_{j=1}^{k_{\sigma(i)}} z_{\sigma(i)\sigma_{\sigma(i)}(j)} \right) \subseteq \sum_{i=1}^{n} \left(\prod_{j=1}^{k_{\sigma(i)}} z_{\sigma(i)\sigma_{\sigma(i)}(j)}^m \right) \subseteq \sum_{i=1}^{n} \left(\prod_{j=1}^{k_{\sigma(i)}} z_{\sigma(i)\sigma_{\sigma(i)}(j)} \right)^m$$

Now, by relation (2), there exists $t_i \in \prod_{j=1}^{k_i} z_{ij}$ for every $1 \leq i \leq n$ such that $x \in \sum_{i=1}^{n} t_i^m$ and $y \in \sum_{i=1}^{n} t_{\sigma(i)}^m$, which implies that $x\xi_m y$. Then $\alpha \subseteq \xi_m$. This completes the proof. $\square$

This shows that the ξ_m^*-relation is a new form of α^*-relation on m-idempotent hyperrings satisfying relation (2). Based on Theorem 4.1, one immediately notices that Theorem 3.6 and Theorem 3.7 are valid, describing the main properties already known for the α^*-relation.

Corollary 4.2. *On m-idempotent hyperrings satisfying relation (2), we have $\gamma_{\cdot} \subseteq \xi_m$ and so $\gamma_{\cdot}^* \subseteq \xi_m^*$.*

Proof. By Theorem 4.1, we have $\alpha = \xi_m$ on m-idempotent hyperrings satisfying relation (2). Hence, the proof is completed. $\square$

Example 4.3. *Consider the hyperring $R = \{a, b, c, d, e, f, g\}$ in Example 3.1. Then we have*

$$\gamma_{\cdot}(a) = \gamma_{\cdot}(b) = \{a, b\} \quad and \quad \gamma_{\cdot}(x) = \{x\} \quad \forall x \in R - \{a, b\}.$$

Hence, $\gamma_{\cdot} = \xi_m$. Note that R is not m-idempotent for $2 \leq m \in \mathbb{N}$.

Example 4.4. *Consider the 2-idempotent Krasner hyperring $R = \{0, a, b, c\}$ defined by the following hyperaddition and multiplication:*

$+$	0	a	b	c
0	0	a	b	c
a	a	$\{0, b\}$	$\{a, c\}$	b
b	b	$\{a, c\}$	$\{0, b\}$	a
c	c	b	a	0

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	a	b	c
b	0	b	b	0
c	0	c	0	c

We have $\xi_2^*(0) = \xi_2^*(b) = \{0, b\}$ and $\xi_2^*(a) = \xi_2^*(c) = \{a, c\}$. Moreover, one obtains that $\gamma_{\cdot}^*(x) = \{x\}$ for all $x \in R$. Hence, $\gamma_{\cdot}^* \subsetneq \xi_2^*$.

Now, let $(R, +, \cdot)$ be a hyperring. Since $\gamma_{\cdot}^*$ is a strongly regular relation on $(R, \cdot)$, we can consider the additive hyperring $(R/\gamma_{\cdot}^*, \uplus, \odot)$, where

$$\gamma_{\cdot}^*(a) \uplus \gamma_{\cdot}^*(b) = \{\gamma_{\cdot}^*(c) \mid c \in \gamma_{\cdot}^*(a) + \gamma_{\cdot}^*(b)\}$$
$$\gamma_{\cdot}^*(a) \odot \gamma_{\cdot}^*(b) = \gamma_{\cdot}^*(d), \ \forall d \in \gamma_{\cdot}^*(a) \cdot \gamma_{\cdot}^*(b).$$

By [11] and Theorem 4.1, we have $R/\xi_m^* = R/\alpha^* \cong (R/\gamma_{\cdot}^*)/\gamma_{\uplus}^*$, on an m-idempotent hyperring satisfying relation (2). In the following we prove this result without using the α^*-relation and we believe it is more understandable with respect to the similar proof for the α-relation given in [11, Theorem 7.1.6].

Theorem 4.5. *Let $(R, +, \cdot)$ be an m-idempotent hyperring satisfying relation (2). Then $R/\xi_m^* \cong (R/\gamma_\cdot^*)/\gamma_\uplus^*$.*

