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On a local version of Jack's lemma

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Abstract

The purpose of this paper is to provide a result which concerns with the boundary behavior of analytic functions. It may be a local version of the well known Jack's lemma when we change the function normalization at the origin.

1 Introduction

Let $\mathcal{H}$ denote the class of analytic functions in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{A}(p)$ denote the class of all functions analytic in the unit disk $\mathbb{D}$ which have the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n}, \quad z \in \mathbb{D}, \tag{1}$$

where p is positive integer. In this section we develop a key lemma that forms the groundwork for many of the results. It is a local version of the following lemma, well known as the Jack's lemma.

Lemma 1.1. *[1] Let $w(z)$ be non-constant and analytic function in the unit disc $\mathbb{D}$ with $w(0) = 0$. If $|w(z)|$ attains its maximum value on the disc $|z| \le r$ at the point z_0, $|z_0| = r$, then $z_0 w'(z_0) = kw(z_0)$ and $k \ge 1$.*

The Jack's lemma has found several of the applications and generalizations in the theory of differential subordinations, see for instance [2], [3] and [4]. In this paper we generalize the following Nunokawa's lemma, [5], see also [6] for its angle version.

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Lemma 1.2. *Let p be analytic function in $|z| < 1$, with $p(0) = 1$. If there exists a point z_0, $|z_0| < 1$, such that $\mathfrak{Re}\{p(z)\} > 0$ for $|z| < |z_0|$ and $p(z_0) = \pm ia$ for some $a > 0$, then we have*

$$\frac{z_0 p'(z_0)}{p(z_0)} = \frac{2ik \arg\{p(z_0)\}}{\pi}, \quad \arg\{p(z_0)\} = \pm\frac{\pi}{2}$$

for some $k \geq (a + a^{-1})/2 \geq 1$.

Lemma 1.3. *Let $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ be analytic in $\mathbb{D}$ with $p(z) \neq 0$ therein. If there exists a point z_1, $0 < |z_1| < 1$ and the sector $S_\delta(z_1)$, for which*

$$\max\{z \in S_\delta(z_1) : |p(z)|\} = |p(z_1)| \tag{2}$$

where $z_1 = |z_1|e^{i\theta_1}$

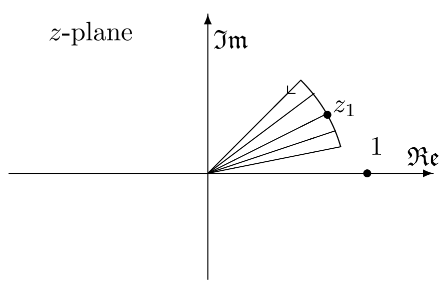
$$S_\delta(z_1) = \{re^{i\theta} : 0 \leq r \leq |z_1|,\ |\theta - \theta_1| \leq \delta\},$$

then we have

$$\frac{z_1 p'(z_1)}{p(z_1)} \in \mathbb{R}, \quad \frac{z_1 p'(z_1)}{p(z_1)} \geq 0, \tag{3}$$

moreover

$$\mathfrak{Re}\left\{1 + \frac{z_1 p''(z_1)}{p'(z_1)}\right\} \geq \frac{z_1 p'(z_1)}{p(z_1)} \geq 0. \tag{4}$$



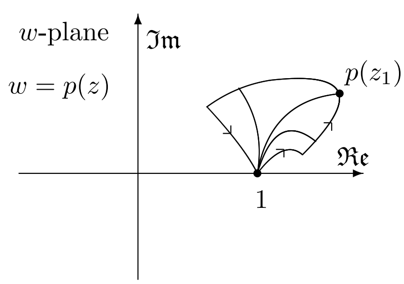


Fig.1. ***z*-plane.** **Fig.2.** ***w*-plane.**

Proof. From the hypothesis, we can have the above pictures, Fig. 1. and Fig. 2. Then it follows that

$$\frac{zp'(z)}{p(z)} = \frac{\mathrm{d}\log|p(z)| + i\mathrm{d}\arg\{p(z)\}}{i\mathrm{d}\theta} = \frac{\mathrm{d}\arg\{p(z)\}}{\mathrm{d}\theta} - i\frac{1}{|p(z)|}\frac{\mathrm{d}|p(z)|}{\mathrm{d}\theta}, \tag{5}$$

where z moves on the arc $z = |z_1|e^{i\theta}$ and $\theta_1 - \delta \le \theta \le \theta_1 + \delta$. From the hypothesis, we have also

$$\left(\frac{\mathrm{d}|p(z)|}{\mathrm{d}\theta}\right)_{z=z_1} = 0 \tag{6}$$

and from geometrical observation, we have

$$\left(\frac{\mathrm{d}\arg\{p(z)\}}{\mathrm{d}\theta}\right)_{z=z_1} \ge 0. \tag{7}$$