Proof. The relation $\gamma_\uplus^*$ is a strongly regular relation on $(R/\gamma_\cdot^*, \uplus)$ and so $(R/\gamma_\cdot^*)/\gamma_\uplus^*$ is a commutative ring. Consider the homomorphism $\varphi : R \longrightarrow (R/\gamma_\cdot^*)/\gamma_\uplus^*$ and the equivalence relation $\theta = \{(a, b) \in R \times R \mid \varphi(a) = \varphi(b)\}$ which is a strongly regular relation on R. By [11, Theorem 2.5.4], $R/\theta \cong (R/\gamma_\cdot^*)/\gamma_\uplus^*$. Clearly, $\xi_m^* \subseteq \theta$ by Theorem 3.7. Hence, it is enough to show that $\theta \subseteq \xi_m^*$. We have

$$\theta(a) = \{z \mid z\theta a\} = \{z \mid \gamma_\uplus^*(\gamma_\cdot^*(a)) = \gamma_\uplus^*(\gamma_\cdot^*(z))\} = \{z \mid \gamma_\cdot^*(a)\gamma_\uplus^*\gamma_\cdot^*(z)\}.$$

This means that if $z \in \theta(a)$, then $\gamma_\cdot^*(z)\gamma_\uplus^*\gamma_\cdot^*(a)$ and so there exist $\sigma \in \mathbb{S}_n$ and $\gamma_\cdot^*(x_1), \ldots, \gamma_\cdot^*(x_n) \in R/\gamma_\cdot^*$ such that

$$\gamma_\cdot^*(a) \in \biguplus_{i=1}^{n} \gamma_\cdot^*(x_i) \quad \text{and} \quad \gamma_\cdot^*(z) \in \biguplus_{i=1}^{n} \gamma_\cdot^*(x_{\sigma(i)}).$$

Therefore,

$$\begin{aligned} &\gamma_\cdot^*(a) = \gamma_\cdot^*(c); \quad c \in \gamma_\cdot^*(x_1) + \ldots + \gamma_\cdot^*(x_n) = \sum_{i=1}^{n} \gamma_\cdot^*(x_i), \\ &\gamma_\cdot^*(z) = \gamma_\cdot^*(d); \quad d \in \sum_{i=1}^{n} \gamma_\cdot^*(x_{\sigma(i)}). \end{aligned}$$

By Corollary 4.2, from $a\gamma_\cdot^*c$ and $z\gamma_\cdot^*d$ we have $\xi_m^*(a) = \xi_m^*(c)$ and $\xi_m^*(z) = \xi_m^*(d)$. Moreover, since R is m-idempotent, it follows that, for every set $A \subseteq R$, there is $A \subseteq A^m$. Hence,

$$c \in \sum_{i=1}^{n} \gamma_\cdot^*(x_i)^m \quad \text{and} \quad d \in \sum_{i=1}^{n} \gamma_\cdot^*(x_{\sigma(i)})^m.$$

According with relation (2), there exist $t_i \in \gamma_\cdot^*(x_i)$ such that $c \in \sum_{i=1}^{n} t_i^m$ and $d \in \sum_{i=1}^{n} t_{\sigma(i)}^m$. Then, $c\xi_m^*d$ and thus $\xi_m^*(c) = \xi_m^*(d)$. Hence, $\xi_m^*(z) = \xi_m^*(a)$ which implies that $\theta \subseteq \xi_m^*$. □

Finally, we determine on which hyperrings the two relations ε_m^* and ξ_m^* coincide. We recall that a weak commutative hyperring is a hyperring $(R, +, \cdot)$ where for all $x, y \in R$ we have $(x + y) \cap (y + x) \neq \emptyset$ and $(x \cdot y) \cap (y \cdot x) \neq \emptyset$.

Theorem 4.6. *On weak commutative m-idempotent hyperrings satisfying relation (2), there is $\varepsilon_m^* = \xi_m^*$.*

Proof. By [19], we know that $\varepsilon_m^* = \gamma^*$ on m-idempotent hyperrings satisfying relation (1). Besides, by Theorem 4.1, $\xi_m^* = \alpha^*$ on m-idempotent hyperrings satisfying relation (2). On the other hand, $\gamma^* = \alpha^*$ on weak commutative hyperrings by [11, Theorem 7.1.13]. Hence, we conclude that, on weak commutative m-idempotent hyperrings satisfying relation (2), there is

$$\varepsilon_m^* = \gamma^* = \alpha^* = \xi_m^*.$$

□

5 Conclusions

On a general hyperring the fundamental relation γ^* was defined [21] such that the related quotient structure is a ring (not always commutative). Later on, by defining the α^*-relation [12], the problem of the commutativity was solved. But, in a recently published work [19], a particular class of hyperrings was introduced, on which γ^* is not the smallest strongly regular relation, but there exists a smaller one, the ε_m-relation, having a similar behaviour as γ^*. Moreover, on m-idempotent hyperrings satisfying a certain condition 1, these two relations are equal. Following the same idea, but with the aim to obtain a commutative quotient ring, in this note we have introduced the ξ_m^*-relation on general hyperrings having commutative multiplicative part and satisfying a certain condition 2. On such hyperrings, ξ_m^* is a strongly regular relation and the corresponding quotient structure is a ring (not commutative in general), that is always commutative on m-idempotent hyperrings satisfying condition 2. In this case, ξ_m is a new form for the α-relation. Moreover, all four fundamental relations γ, α, ε_m and ξ_m coincide on m-idempotent Krasner hyperrings, while on weak commutative m-idempotent hyperrings satisfying condition 2, there is $\varepsilon_m^* = \gamma^* = \alpha^* = \xi_m^*$.

References

[1] H. Aghabozorgi, B. Davvaz, M. Jafarpour, *Nilpotent groups derived from hypergroups*, J. Algebra, 382 (2013) 177-184.

[2] R. Ameri, E. Mohammadzadeh, *Engel groups derived from hypergroups*, European J. Combin., 44 (2015) 191-197.

[3] R. Ameri, T. Nozari, *A new characterization of fundamental relation on hyperrings,* Int. J. Contemp. Math. Sci. 5 (15) (2010) 721-738.

[4] S.M. Anvariyeh, B. Davvaz, *Strongly transitive geometric spaces associated to hypermodules*, J. Algebra, 322 (2009) 1340-1359.

[5] S.M. Anvariyeh, S. Mirvakili, B. Davvaz, *θ-relation on hypermodules and fundamental modules over commutative fundamental rings*, Comm. Algebra, 36 (2) (2008) 622-631.

[6] A. Asokkumar, *Derivations in hyperrings and prime hyperrings,* Iran. J. Math. Sci. Inform., 8 (1) (2013) 1-13.

[7] S.P. Borgatti, M.G. Everett, *The class of all regular equivalences: algebraic structures and computation*, Soc. Networks, 11(1989), 65-88.

[8] P. Corsini, *Prolegomena of Hypergroup Theory,* Aviani Editore,1993.

[9] I. Cristea, M. Stefanescu, C. Angeluta, *About the fundamental relations defined on hypergroupoids associated with binary relations,* European. J. Combin., 32 (1) (2011) 72-81.

[10] B. Davvaz, P. Ghiasvand, S. Mirvakili, *Boolean rings obtained from hyperrings with $\eta^*_{1,m}$-relations,* Iran. J. Sci. Technol. Trans. A Sci., In press.

[11] B. Davvaz, V. Leoreanu-Fotea, *Hyperring Theory and Applications,* International Academic Press, USA, 2007.

[12] B. Davvaz and T. Vougiouklis, *Commutative rings obtained from hyperrings (Hv-rings) with α^*-relations,* Comm. Algebra, 35 (11) (2007) 3307-3320.

[13] D. Freni, *A new characterization of the derived hypergroup via strongly regular equivalences,* Comm. Algebra, 30 (8) (2002) 3977-3989.

[14] K. Hila and B. Davvaz, *An introduction to the theory of algebraic multihyperring spaces,* An. Şt. Univ. Ovidius Constanţa, 22 (3) (2014) 59-72.

[15] M. Jafarpour, H. Aghabozorgi, B. Davvaz, *Solvable groups derived from hypergroups*, J. Algebra Appl. 15, 1650067 (2016) (9 pages).

[16] J. Jantosciak, *Algebraic hyperstructures and applications,* (Xanthi, 1990), 119-122, World Sci. Publ., Teaneck, NJ, 1991.

[17] M. Koskas, *Groupoids, demi-hypergroupes et hypergroupes,* J. Math. Pures Appl., 49 (9) (1970) 155-192.

[18] M. Krasner, *A class of hyperrings and hyperfields,* Int. J. Math. Math. Sci., 6 (2) (1983) 307-311.

[19] M. Norouzi, I. Cristea, *Fundamental relation on m-idempotent hyperrings*, Open. Math., 15 (2017) 1558-1567.

[20] M. Norouzi, I. Cristea, *Transitivity of the ε_m-relation on (m-idempotent) hyperrings*, Open. Math., 16 (2018) 1012-1021.

[21] T. Vougiouklis, *The fundamental relation in hyperrings. The general hyperfield,* Algebraic hyperstructures and applications (Xanthi, 1990), 203-211, World Sci. Publishing, Teaneck, NJ, 1991.

[22] T. Vougiouklis, *Hyperstructures and their representations,* Hadronic Press Inc., Palm Harbor, 1994.

Azam ADINEH ZADEH,
Department of Mathematics,
Faculty of Basic Sciences,
University of Bojnord,
Bojnord, Iran.
Email: a.adinezadeh@gmail.com

Morteza NOROUZI,
Department of Mathematics,
Faculty of Basic Sciences,
University of Bojnord,
Bojnord, Iran.
Email: m.norouzi@ub.ac.ir, m.norouzi65@yahoo.com

Irina CRISTEA,
Center for Information Technologies and Applied Mathematics,
University of Nova Gorica,
Nova Gorica, Slovenia.
Email: irinacri@yahoo.co.uk, irina.cristea@ung.si