It completes the proof of (3). To prove (4) let us put

$$q(z) = \frac{zp'(z)}{p(z)}, \quad q(0) = 0. \tag{8}$$

From the hypothesis, $q(z)$ is analytic in $\mathbb{D}$ and

$$q(z) \ne 0, \quad z \in S_\delta(z_1).$$

Then it follows that

$$q(z) = \frac{zp'(z)}{p(z)} = \frac{\mathrm{d}\arg\{p(z)\}}{\mathrm{d}\theta} - i\frac{1}{|p(z)|}\frac{\mathrm{d}|p(z)|}{\mathrm{d}\theta},$$

where $z = |z_1|e^{i\theta}$ and $\theta_1 - \delta \le \theta \le \theta_1 + \delta$. Then, from the above picture, we have

$$\frac{\mathrm{d}|p(z)|}{\mathrm{d}\theta} \ge 0, \quad \theta_1 - \delta \le \theta \le \theta_1$$

and

$$\frac{\mathrm{d}|p(z)|}{\mathrm{d}\theta} \le 0, \quad \theta_1 \le \theta \le \theta_1 + \delta.$$

Therefore, we have

$$\begin{aligned} \mathfrak{Im}\{q(z)\}) &< 0 \text{ for } \theta_1 - \delta \le \theta \le \theta_1, \\ \mathfrak{Im}\{q(z)\}) &= 0 \text{ for } \theta = \theta_1, \\ \mathfrak{Im}\{q(z)\}) &> 0 \text{ for } \theta_1 \le \theta \le \theta_1 + \delta. \end{aligned}$$

This shows that

$$\begin{aligned} \left(\frac{\mathrm{d}\arg\{q(z)\}}{\mathrm{d}\theta}\right)_{z=z_1} &= \mathfrak{Re}\left\{\frac{zq'(z)}{q(z)}\right\}_{z=z_1} \\ &= \mathfrak{Re}\left\{1 + \frac{zp''(z)}{p'(z)} - \frac{zp'(z)}{p(z)}\right\}_{z=z_1} \\ &\ge 0 \end{aligned}$$

This shows that

$$1+\mathfrak{Re}\left\{1+\frac{z_1p''(z_1)}{p'(z_1)}\right\}\geq \mathfrak{Re}\left\{\frac{z_1p'(z_1)}{p(z_1)}\right\}=\frac{z_1p'(z_1)}{p(z_1)}.$$

It completes the proof of (4).□

Remark The results of Lemma 1.3 and Theorem 2.1 below, hold to be correct not only for the case $|p(z)|$ and $|f(z)|$ take its local maximum value at the point $z=z_0$ in the domain $|z|\leq|z_0|$ but at the point z_1 in the subset $S_\delta(z_1)\subset\mathbb{D}$. It is an improvement of the known results from [1] and [4]. Lemma 1.3 is applicable for the points $z=\alpha$ and not for $z=\beta$, Fig. 3.

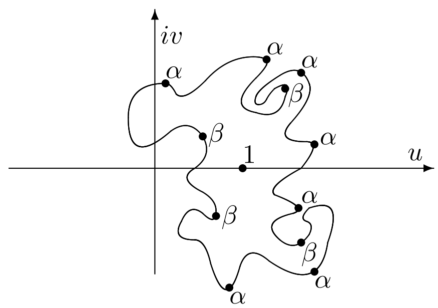


Fig.3. $p(|z|\leq|z_1|)$.

2 Applications

Theorem 2.1. *Let $f(z)=z^p+\sum_{n=p+1}^{\infty}a_nz^n$, $1\leq p$, be analytic and p-valent in $\mathbb{D}$. If there exists a point z_1, $0<|z_1|<1$ and the sector $S_\delta(z_1)$, for which*

$$\max\{z\in S_\delta(z_1):|f(z)|\}=|f(z_1)|, \tag{9}$$

where $z_1=|z_1|e^{i\theta_1}$ and

$$S_\delta(z_1)=\{re^{i\theta}:0\leq r\leq|z_1|,\ |\theta-\theta_1|<\delta\},$$

then we have

$$\frac{z_1f'(z_1)}{f(z_1)}\in\mathbb{R},\quad \frac{z_1f'(z_1)}{f(z_1)}\geq p, \tag{10}$$

moreover

$$\mathfrak{Re}\left\{1+\frac{z_1 f''(z_1)}{f'(z_1)}\right\} \geq \mathfrak{Re}\left\{\frac{z_1 f'(z_1)}{f'(z_1)}\right\} = \frac{z_1 f'(z_1)}{f'(z_1)} \geq p. \tag{11}$$

Proof. For the proof of (10), let us put

$$p(z) = \frac{f(z)}{z^p}, \quad p(0) = 1.$$

From the hypothesis, we have that $p(z)$ is analytic in $\mathbb{D}$ and $p(z) \neq 0$ in $\mathbb{D}$ since $f(z)$ is p-valent in $\mathbb{D}$. Then it follows that $|p(z)|$ takes its maximum value at the point $z = z_1$ in the sector $S_\delta(z_1)$. Therefore, applying Lemma 1.3, we have

$$\begin{aligned} \frac{z_1 p'(z_1)}{p(z_1)} &= \mathfrak{Re}\left\{\frac{z_1 p'(z_1)}{p(z_1)}\right\} \\ &= \frac{z_1 f'(z_1)}{f(z_1)} - p \\ &= \mathfrak{Re}\left\{\frac{z_1 f'(z_1)}{f(z_1)}\right\} - p \\ &\geq 0. \end{aligned}$$

It completes the proof of (10).

For the proof of (11), let us put

$$q(z) = \frac{zf'(z)}{pf(z)}, \quad q(0) = 1.$$

From the hypothesis, and from (10), $q(z)$ is analytic in $\mathbb{D}$ and

$$\frac{z_1 f'(z_1)}{f(z_1)} \geq p^2 > 0.$$

Applying Lemma 1.3, we have

$$\begin{aligned} \frac{z_1 q'(z_1)}{q(z_1)} &= \mathfrak{Re}\left\{1+\frac{z_1 f''(z_1)}{f'(z_1)} - \frac{z_1 f'(z_1)}{f(z_1)}\right\} \\ &= \mathfrak{Re}\left\{1+\frac{z_1 f''(z_1)}{f'(z_1)}\right\} - \mathfrak{Re}\left\{\frac{z_1 f'(z_1)}{f(z_1)}\right\} \\ &\geq 0. \end{aligned}$$

this shows that

$$1+\mathfrak{Re}\left\{\frac{z_1 f''(z_1)}{f'(z_1)}\right\} \geq \mathfrak{Re}\left\{\frac{z_1 f'(z_1)}{f(z_1)}\right\} = \frac{z_1 f'(z_1)}{f(z_1)} \geq 0.$$

It completes the proof of (11). □

Lemma 2.2. *Let $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ be analytic in $\mathbb{D}$ with $p(z) \neq 0$ with $p(z) \neq 0$ therein. If there exists a point z_1, $0 < |z_1| < 1$ and the sector $S_\delta(z_1)$, for which*

$$\min\{|z| \le r < 1 : |p(z)|\} = |p(z_1)| \tag{12}$$

where $|z_1| = r < 1$. Then we have

$$\frac{z_1 p'(z_1)}{p(z_1)} \in \mathbb{R}, \quad \frac{z_1 p'(z_1)}{p(z_1)} \le 0, \tag{13}$$

moreover

$$\mathfrak{Re}\left\{1 + \frac{z_1 p''(z_1)}{p'(z_1)}\right\} \le \frac{z_1 p'(z_1)}{p(z_1)} \le 0. \tag{14}$$

Proof. then we have

$$\begin{aligned}
\frac{z_1 p'(z_1)}{p(z_1)} &= \left.\frac{\mathrm{d}\log p(z)}{\mathrm{d}\log z}\right|_{z=z_1} \\
&= \left.\frac{\mathrm{d}\log|p(z)| + i\mathrm{d}\arg\{p(z)\}}{i\mathrm{d}\varphi}\right|_{z=z_1} \\
&= \left.\frac{\mathrm{d}\arg\{p(z)\}}{\mathrm{d}\varphi} - \frac{i}{|p(z)|}\frac{\mathrm{d}|p(z)|}{\mathrm{d}\varphi}\right|_{z=z_1} \\
&= \left.\frac{\mathrm{d}\arg\{p(z)\}}{\mathrm{d}\varphi}\right|_{z=z_1} \\
&\le 0,
\end{aligned} \tag{15}$$

because of (12). This gives (13). For the proof of (14) consider

$$\begin{aligned}
\frac{\mathrm{d}\log\left(\frac{zp'(z)}{p(z)}\right)}{\mathrm{d}\log\{z\}} &= \frac{\mathrm{d}\log\left|\frac{zp'(z)}{p(z)}\right|}{i\mathrm{d}\theta} - \frac{i}{i\mathrm{d}\theta}\left(\frac{1}{|p(z)|}\frac{\mathrm{d}|p(z)|}{\mathrm{d}\theta}\right) \\
&= -\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{1}{|p(z)|}\frac{\mathrm{d}|p(z)|}{\mathrm{d}\theta}\right) - i\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{\mathrm{d}\arg\{p(z)\}}{\mathrm{d}\theta}\right) \\
&= \frac{1}{|p(z)|^2}\left(\frac{\mathrm{d}|p(z)|}{\mathrm{d}\theta}\right)^2 - \frac{1}{|p(z)|}\left(\frac{\mathrm{d}^2|p(z)|}{\mathrm{d}\theta^2}\right) - i\frac{\mathrm{d}^2\arg\{p(z)\}}{\mathrm{d}\theta^2} \\
&= 1 + \frac{zp''(z)}{p'(z)} - \frac{zp'(z)}{p(z)},
\end{aligned}$$

where $z = re^{i\theta}$ and $0 \le \theta \le 2\pi$. If we put $z = z_1$, then we have

$$\begin{aligned} & 1 + \frac{z_1 p''(z_1)}{p'(z_1)} - \frac{z_1 p'(z_1)}{p(z_1)} \\ = \quad & \frac{1}{|p(z)|^2}\left(\frac{\mathrm{d}|p(z)|}{\mathrm{d}\theta}\right)^2_{z=z_1} - \frac{1}{|p(z)|}\left(\frac{\mathrm{d}^2|p(z)|}{\mathrm{d}\theta^2}\right)_{z=z_1} - i\left(\frac{\mathrm{d}^2 \arg\{p(z)\}}{\mathrm{d}\theta^2}\right)_{z=z_1} \\ = \quad & -\frac{1}{|p(z)|}\left(\frac{\mathrm{d}^2|p(z)|}{\mathrm{d}\theta^2}\right)_{z=z_1} - i\left(\frac{\mathrm{d}^2 \arg\{p(z)\}}{\mathrm{d}\theta^2}\right)_{z=z_1} \end{aligned}$$

because of (12). Therefore,

$$\begin{aligned} & \mathfrak{Re}\left\{1 + \frac{z_1 p''(z_1)}{p'(z_1)} - \frac{z_1 p'(z_1)}{p(z_1)}\right\} \\ = \quad & -\frac{1}{|p(z)|}\left(\frac{\mathrm{d}^2|p(z)|}{\mathrm{d}\theta^2}\right)_{z=z_1} \\ \le \quad & 0 \end{aligned}$$

because $|p(z)|$ attains its minimum value at $z = z_1$, and from the known geometric property, we have

$$\left(\frac{\mathrm{d}^2|p(z)|}{\mathrm{d}\theta^2}\right)_{z=z_1} \ge 0.$$

It completes the proof of (14).□

Applying Lemma 2.2 and the same method as in the proof of Theorem 2.1 we can proof the following theorem.

Theorem 2.3. *Let $f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n$, $1 \le p$, be analytic and p-valent in $\mathbb{D}$. If there exists a point z_1, $0 < |z_1| < 1$ and the sector $S_\delta(z_1)$, for which*

$$\max\{z \in S_\delta(z_1) : |f(z)|\} = |f(z_1)|, \tag{16}$$

where $z_1 = |z_1|e^{i\theta_1}$ and

$$S_\delta(z_1) = \{re^{i\theta} : 0 \le r \le |z_1|,\ |\theta - \theta_1| < \delta\},$$

then we have

$$\frac{z_1 f'(z_1)}{f(z_1)} \in \mathbb{R}, \quad \frac{z_1 f'(z_1)}{f(z_1)} \le p, \tag{17}$$

moreover

$$\mathfrak{Re}\left\{1 + \frac{z_1 f''(z_1)}{f'(z_1)}\right\} \le \mathfrak{Re}\left\{\frac{z_1 f'(z_1)}{f'(z_1)}\right\} = \frac{z_1 f'(z_1)}{f'(z_1)} \le p. \tag{18}$$

For some related results we refer to [7, 8, 9].

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